INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

By W.E. Diewert. April, 2006.

CHAPTER 10: Elementary Indexes[1]

1. Introduction

In all countries, the calculation of a Consumer Price Index proceeds in two (or more) stages. In the first stage of calculation, *elementary price indexes* are estimated for the *elementary expenditure aggregates* of a CPI. In the second and higher stages of aggregation, these elementary price indexes are combined to obtain higher level indexes using information on the expenditures on each elementary aggregates as weights. An elementary aggregate consists of the expenditures on a small and relatively homogeneous set of products defined within the consumption classification used in the CPI. Samples of prices are collected within each elementary aggregate, so that elementary aggregates serve as strata for sampling purposes.

Data on the expenditures, or quantities, of the different goods and services are typically not available within an elementary aggregate. As there are no quantity or expenditure weights, most of the index number theory outlined in the previous chapters is not directly applicable. An elementary price index is a more primitive concept that relies on price data only.

The question of what is the most appropriate formula to use to estimate an elementary price index is considered in this Chapter. The quality of a CPI depends heavily on the quality of the elementary indexes, which are the basic building blocks from which CPIs are constructed.

CPI compilers have to select *representative products* within an elementary aggregate and then collect a sample of prices for each of the representative products, usually from a sample of different outlets. The individual products whose prices are actually collected are described as the *sampled products*. Their prices are collected over a succession of time periods. An elementary price index is therefore typically calculated from two sets of matched price observations. It is assumed in this chapter that there are no missing observations and no changes in the quality of the products sampled so that the two sets of prices are perfectly matched. The treatment of new and disappearing products, and of quality change, is a separate and complex issue that is discussed in detail in a later chapter.

Even though quantity or expenditure weights are usually not available to weight the individual elementary price quotes, it is useful to consider an *ideal framework* where expenditure information is available. This is done in section 2 below. The problems involved in aggregating narrowly defined price quotes over *time* are also discussed in this

[1] Bert Balk provided a considerable amount of help in drafting this chapter.

section. Thus the discussion in section 2 provides a theoretical target for "practical" elementary price indexes that are constructed using only information on prices.

Section 3 introduces the main elementary index formulae that are used in practice and section 4 develops some numerical relationships between the various indexes.

Previous chapters developed the various approaches to index number theory when information on both prices and quantities was available. It is also possible to develop axiomatic, economic or sampling approaches to elementary indexes and these three approaches are discussed below in sections 4, 5 and 6 respectively. Section 8 develops a simple statistical approach to elementary indexes that resembles a highly simplified hedonic regression model.

Section 9 concludes with an overview of the various results.[2]

2. Ideal Elementary Indexes

The aggregates covered by a CPI or a PPI are usually arranged in the form of a tree like hierarchy (such as COICOP or NACE). Any *aggregate* is a set of economic transactions pertaining to a set of commodities over a specified time period. Every economic transaction relates to the change of ownership of a specific, well defined commodity (good or service) at a particular place and date, and comes with a quantity and a price. The price index for an aggregate is calculated as a weighted average of the price indexes for the subaggregates, the (expenditure or sales) weights and type of average being determined by the index formula. One can descend in such a hierarchy as far as available information allows the weights to be decomposed. The lowest level aggregates are called *elementary* aggregates. They are basically of two types:

- those for which all detailed price and quantity information is available;
- those for which the statistician, considering the operational cost and/or the response burden of getting detailed price and quantity information about all the transactions, decides to make use of a representative sample of commodities and/or respondents.

The practical relevance of studying this topic is large. Since the elementary aggregates form the building blocks of a CPI or a PPI, the choice of an inappropriate formula at this level can have a tremendous impact on the overall index.

In this section, it will be assumed that detailed price and quantity information for all transactions pertaining to the elementary aggregate for the two time periods under consideration is available. This assumption allows us to define an *ideal elementary aggregate*. Subsequent sections will relax this strong assumption about the availability of detailed price and quantity data on transactions but it is necessary to have a theoretically ideal target for the "practical" elementary index.

[2] This Chapter draws heavily on the recent contributions of Dalén (1992), Balk (1994) (1998) (2002) and Diewert (1995) (2002).

The detailed price and quantity data, although perhaps not available to the statistician, is in principle, available in the outside world. It is frequently the case, that at the respondent level (i.e., at the outlet or firm level), some aggregation of the individual transactions information has been executed, usually in a form that suits the respondent's financial or management information system. This respondent determined level of information could be called the *basic information level*. This is, however, not necessarily the finest level of information that could be made available to the price statistician. One could always ask the respondent to provide more disaggregated information. For instance, instead of monthly data one could ask for weekly data; or, whenever appropriate, one could ask for regional instead of global data; or, one could ask for data according to a finer commodity classification. The only natural barrier to further disaggregation is the individual transaction level.[3]

It is now necessary to discuss a problem that arises when detailed data on *individual transactions* are available, either at the level of the individual household or at the level of an individual outlet. Recall that in previous chapters, the price and quantity indexes, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, were introduced. These (bilateral) price and quantity indexes decomposed the value ratio V^1/V^0 into a price change part $P(p^0,p^1,q^0,q^1)$ and a quantity change part $Q(p^0,p^1,q^0,q^1)$. In this framework, it was taken for granted that the period t price and quantity for commodity i, p_i^t and q_i^t respectively, were well defined. However, these definitions are not straightforward since individual consumers may purchase the *same* item during period t *at different prices*. Similarly, if we look at the sales of a particular shop or outlet that sells to consumers*, the same item may sell at very different prices during the course of the period*. Hence before a traditional bilateral price index of the form $P(p^0,p^1,q^0,q^1)$ considered in previous chapters can be applied, there is a non trivial *time aggregation problem* that must be solved in order to obtain the basic prices p_i^t and q_i^t that are the components of the price vectors p^0 and p^1 and the quantity vectors q^0 and q^1.

Walsh[4] and Davies (1924) (1932), suggested a solution to this time aggregation problem: the appropriate quantity at this very first stage of aggregation is the *total quantity purchased* of the narrowly defined item and the corresponding price is the value of purchases of this item divided by the total amount purchased, which is a *narrowly defined unit value*. In more recent times, other researchers have adopted the Walsh and Davies

[3] See Balk (1994) for a similar approach.

[4] Walsh explained his reasoning as follows: "Of all the prices reported of the same kind of article, the average to be drawn is the arithmetic; and the prices should be weighted according to the relative mass quantities that were sold at them." Correa Moylan Walsh (1901; 96). "Some nice questions arise as to whether only what is consumed in the country, or only what is produced in it, or both together are to be counted; and also there are difficulties as to the single price quotation that is to be given at each period to each commodity, since this, too, must be an average. Throughout the country during the period a commodity is not sold at one price, nor even at one wholesale price in its principal market. Various quantities of it are sold at different prices, and the full value is obtained by adding all the sums spent (at the same stage in its advance towards the consumer), and the average price is found by dividing the total sum (or the full value) by the total quantities." Correa Moylan Walsh (1921; 88).

solution to the time aggregation problem.[5] Note that this solution to the time aggregation problem has the following advantages:

- The quantity aggregate is intuitively plausible, being the total quantity of the narrowly defined item purchased by the household (or sold by the outlet) during the time period under consideration;
- The product of the price times quantity equals the total value purchased by the household (or sold by the outlet) during the time period under consideration.

We will adopt this solution to the time aggregation problem as our concept for the price and quantity at this very first stage of aggregation.

Having decided on an appropriate theoretical definition of price and quantity for an item at the very lowest level of aggregation (i.e., a narrowly defined unit value and the total quantity sold of that item at the individual outlet), we now consider how to aggregate these narrowly defined elementary prices and quantities into an overall elementary aggregate. Suppose that there are M lowest level items or specific commodities in this chosen elementary category. Denote the period t quantity of item m by q_m^t and the corresponding time aggregated unit value by p_m^t for t = 0,1 and for items m = 1,2,...,M. Define the period t quantity and price vectors as $q^t \equiv [q_1^t,q_2^t,...,q_M^t]$ and $p^t \equiv [p_1^t,p_2^t,...,p_M^t]$ for t = 0,1. It is now necessary to choose a theoretically ideal index number formula $P(p^0,p^1,q^0,q^1)$ that will aggregate the individual item prices into an overall aggregate price relative for the M items in the chosen elementary aggregate. However, this problem of choosing a functional form for $P(p^0,p^1,q^0,q^1)$ is identical to the overall index number problem that was addressed in previous chapters. In these previous chapters, four different approaches to index number theory were studied that led to specific index number formulae as being "best" from each perspective. From the viewpoint of *fixed basket approaches*, it was found that the Fisher (1922) and Walsh (1901) price indexes, P_F and P_W, appeared to be "best". From the viewpoint of the *test approach*, the Fisher index appeared to be "best". From the viewpoint of the *stochastic approach* to index number theory, the Törnqvist Theil (1967) index number formula P_T emerged as being "best". Finally, from the viewpoint of the *economic approach* to index number theory, the Walsh price index P_W, the Fisher ideal index P_F and the Törnqvist Theil index number formula P_T were all regarded as being equally desirable. It was also shown that the same three index number formulae numerically approximate each other very closely and so it will not matter very much which of these alternative indexes is chosen.[6] Hence, the *theoretically ideal elementary index number formula* is taken to be one of the three formulae $P_F(p^0,p^1,q^0,q^1)$, $P_W(p^0,p^1,q^0,q^1)$ or $P_T(p^0,p^1,q^0,q^1)$ where the period t quantity of item m, q_m^t, is the total quantity of that narrowly defined item purchased by the household during period t (or sold by the outlet during period t) and the corresponding

[5] See for example Szulc (1987; 13), Dalén (1992; 135) , Reinsdorf (1994), Diewert (1995; 20-21), Reinsdorf and Moulton (1997) and Balk (2002).

[6] Theorem 5 in Diewert (1978; 888) showed that P_F, P_T and P_W will approximate each other to the second order around an equal price and quantity point; see Diewert (1978; 894), Hill (2004) and Chapter 11 for some empirical results.

price for item m is p_m^t, the time aggregated unit value, for t = 0,1 and for items m = 1,2,...,M.

In the following section, various "practical" elementary price indexes will be defined. These indexes do not have quantity weights and thus are functions only of the price vectors p^0 and p^1, which contain time aggregated unit values for the M items in the elementary aggregate for periods 0 and 1. Thus when a practical elementary index number formula, say $P_E(p^0,p^1)$, is compared to an ideal elementary price index, say the Fisher price index $P_F(p^0,p^1,q^0,q^1)$, then obviously P_E will differ from P_F because the prices are not weighted according to their economic importance in the practical elementary formula. Call this difference between the two index number formulae *formula approximation error*.

Practical elementary indexes are subject to two other types of error:

- The statistical agency may not be able to collect information on all M prices in the elementary aggregate; i.e., only a *sample* of the M prices may be collected. Call the resulting divergence between the incomplete elementary aggregate and the theoretically ideal elementary index, the *sampling error*.
- Even if a price for a narrowly defined item is collected by the statistical agency, it may not be equal to the theoretically appropriate time aggregated unit value price. This use of an inappropriate price at the very lowest level of aggregation gives rise to *time aggregation error*.[7]

In section 7 below, a sampling framework for the collection of prices that can reduce the above three types of error will be discussed. In section 3 below, the five main elementary index number formulae are defined and in section 4, various numerical relationships between these five indexes are developed. Sections 5 and 6 develop the axiomatic and economic approaches to elementary indexes and the five main elementary formulae used in practice will be evaluated in the light of these approaches.

3. Elementary Indexes used in Practice

Suppose that there are M lowest level items or specific commodities in a chosen elementary category. Denote the period t price of item m by p_m^t for t = 0,1 and for items m = 1,2,...,M. Define the period t price vector as $p^t \equiv [p_1^t,p_2^t,...,p_M^t]$ for t = 0,1.

The first widely used elementary index number formula is due to the French economist Dutot (1738):

$$(1)\quad P_D(p^0,p^1) \equiv [\textstyle\sum_{m=1}^M (1/M)\, p_m^1]/[\sum_{m=1}^M (1/M)\, p_m^0] = [\sum_{m=1}^M p_m^1]/[\sum_{m=1}^M p_m^0].$$

[7] Many statistical agencies send price collectors to various outlets on certain days of the month to collect list prices of individual items. These collected prices can be regarded as approximations to the time aggregated unit values for those items but they are only approximations.

Thus the Dutot elementary price index is equal to the arithmetic average of the M period 1 prices divided by the arithmetic average of the M period 0 prices.

The second widely used elementary index number formula is due to the Italian economist Carli (1764):

(2) $P_C(p^0,p^1) \equiv \sum_{m=1}^{M} (1/M)(p_m^1/p_m^0)$.

Thus the Carli elementary price index is equal to the *arithmetic* average of the M item price ratios or price relatives, p_m^1/p_m^0.

The third widely used elementary index number formula is due to the English economist Jevons (1863):

(3) $P_J(p^0,p^1) \equiv \prod_{m=1}^{M} (p_m^1/p_m^0)^{1/M}$.

Thus the Jevons elementary price index is equal to the *geometric* average of the M item price ratios or price relatives, p_m^1/p_m^0.

The fourth elementary index number formula P_H is the *harmonic* average of the M item price relatives and it was first suggested in passing as an index number formula by Jevons (1865; 121) and Coggeshall (1887):

(4) $P_H(p^0,p^1) \equiv [\sum_{m=1}^{M} (1/M)(p_m^1/p_m^0)^{-1}]^{-1}$.

Finally, the fifth elementary index number formula is the geometric average of the Carli and harmonic formulae; i.e., it is *the geometric mean of the arithmetic and harmonic means of the M price relatives*:

(5) $P_{CSWD}(p^0,p^1) \equiv [P_C(p^0,p^1)\, P_H(p^0,p^1)]^{1/2}$.

This index number formula was first suggested by Fisher (1922; 472) as his formula 101. Fisher also observed that, empirically for his data set, P_{CSWD} was very close to the Jevons index, P_J, and these two indexes were his "best' unweighted index number formulae. In more recent times, Carruthers, Sellwood and Ward (1980; 25) and Dalén (1992; 140) also proposed P_{CSWD} as an elementary index number formula.

Having defined the most commonly used elementary formulae, the question now arises: which formula is "best"? Obviously, this question cannot be answered until desirable properties for elementary indexes are developed. This will be done in a systematic manner in section 5 below but in the present section, one desirable property for an elementary index will be noted. This is the *time reversal test*, which was noted in Chapter 1. In the present context, this test for the elementary index $P(p^0,p^1)$ becomes:

(6) $P(p^0,p^1)\, P(p^1,p^0) = 1$.

This test says that if the prices in period 2 revert to the initial prices of period 0, then the product of the price change going from period 0 to 1, $P(p^0,p^1)$, times the price change going from period 1 to 2, $P(p^1,p^0)$, should equal unity; i.e., under the stated conditions, we should end up where we started. It can be verified that the Dutot, Jevons and Carruthers, Sellwood, Ward and Dalén indexes, P_D, P_J and P_{CSWD}, all satisfy the time reversal test but that the Carli and Harmonic indexes, P_C and P_H, fail this test. In fact, these last two indexes fail the test in the following *biased* manner:

(7) $P_C(p^0,p^1)\ P_C(p^1,p^0) \geq 1$;
(8) $P_H(p^0,p^1)\ P_H(p^1,p^0) \leq 1$

with strict inequalities holding in (7) and (8) provided that the period 1 price vector p^1 is not proportional to the period 0 price vector p^0.[8] Thus the Carli index will generally have an upward bias while the Harmonic index will generally have a downward bias. Fisher (1922; 66 and 383) seems to have been the first to establish the upward bias of the Carli index[9] and he made the following observations on its use by statistical agencies:

"In fields other than index numbers it is often the best form of average to use. But we shall see that the simple arithmetic average produces one of the very worst of index numbers. And if this book has no other effect than to lead to the total abandonment of the simple arithmetic type of index number, it will have served a useful purpose." Irving Fisher (1922; 29-30).

In the following section, some numerical relationships between the five elementary indices defined in this section will be established. Then in the subsequent section, a more comprehensive list of desirable properties for elementary indexes will be developed and the five elementary formulae will be evaluated in the light of these properties or tests.

4. Numerical Relationships between the Frequently Used Elementary Indexes

It can be shown[10] that the Carli, Jevons and Harmonic elementary price indexes satisfy the following inequalities:

(9) $P_H(p^0,p^1) \leq P_J(p^0,p^1) \leq P_C(p^0,p^1)$;

i.e., the Harmonic index is always equal to or less than the Jevons index which in turn is always equal to or less than the Carli index. In fact, the strict inequalities in (9) will hold provided that the period 0 vector of prices, p^0, is not proportional to the period 1 vector of prices, p^1.

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[8] These inequalities follow from the fact that a harmonic mean of M positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901;517) or Fisher (1922; 383-384). This inequality is a special case of Schlömilch's Inequality; see Hardy, Littlewood and Polya (1934; 26).

[9] See also Pigou (1924; 59 and 70), Szulc (1987; 12) and Dalén (1992; 139). Dalén (1994; 150-151) provides some nice intuitive explanations for the upward bias of the Carli index.

[10] Each of the three indexes P_H, P_J and P_C is a mean of order r where r equals −1, 0 and 1 respectively and so the inequalities follow from Schlömilch's inequality; see Hardy, Littlewood and Polya (1934; 26).

The inequalities (9) do not tell us by how much the Carli index will exceed the Jevons index and by how much the Jevons index will exceed the Harmonic index. Hence, in the remainder of this section, some approximate relationships between the five indexes defined in the previous section will be developed that will provide some practical guidance on the relative magnitudes of each of the indexes.

The first approximate relationship that will be derived is between the Carli index P_C and the Dutot index P_D. For each period t, define the *arithmetic mean of the M prices* pertaining to that period as follows:

(10) $p^{t*} \equiv \sum_{m=1}^{M} (1/M)\, p_m^{\,t}$; $\qquad t = 0,1.$

Now define the *multiplicative deviation of the mth price in period t relative to the mean price in that period*, $e_m^{\,t}$, as follows:

(11) $p_m^{\,t} = p^{t*}(1+e_m^{\,t})$; $\qquad m = 1,...,M$; $t = 0,1.$

Note that (10) and (11) imply that the deviations $e_m^{\,t}$ sum to zero in each period; i.e., we have:

(12) $\sum_{m=1}^{M} e_m^{\,t} = 0$; $\qquad t = 0,1.$

Note that the Dutot index can be written as the ratio of the mean prices, p^{1*}/p^{0*}; i.e., we have:

(13) $P_D(p^0,p^1) = p^{1*}/p^{0*}$.

Now substitute equations (11) into the definition of the Jevons index, (3):

(14) $P_J(p^0,p^1) = \prod_{m=1}^{M} [p^{1*}(1+e_m^{\,1})/p^{0*}(1+e_m^{\,0})]^{1/M}$
$= [p^{1*}/p^{0*}] \prod_{m=1}^{M} [(1+e_m^{\,1})/(1+e_m^{\,0})]^{1/M}$
$= P_D(p^0,p^1)\, f(e^0,e^1)$ $\qquad$ using (13)

where $e^t \equiv [e_1^{\,t},...,e_M^{\,t}]$ for t = 0 and 1 and the function f is defined as follows:

(15) $f(e^0,e^1) \equiv \prod_{m=1}^{M} [(1+e_m^{\,1})/(1+e_m^{\,0})]^{1/M}$.

Expand $f(e^0,e^1)$ by a second order Taylor series approximation around $e^0 = 0_M$ and $e^1 = 0_M$. Using (12), it can be verified[11] that we obtain the following second order approximate relationship between P_J and P_D:

(16) $P_J(p^0,p^1) \approx P_D(p^0,p^1)[1 + (1/2M)e^0\cdot e^0 - (1/2M)e^1\cdot e^1]$
$= P_D(p^0,p^1)[1 + (1/2)\mathrm{var}(e^0) - (1/2)\mathrm{var}(e^1)]$

[11] This approximate relationship was first obtained by Carruthers, Sellwood and Ward (1980; 25).

where var(e^t) is the variance of the period t multiplicative deviations; i.e., for t = 0,1:

$$(17)\ \mathrm{var}(e^t) \equiv (1/M)\textstyle\sum_{m=1}^{M}(e_m^{\,t} - e^{t*})^2$$
$$= (1/M)\textstyle\sum_{m=1}^{M}(e_m^{\,t})^2 \qquad \text{since } e^{t*} = 0 \text{ using (12)}$$
$$= (1/M)\, e^t \cdot e^t .$$

Under normal conditions[12], the variance of the deviations of the prices from their means in each period is likely to be approximately constant and so under these conditions, the Jevons price index will approximate the Dutot price index to the second order.

Note that with the exception of the Dutot formula, the remaining four elementary indexes defined in section 3 are functions of the relative prices of the M items being aggregated. This fact is used in order to derive some approximate relationships between these four elementary indexes. Thus define the *mth price relative* as

$$(18)\ r_m \equiv p_m^{\,1}/p_m^{\,0}\ ; \qquad m = 1,\ldots,M.$$

Define the *arithmetic mean of the m price relatives* as

$$(19)\ r^* \equiv (1/M)\textstyle\sum_{m=1}^{M} r_m = P_C(p^0,p^1)$$

where the last equality follows from the definition (2) of the Carli index. Finally, define the *deviation* e_m *of the mth price relative* r_m *from the arithmetic average of the M price relatives* r* as follows:

$$(20)\ r_m = r^*(1+e_m)\ ; \qquad m = 1,\ldots,M.$$

Note that (19) and (20) imply that the deviations e_m sum to zero; i.e., we have:

$$(21)\ \textstyle\sum_{m=1}^{M} e_m = 0.$$

Now substitute equations (20) into the definitions of P_C, P_J, P_H and P_{CSWD}, (2)-(5) above, in order to obtain the following representations for these indexes in terms of the vector of deviations, $e \equiv [e_1,\ldots,e_M]$:

$$(22)\ P_C(p^0,p^1) = \textstyle\sum_{m=1}^{M} (1/M)r_m = r^*\,1 \equiv r^* f_C(e)\ ;$$
$$(23)\ P_J(p^0,p^1) = \textstyle\prod_{m=1}^{M} r_m^{\,1/M} = r^* \textstyle\prod_{m=1}^{M} (1+e_m)^{1/M} \equiv r^* f_J(e)\ ;$$
$$(24)\ P_H(p^0,p^1) = [\textstyle\sum_{m=1}^{M} (1/M)(r_m)^{-1}]^{-1} = r^* [\textstyle\sum_{m=1}^{M} (1/M)(1+e_m)^{-1}]^{-1} \equiv r^* f_H(e)\ ;$$
$$(25)\ P_{CSWD}(p^0,p^1) = [P_C(p^0,p^1)P_H(p^0,p^1)]^{1/2} = r^*[f_C(e)f_H(e)]^{1/2} \equiv r^* f_{CSWD}(e)$$

[12] If there are significant changes in the overall inflation rate, some studies indicate that the variance of deviations of prices from their means can also change. Also if M is small, then there will be sampling fluctuations in the variances of the prices from period to period.

where the last equation in (22)-(25) serves to define the deviation functions, $f_C(e)$, $f_J(e)$, $f_H(e)$ and $f_{CSWD}(e)$. The second order Taylor series approximations to each of these functions[13] around the point $e = 0_M$ are:

(26) $f_C(e) \approx 1$;
(27) $f_J(e) \approx 1 - (1/2M)e{\cdot}e = 1 - (1/2)\text{var}(e)$;
(28) $f_H(e) \approx 1 - (1/M)e{\cdot}e = 1 - \text{var}(e)$;
(29) $f_{CSWD}(e) \approx 1 - (1/2M)e{\cdot}e = 1 - (1/2)\text{var}(e)$

where we have made repeated use of (21) in deriving the above approximations.[14] Thus to the second order, the Carli index P_C will *exceed* the Jevons and Carruthers Sellwood Ward Dalén indexes, P_J and P_{CSWD}, by (1/2)r*var(e), which is r* times one half the variance of the M price relatives p_m^1/p_m^0. Similarly, to the second order, the Harmonic index P_H will *lie below* the Jevons and Carruthers Sellwood Ward Dalén indexes, P_J and P_{CSWD}, by r* times one half the variance of the M price relatives p_m^1/p_m^0.

Thus empirically, it is expected that the Jevons and Carruthers Sellwood Ward and Dalén indexes will be very close to each other. Using the previous approximation result (16), it is expected that the Dutot index P_D will also be fairly close to P_J and P_{CSWD}, with some fluctuations over time due to changing variances of the period 0 and 1 deviation vectors, e^0 and e^1. Thus it is expected that these three elementary indexes will give much the same numerical answers in empirical applications. On the other hand, the Carli index can be expected to be substantially *above* these three indices, with the degree of divergence growing as the variance of the M price relatives grows. Similarly, the Harmonic index can be expected to be substantially *below* the three middle indices, with the degree of divergence growing as the variance of the M price relatives grows.

5. The Axiomatic Approach to Elementary Indexes

Recall that in Chapter 3, the axiomatic approach to bilateral price indexes $P(p^0,p^1,q^0,q^1)$ was developed. In the present Chapter, the elementary price index $P(p^0,p^1)$ depends only on the period 0 and 1 price vectors, p^0 and p^1 respectively so that the elementary price index does not depend on the period 0 and 1 quantity vectors, q^0 and q^1. One approach to obtaining new tests or axioms for an elementary index is to look at the twenty or so axioms that were listed in Chapter 3 for bilateral price indices $P(p^0,p^1,q^0,q^1)$ and adapt those axioms to the present context; i.e., use the old bilateral tests for $P(p^0,p^1,q^0,q^1)$ that do not depend on the quantity vectors q^0 and q^1 as tests for an elementary index $P(p^0,p^1)$.[15] This approach will be utilized in the present section.

The first eight tests or axioms are reasonably straightforward and uncontroversial,

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[13] From (22), it can be seen that $f_C(e)$ is identically equal to 1 so that (26) will be an exact equality rather than an approximation.
[14] These second order approximations are due to Dalén (1992; 143) for the case r* =1 and to Diewert (1995; 29) for the case of a general r*.
[15] This was the approach used by Diewert (1995; 5-17), who drew on the earlier work of Eichhorn (1978; 152-160) and Dalén (1992).

T1: *Continuity*: $P(p^0,p^1)$ is a continuous function of the M positive period 0 prices $p^0 \equiv [p_1^0,...,p_M^0]$ and the M positive period 1 prices $p^1 \equiv [p_1^1,...,p_M^1]$.

T2: *Identity*: $P(p,p) = 1$; i.e., the period 0 price vector equals the period 1 price vector, then the index is equal to unity.

T3: *Monotonicity in Current Period Prices*: $P(p^0,p^1) < P(p^0,p)$ if $p^1 < p$; i.e., if any period 1 price increases, then the price index increases.

T4: *Monotonicity in Base Period Prices*: $P(p^0,p^1) > P(p,p^1)$ if $p^0 < p$; i.e., if any period 0 price increases, then the price index decreases.

T5: *Proportionality in Current Period Prices*: $P(p^0,\lambda p^1) = \lambda P(p^0,p^1)$ if $\lambda > 0$; i.e., if all period 1 prices are multiplied by the positive number λ, then the initial price index is also multiplied by λ.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1) = \lambda^{-1} P(p^0,p^1)$ if $\lambda > 0$; i.e., if all period 0 prices are multiplied by the positive number λ, then the initial price index is multiplied by $1/\lambda$.

T7: *Mean Value Test*: $\min_m \{p_m^1/p_m^0 : m = 1,...,M\} \leq P(p^0,p^1) \leq \max_m \{p_m^1/p_m^0 : m = 1,...,M\}$; i.e., the price index lies between the smallest and largest price relatives.

T8: *Symmetric Treatment of Outlets*: $P(p^0,p^1) = P(p^0*,p^1*)$ where p^0* and p^1* denote the *same* permutation of the components of p^0 and p^1; i.e., if we change the ordering of the outlets (or households) from which we obtain the price quotations for the two periods, then the elementary index remains unchanged.

Eichhorn (1978; 155) showed that Tests 1, 2, 3 and 5 imply Test 7, so that not all of the above tests are logically independent.

The following tests are more controversial and are not necessarily accepted by all price statisticians.

T9: *The Price Bouncing Test*: $P(p^0,p^1) = P(p^0*,p^1**)$ where p^0* and p^1** denote possibly *different* permutations of the components of p^0 and p^1; i.e., if the ordering of the price quotes for both periods is changed in possibly different ways, then the elementary index remains unchanged.

Obviously, T8 is a special case of T9 where the two permutations of the initial ordering of the prices are restricted to be the same. Thus T9 implies T8. Test T9 is due to Dalén (1992; 138). He justified this test by suggesting that the price index should remain unchanged if outlet prices "bounce" in such a manner that the outlets are just exchanging prices with each other over the two periods. While this test has some intuitive appeal, it

is not consistent with the idea that outlet prices should be matched to each other in a one to one manner across the two periods.

The following test was also proposed by Dalén (1992) in the elementary index context:

T10: *Time Reversal*: $P(p^1,p^0) = 1/P(p^0,p^1)$; i.e., if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index.

Since many price statisticians approve of the Laspeyres price index in the bilateral index context and this index does not satisfy the time reversal test, it is obvious that not all price statisticians would regard the time reversal test in the elementary index context as being a fundamental test that must be satisfied. Nevertheless, many other price statisticians do regard this test as a fundamental one since it is difficult to accept an index that gives a different answer if the ordering of time is reversed.[16]

T11: *Circularity*: $P(p^0,p^1)P(p^1,p^2) = P(p^0,p^2)$; i.e., the price index going from period 0 to 1 times the price index going from period 1 to 2 equals the price index going from period 0 to 2 directly.

The circularity and identity tests imply the time reversal test; (just set $p^2 = p^0$). Thus the circularity test is essentially a strengthening of the time reversal test and so price statisticians who did not accept the time reversal test are unlikely to accept the circularity test. However, if there are no obvious drawbacks to accepting the circularity test, it would seem to be a very desirable property: it is a generalization of a property that holds for a single price relative.

T12: *Commensurability*: $P(\lambda_1 p_1^0,..., \lambda_M p_M^0; \lambda_1 p_1^1,..., \lambda_M p_M^1) = P(p_1^0,...,p_M^0; p_1^1,...,p_M^1) = P(p^0,p^1)$ for all $\lambda_1 > 0, ... , \lambda_M > 0$; i.e., if we change the units of measurement for each commodity in each outlet, then the elementary index remains unchanged.

In the bilateral index context, virtually every price statistician accepts the validity of this test. However, in the elementary context, this test is more controversial. If the M items in the elementary aggregate are all very homogeneous, then it makes sense to measure all of the items in the same units. Hence, if we change the unit of measurement, then test T12 should restrict all of the λ_m to be the same number (say λ) and the test T12 becomes

(30) $P(\lambda p^0,\lambda p^1) = P(p^0,p^1)$; $\lambda > 0$.

[16] Price statisticians do not seem to resist the time reversal test in the elementary index context as strongly as in the regular bilateral approach to index number theory where quantity or expenditure weights are available. This may be due to a feeling that the Laspeyres or Lowe indexes are really the "right" indexes to use in the weighted context, given the practical difficulties in securing up to date weights. Since the Laspeyres index does not satisfy the time reversal test, the importance of this test is played down somewhat. However, in the unweighted context, there is less resistance to using an elementary index that satisfies the time reversal test since there is not a clear a priori preference for any particular elementary formula.

Note that this modified test 12 will be satisfied if tests T5 and T6 are satisfied. Thus if the items in the elementary aggregate are very homogeneous, then there is no need for test T12.

However, in actual practice, there will usually be thousands of individual items in each elementary aggregate and the hypothesis of item homogeneity is not warranted. Under these circumstances, it is important that the elementary index satisfy the commensurability test, since the units of measurement of the heterogeneous items in the elementary aggregate are arbitrary and hence *the price statistician can change the index simply by changing the units of measurement for some of the items*.

This completes the listing of the tests for an elementary index. There remains the task of evaluating how many tests are passed by each of the five elementary indexes defined in section 3 above.

Problems

1. Show that the Jevons elementary index P_J satisfies *all* of the above tests.

2. Show that the Dutot index P_D satisfies all of the tests with the important exception of the Commensurability Test T12, which it fails.

3. Show that the Carli and Harmonic elementary indexes, P_C and P_H, fail the price bouncing test T9, the time reversal test T10 and the circularity test T11 but pass the other tests.

4. Show that the geometric mean of the Carli and Harmonic elementary indexes, P_{CSWD}, fails only the price bouncing test T9 and the circularity test T11.

Thus the Jevons elementary index P_J satisfies *all* of the tests and hence emerges as being "best" from the viewpoint of the axiomatic approach to elementary indexes.

The Dutot index P_D satisfies all of the tests with the important exception of the Commensurability Test T12, which it fails. If there are heterogeneous items in the elementary aggregate, this is a rather serious failure and hence price statisticians should be careful in using this index under these conditions.

The geometric mean of the Carli and Harmonic elementary indexes fails only the price bouncing test T9 and the circularity test T11. The failure of these two tests is probably not a fatal failure and so this index could be used by price statisticians (who used the test approach for guidance in choosing an index formula), if for some reason, it was decided not to use the Jevons formula. As was observed in section 4 above, numerically, P_{CSWD} will be very close to P_J.

The Carli and Harmonic elementary indices, P_C and P_H, fail the price bouncing test T9, the time reversal test T10 and the circularity test T11 and pass the other tests. The failure

of T9 and T11 is again not a fatal failure but the failure of the time reversal test T10 is a rather serious failure and so price statisticians should avoid using these indexes.

6. The Economic Approach to Elementary Indexes

Recall the notation and discussion in section 2 above. Suppose that each purchaser of the items in the elementary aggregate has preferences over a vector of purchases $q \equiv [q_1,...,q_M]$ that can be represented by the linearly homogeneous aggregator function f(q). Further assume that each purchaser engages in cost minimizing behavior in each period. Then as was seen in Chapter 4, it can be shown that certain specific functional forms for the aggregator or utility function f(q) or its dual unit cost function c(p)[17] lead to specific functional forms for the price index, $P(p^0,p^1,q^0,q^1)$, with

(31) $P(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$.

Suppose that the purchasers have aggregator functions f defined as follows[18]:

(32) $f(q_1,...,q_M) \equiv \min_m \{q_m/\alpha_m : m = 1,...,M\}$

where the α_m are positive constants. Then under these assumptions, it can be shown that equation (31) becomes[19]:

(33) $c(p^1)/c(p^0) = p^1\cdot q^0/p^0\cdot q^0 = p^1\cdot q^1/p^0\cdot q^1$

and the quantity vectors of purchases during the two periods must be proportional; i.e.,

(34) $q^1 = \lambda q^0$ for some $\lambda > 0$.

From the first equation in (33), we see that the true cost of living index, $c(p^1)/c(p^0)$, under assumptions (32) about the aggregator function f, is equal to the Laspeyres price index, $P_L(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^0/p^0\cdot q^0$. We now show how various elementary formulae can estimate this Laspeyres formula under alternative assumptions about the sampling of prices.

In order to provide a justification for the use of the Dutot elementary formula, write the Laspeyres index number formula as follows:

(35) $$P_L(p^0,p^1,q^0,q^1) \equiv \sum_{m=1}^M p_m^1 q_m^0 / \sum_{m=1}^M p_m^0 q_m^0 = \sum_{m=1}^M \rho_m^0 p_m^1 / \sum_{m=1}^M \rho_m^0 p_m^0$$

where the *base period item probabilities* ρ_m^0 are defined as follows:

[17] The unit cost function is defined as $c(p) \equiv \min_q \{p\cdot q : f(q) = 1\}$.
[18] The preferences which correspond to this f are known as Leontief (1936) or no substitution preferences.
[19] See Pollak (1983).

(36) $\rho_m^0 \equiv q_m^0 / \sum_{i=1}^M q_i^0$; $\qquad$ m = 1,...,M.

Thus the base period probability for item m, ρ_m^0, is equal to the purchases of item m in the base period relative to total purchases of all items in the commodity class in the base period. *Note that these definitions require that all items in the commodity class have the same units*.[20]

Now it is easy to see how formula (35) could be turned into a rigorous sampling framework for sampling prices in the particular commodity class under consideration.[21] If item prices in the commodity class were sampled proportionally to their base period probabilities ρ_m^0, then the Laspeyres index (35) could be estimated by a probability weighted Dutot index defined by the second line in (35). In general, with an appropriate sampling scheme, the use of the Dutot formula at the elementary level of aggregation *for homogeneous items* can be perfectly consistent with a Laspeyres index concept.

The Dutot formula can also be consistent with a Paasche index concept. If we use the Paasche formula at the elementary level of aggregation, we obtain the following formula:

$$(37)\quad P_P(p^0,p^1,q^0,q^1) \equiv \sum_{m=1}^M p_m^1 q_m^1 / \sum_{m=1}^M p_m^0 q_m^1 = \sum_{m=1}^M \rho_m^1 p_m^1 / \sum_{m=1}^M \rho_m^1 p_m^0$$

where the *period one item probabilities* ρ_m^1 are defined as follows:

(38) $\rho_m^1 \equiv q_m^1 / \sum_{i=1}^M q_i^1$; $\qquad$ m = 1,...,M.

Thus the period one probability for item m, ρ_m^1, is equal to the quantity purchased of item m in period one relative to total purchases of all items in the commodity class in that period.

Again, it is easy to see how formula (37) could be turned into a rigorous sampling framework for sampling prices in the particular commodity class under consideration. If item prices in the commodity class were sampled proportionally to their period one probabilities ρ_m^1, then the Paasche index (37) could be estimated by the probability weighted Dutot index defined by the second line of (37). In general, with an appropriate sampling scheme, the use of the Dutot formula at the elementary level of aggregation (*for a homogeneous elementary aggregate*) can be perfectly consistent with a Paasche index concept.

Rather than use the fixed basket representations for the Laspeyres and Paasche indexes, we could use the expenditure share representations for the Laspeyres and Paasche indexes and use the expenditure shares s_m^0 or s_m^1 as probability weights for price relatives. Thus if the relative prices of items in the commodity class under consideration are sampled

[20] The probabilities defined by (36) are meaningless unless the items are homogeneous.
[21] For the details, see Balk (2002; 7).

using weights that are proportional to their base period expenditure shares in the commodity class, then the following *probability weighted Carli index*

(39) $P_C(p^0,p^1,s^0) \equiv \sum_{m=1}^{M} s_m^0 (p_m^1/p_m^0)$

will be equal to the Laspeyres index.[22] Of course, formula (39) does not require the assumption of homogeneous items as did (35) and (37) above. On the other hand, if the relative prices of items in the commodity class under consideration are sampled using weights that are proportional to their period one expenditure shares in the commodity class, then the following *probability weighted harmonic index*

(40) $P_H(p^0,p^1,s^1) \equiv \{\sum_{m=1}^{M} s_m^1 [p_m^1/p_m^0]^{-1}\}^{-1}$

will be equal to the Paasche index.

The above results show that the Dutot elementary index can be justified as an approximation to an underlying Laspeyres or Paasche price index for a homogeneous elementary aggregate *under appropriate price sampling schemes*. The above results also show that the Carli and Harmonic elementary indexes can be justified as approximations to an underlying Laspeyres or Paasche price index for a heterogeneous elementary aggregate *under appropriate price sampling schemes*.

Recall that assumption (32) on f justified the Laspeyres and Paasche indexes as being the "true" elementary aggregate from the viewpoint of the economic approach to elementary indices. Suppose now that assumption (32) is replaced by the following assumption of *Cobb Douglas* (1928) *preferences*:

(41) $f(q_1,...,q_M) \equiv \prod_{m=1}^{M} q_m^{\beta_m}$; $\beta_m > 0$ for m = 1,...,M and $\sum_{m=1}^{M} \beta_m = 1$.

Under assumption (41), the true economic elementary price index is[23]:

(42) $c(p^1)/c(p^0) = \prod_{m=1}^{M} (p_m^1/p_m^0)^{\beta_m}$.

It turns out if purchasers have the above Cobb Douglas preferences, then item expenditures will be proportional over the two periods so that:

(43) $p_m^1 q_m^1 = \lambda\, p_m^0 q_m^0$ for m = 1,...,M and for some $\lambda > 0$.

Under these conditions, the base period expenditure shares s_m^0 will equal the corresponding period 1 expenditure shares s_m^1 as well as the corresponding β_m; i.e., (41) implies:

(44) $s_m^0 = s_m^1 \equiv \beta_m$; m = 1,...,M.

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[22] For a rigorous derivation of a sampling framework, see Balk (2002; 10).
[23] See Pollak (1983).

Thus if the relative prices of items in the commodity class under consideration are sampled using weights that are proportional to their base period expenditure shares in the commodity class, then the following *probability weighted Jevons index*

(45) $\ln P_J(p^0,p^1,s^0) \equiv \sum_{m=1}^{M} s_m^0 \ln(p_m^1/p_m^0)$

will be equal to the logarithm of the true elementary price aggregate defined by (42).[24]

The above results show that the Jevons elementary index can be justified as an approximation to an underlying Cobb Douglas price index for a heterogeneous elementary aggregate *under an appropriate price sampling scheme*.

The assumption of Leontief preferences implies that the quantity vectors pertaining to the two periods under consideration will be proportional; recall (34) above. On the other hand, the assumption of Cobb Douglas preferences implies that expenditures will be proportional over the two periods; recall (43) above. Index number theorists have been debating the relative merits of the proportional quantities versus proportional expenditures assumption for a long time. Authors who thought that the proportional expenditures assumption was more likely empirically include Jevons (1865; 295) and Ferger (1931; 39) (1936; 271). These early authors did not have the economic approach to index number theory at their disposal but they intuitively understood, along with Pierson (1895; 332), that substitution effects occurred and hence the proportional expenditures assumption was more plausible than the proportional quantities assumption.

The results in the previous section gave some support for the use of the *unweighted* Jevons elementary index over the use of the *unweighted* Dutot, Carli and Harmonic indices. The results in this section give some support for the use of an appropriately *weighted* Jevons index over the other *weighted* indexes, since from the economic perspective, cross item elasticities of substitution are much more likely to be close to unity (this corresponds to the case of Cobb Douglas preferences) than to zero (this corresponds to the case of Leontief preferences). If the probability weights in the weighted Jevons index are taken to be the arithmetic average of the period 0 and 1 item expenditure shares and narrowly defined unit values are used as the price concept, then the weighted Jevons index becomes an ideal type of elementary index discussed in section 2 above.

7. The Sampling Approach to Elementary Indexes

In the previous section, it was shown that appropriately *weighted elementary indexes* were capable of approximating various economic population elementary indexes, with the approximation becoming exact as the sampling approached complete coverage. Conversely, it can be seen that, in general, it will be impossible for an *unweighted elementary price index* of the type defined in section 3 to approach the theoretically ideal elementary price index defined in section 2, even if all item prices in the elementary

[24] See Balk (2002; 9) for a rigorous derivation.

aggregate were sampled.[25] Hence, rather than just sampling prices, it will be necessary for the price statistician to collect information on the *transaction values* (or quantities) associated with the sampled prices in order to form sample elementary aggregates that will approach the target ideal elementary aggregate as the sample size becomes large. Thus instead of just collecting a sample of prices, it will be necessary to collect corresponding sample quantities (or values) so that a sample Fisher, Törnqvist or Walsh price index can be constructed. This sample based superlative elementary price index will approach the population ideal elementary index as the sample size becomes large. This approach to the construction of elementary indices in a sampling context was recommended by Pigou (1924; 66-67), Fisher (1922; 380), Diewert (1995; 25) and Balk (2002).[26] In particular, Pigou (1924; 67) suggested that the sample based Fisher ideal price index be used to deflate the value ratio for the aggregate under consideration in order to obtain an estimate of the quantity ratio for the aggregate under consideration.

Until fairly recently, it was not possible to determine how close an unweighted elementary index defined in section 3 was to an ideal elementary aggregate. However, with the availability of *scanner data* (i.e., of detailed data on the prices and quantities of individual items that are sold in retail outlets), it has been possible to compute ideal elementary aggregates for some item strata and compare the results with statistical agency estimates of price change for the same class of items. Of course, the statistical agency estimates of price change are usually based on the use of the Dutot, Jevons or Carli formulae. The following quotations summarize many of these scanner data studies:

"A second major recent development is the willingness of statistical agencies to experiment with scanner data, which are the electronic data generated at the point of sale by the retail outlet and generally include transactions prices, quantities, location, date and time of purchase and the product described by brand, make or model. Such detailed data may prove especially useful for constructing better indexes at the elementary level. Recent studies that use scanner data in this way include Silver (1995), Reinsdorf (1996), Bradley, Cook, Leaver and Moulton (1997), Dalén (1997), de Haan and Opperdoes (1997) and Hawkes (1997). Some estimates of elementary index bias (on an annual basis) that emerged from these studies were: 1.1 percentage points for television sets in the United Kingdom; 4.5 percentage points for coffee in the United States; 1.5 percentage points for ketchup, toilet tissue, milk and tuna in the United States; 1 percentage point for fats, detergents, breakfast cereals and frozen fish in Sweden; 1 percentage point for coffee in the Netherlands and 3 percentage points for coffee in the United States respectively. These bias estimates incorporate both elementary and outlet substitution biases and are significantly higher than our earlier ballpark estimates of .255 and .41 percentage points. On the other hand, it is unclear to what extent these large bias estimates can be generalized to other commodities." W. Erwin Diewert (1998; 54-55).

"Before considering the results it is worth commenting on some general findings from scanner data. It is stressed that the results here are for an experiment in which the same data were used to compare different methods. The results for the U.K. Retail Prices Index can not be fairly compared since they are based on quite different practices and data, their data being collected by price collectors and having strengths as well as weaknesses (Fenwick, Ball, Silver and Morgan (2002)). Yet it is worth following up on Diewert's (2002) comment on the U.K. Retail Prices Index electrical appliances section, which includes a wide variety of appliances, such as irons, toasters, refrigerators, etc. which went from 98.6 to 98.0,a drop of 0.6 percentage points from January 1998 to December 1998. He compares these results with those for washing machines and notes that '..it may be that the non washing machine components of the electrical appliances index increased in price enough over this period to cancel out the large apparent drop in the price of

[25] The numerical example given at the end of this section illustrates this point.
[26] Balk (2002) provides the details for this sampling framework.

washing machines but I think that this is somewhat unlikely.' A number of studies on similar such products have been conducted using scanner data for this period. Chained Fishers indices have been calculated from the scanner data, (the RPI (within year) indices are fixed base Laspeyres ones), and have been found to fall by about 12% for televisions (Silver and Heravi, 2001a), 10% for washing machines (Table 7 below), 7.5% for dishwashers, 15% for cameras and 5% for vacuum cleaners (Silver and Heravi (2001b). These results are quite different from those for the RPI section and suggest that the washing machine disparity, as Diewert notes, may not be an anomaly. Traditional methods and data sources seem to be giving much higher rates for the CPI than those from scanner data, though the reasons for these discrepancies were not the subject of this study." Mick Silver and Saeed Heravi (2001c; 25).

The above quotations summarize the results of many elementary aggregate index number studies that are based on the use of scanner data. These studies indicate that when detailed price and quantity data are used in order to compute superlative indexes or hedonic indexes for an expenditure category, the resulting measures of price change are often below the corresponding official statistical agency estimates of price change for that category. Sometimes the measures of price change based on the use of scanner data are *considerably below* the corresponding official measures.[27] These results indicate that there may be large gains in the precision of elementary indexes if a *weighted* sampling framework is adopted.

Is there a simple intuitive explanation for the above empirical results? A partial explanation may be possible by looking at the dynamics of item demand. In any market economy, there are firms and outlets that sell items that are either declining or increasing in price. Usually, the items that decline in price experience an increase in their volume of sales. Thus the expenditure shares that are associated with items that are declining in price usually increase and conversely for the items that increase in price. Unfortunately, elementary indexes cannot pick up the effects of this negative correlation between price changes and the induced changes in expenditure shares because elementary indexes depend only on prices and not on expenditure shares.

An example can illustrate the above point. Suppose that there are only three items in the elementary aggregate and that in period 0, the price of each item is $p_m^0 = 1$ and the expenditure share for each item is equal so that $s_m^0 = 1/3$ for m = 1,2,3. Suppose that in period 1, the price of item 1 increases to $p_1^1 = 1 + i$, the price of item 2 remains constant at $p_2^1 = 1$ and the price of item 3 decreases to $p_3^1 = (1 + i)^{-1}$ where the item 1 rate of increase in price is $i > 0$. Suppose further that the expenditure share of item 1 decreases to $s_1^1 = (1/3) - \sigma$ where σ is a small number between 0 and 1/3 and the expenditure share of item 3 increases to $s_3^1 = (1/3) + \sigma$. The expenditure share of item 2 remains constant at $s_2^1 = 1/3$. The 5 elementary indexes defined in section 3 can all be written as functions of the item 1 inflation rate i (which is also the item 3 deflation rate) as follows:

$$(46)\ P_J(p^0,p^1) = [(1+i)(1+i)^{-1}]^{1/3} = 1 \equiv f_J(i)\ ;$$

[27] However, scanner data studies do not always show large potential biases in official CPIs. Masato Okamoto has informed us that a large scale comparative study in Japan is underway. Using scanner data for about 250 categories of processed food and daily necessities collected over the period 1997 to 2000, it was found that the indexes based on scanner data averaged only about 0.2 percentage points below the corresponding official indexes per year. Japan uses the Dutot formula at the elementary level in its official CPI.

(47) $P_C(p^0,p^1) = (1/3)(1+i) + (1/3) + (1/3)(1+i)^{-1} \equiv f_C(i)$;
(48) $P_H(p^0,p^1) = [(1/3)(1+i)^{-1} + (1/3) + (1/3)(1+i)]^{-1} \equiv f_H(i)$;
(49) $P_{CSW}(p^0,p^1) = [P_C(p^0,p^1)\, P_H(p^0,p^1)]^{1/2} \equiv f_{CSW}(i)$;
(50) $P_D(p^0,p^1) = (1/3)(1+i) + (1/3) + (1/3)(1+i)^{-1} \equiv f_D(i)$.

Note that in this particular example, the Dutot index $f_D(i)$ turns out to equal the Carli index $f_C(i)$. The second order Taylor series approximations to the 5 elementary indexes (46)-(50) are given by (51)-(55) below:

(51) $f_J(i) \approx 1$;
(52) $f_C(i) \approx 1 + (1/3)i^2$;
(53) $f_H(i) \approx 1 - (1/3)i^2$;
(54) $f_{CSW}(i) \approx 1$;
(55) $f_D(i) \approx 1 + (1/3)i^2$.

Thus for small i, the Carli and Dutot indexes will be slightly greater than 1[28], the Jevons and Carruthers, Sellwood and Ward indexes will be approximately equal to 1 and the Harmonic index will be slightly less than 1. Note that the first order Taylor series approximation to all 5 indexes is 1; i.e., to the accuracy of a first order approximation, all 5 indexes equal unity.

Now calculate the Laspeyres, Paasche and Fisher indexes for the elementary aggregate:

(56) $P_L = (1/3)(1+i) + (1/3) + (1/3)(1+i)^{-1} \equiv f_L(i)$;
(57) $P_P = \{[(1/3) - \sigma](1+i)^{-1} + (1/3) + [(1/3) + \sigma](1+i)\}^{-1} \equiv f_P(i)$;
(58) $P_F = [P_L P_P]^{1/2} \equiv f_F(i)$.

First order Taylor series approximations to the above indexes (56)-(58) around i = 0 are given by (59)-(61):

(59) $f_L(i) \approx 1$;
(60) $f_P(i) \approx 1 - 2\sigma i$;
(61) $f_F(i) \approx 1 - \sigma i$.

An ideal elementary index for the three items is the Fisher ideal index $f_F(i)$. The approximations (51)-(55) and (61) show that the Fisher index will lie below all 5 elementary indexes by the amount σi using first order approximations to all 6 indexes. *Thus all 5 elementary indexes will have an approximate upward bias equal to* σi *compared to an ideal elementary aggregate*.

Suppose that the annual item inflation rate for the item rising in price is equal to 10 % so that i = .10 (and hence the rate of price decrease for the item decreasing in price is

[28] Recall the approximate relationship (16) in section 3 above between the Dutot and Jevons indexes. In our example, $var(e^0) = 0$ whereas $var(e^1) > 0$. This explains why the Dutot index is not approximately equal to the Jevons index in our example.

approximately 10 % as well). If the expenditure share of the increasing price item declines by 5 percentage points, then $\sigma = .05$ and the annual approximate upward bias in all 5 elementary indexes is $\sigma i = .05 \times .10 = .005$ or one half of a percentage point. If i increases to 20 % and σ increases to 10 %, then the approximate bias increases to $\sigma i = .10 \times .20 = .02$ or 2 percent.

The above example is highly simplified but more sophisticated versions of it are capable of explaining at least some of the discrepancy between official elementary indexes and superlative indexes calculated by using scanner data for an expenditure class.[29] Basically, elementary indexes defined without using associated quantity or value weights are incapable of picking up shifts in expenditure shares that are induced by fluctuations in item prices.[30] In order to eliminate this problem, it will be necessary to sample values along with prices in both the base and comparison periods.

In the following section, a simple regression based approach to the construction of elementary indexes is outlined and again, the importance of weighting the price quotes will emerge from the analysis.

8. A Simple Stochastic Approach to Elementary Indexes

Recall the notation used in section B above. Suppose the prices of the M items for period 0 and 1 are approximately equal to the right hand sides of (62) and (63) below:

(62) $p_m^0 \approx \beta_m$; $\quad m = 1,...,M;$
(63) $p_m^1 \approx \alpha\beta_m$; $\quad m = 1,...,M$

where α and the β_m are positive parameters. Note that there are 2M prices on the left hand sides of equations (62) and (63) but only M + 1 parameters on the right hand sides of these equations. The basic hypothesis in (62) and (63) is that the two price vectors p^0 and p^1 are proportional (with $p^1 = \alpha p^0$ so that α is the factor of proportionality) except for random multiplicative errors and hence α represents the underlying elementary price aggregate. If we take logarithms of both sides of (62) and (63) and add some random

[29] Note that to sustain the numerical example bias over time, some firms have to continue to make productivity improvements over time (or purchase inputs from firms that make productivity improvements) or new entrants must enter the industry with lower costs than the incumbants. A recent newspaper article noted that the U.S. retailer Wal-Mart pressures suppliers to reduce costs and prices: "Every year, Wal-Mart mandates a 5 per cent drop in its suppliers' prices for standardized products— a reduction that it passes on to its consumers. ... Already the single largest private sector employer in the United States and the third largest employer behind the federal government and the government of California, Wal-Mart's imprint on labour markets extends well beyond its more than one million payroll. Its far greater impact is on the wages of its competitors as Wal-Mart's cost structure immediately sets the benchmark for competitiveness in whatever market it enters. ... Last year, the average 3 per cent decline in all of Wal-Mart's retail prices, when applied to its 8 per cent share of total consumer spending, chopped roughly a quarter point from the U.S. consumer price index (CPI) annual inflation rate. And that of course, does not begin to measure the impact that it has had on its competitors' prices." Jeffrey Rubin (2004).

[30] Put another way, elementary indexes are subject to substitution or representativity bias.

errors e_m^0 and e_m^1 to the right hand sides of the resulting equations, we obtain the following *linear regression model*:

(64) $\ln p_m^0 = \delta_m + e_m^0$; $\quad m = 1,...,M$;
(65) $\ln p_m^1 = \gamma + \delta_m + e_m^1$; $\quad m = 1,...,M$

where

(66) $\gamma \equiv \ln \alpha$ and $\delta_m \equiv \ln \beta_m$; $\quad m = 1,...,M$.

Note that (64) and (65) can be interpreted as a highly simplified *hedonic regression model*.[31] The only characteristic of each commodity is the commodity itself. This model is also a special case of the *country product dummy method* for making international comparisons between the prices of different countries.[32] A major advantage of this regression method for constructing an elementary price index is that *standard errors* for the index number α can be obtained. This advantage of the stochastic approach to index number theory was stressed by Selvanathan and Rao (1994).

It can be verified that the least squares estimator for γ is

(67) $\gamma^* \equiv \sum_{m=1}^{M} (1/M) \ln(p_m^1/p_m^0)$.

If γ^* is exponentiated, then the following estimator for the elementary aggregate α is obtained:

(68) $\alpha^* \equiv \prod_{m=1}^{M} [p_m^1/p_m^0]^{1/M} \equiv P_J(p^0,p^1)$

where $P_J(p^0,p^1)$ is the *Jevons elementary price index* defined in section 3 above. Thus we have obtained a regression model based justification for the use of the Jevons elementary index.

Consider the following unweighted *least squares model*:

(69) $\min_{\gamma,\, \delta\text{'s}} \sum_{m=1}^{M} [\ln p_m^0 - \delta_m]^2 + \sum_{m=1}^{M} [\ln p_m^1 - \gamma - \delta_m]^2$.

It can be verified that the γ solution to the unconstrained minimization problem (69) is the γ^* defined by (67).

There is a problem with the unweighted least squares model defined by (69): namely that the logarithm of each price quote is given exactly the *same weight* in the model no matter what the expenditure on that item was in each period. This is obviously unsatisfactory since a price that has very little economic importance (i.e., a low expenditure share in

[31] Hedonic regression models will be studied in more detail in a later chapter.
[32] See Summers (1973). In our special case, there are only two "countries" which are the two observations on the prices of the elementary aggregate for two periods.

each period) is given the same weight in the regression model compared to a very important item. Thus it is useful to consider the following *weighted least squares model*:

(70) $\min_{\gamma,\delta's} \sum_{m=1}^{M} s_m^0 [\ln p_m^0 - \delta_m]^2 + \sum_{m=1}^{M} s_m^1 [\ln p_m^1 - \gamma - \delta_m]^2$

where the period t expenditure share on commodity m is defined in the usual manner as

(71) $s_m^t \equiv p_m^t q_m^t / \sum_{k=1}^{M} p_k^t q_k^t$; $t = 0,1$; $m = 1,...,M$.

Thus in the model (70), the logarithm of each item price quotation in each period is weighted by its expenditure share in that period. Note that weighting prices by their economic importance is consistent with Theil's (1967; 136-138) stochastic approach to index number theory.[33]

The γ solution to (70) is

(72) $\gamma^{**} = [\sum_{m=1}^{M} h(s_m^0,s_m^1) \ln(p_m^1/p_m^0)]/[\sum_{i=1}^{M} h(s_i^0,s_i^1)]$

where

(73) $h(a,b) \equiv [(1/2)a^{-1} + (1/2)b^{-1}]^{-1} = 2ab/[a + b]$

and h(a,b) is the *harmonic mean* of the numbers a and b. Thus γ^{**} is a share weighted average of the logarithms of the price ratios p_m^1/p_m^0. If γ^{**} is exponentiated, then an estimator α^{**} for the elementary aggregate α is obtained.

How does α^{**} compare to the three ideal elementary price indexes defined in section 2 above? It can be shown[34] that α^{**} approximates those three indexes to the second order around an equal price and quantity point; i.e., for most data sets, α^{**} will be very close to the Fisher, Törnqvist and Walsh elementary indexes.

In fact, a slightly different weighted least squares problem that is similar to (70) will generate exactly the Törnqvist elementary index. Thus consider the following *weighted least squares model*:

(74) $\min_{\gamma,\delta's} \sum_{m=1}^{M}(1/2)(s_m^0+s_m^1)[\ln p_m^0 - \delta_m]^2 + \sum_{m=1}^{M}(1/2)(s_m^0+s_m^1)[\ln p_m^1 - \gamma - \delta_m]^2$.

Thus in the model (74), the logarithm of each item price quotation in each period is weighted by the arithmetic average of its expenditure shares in the two periods under consideration.

The γ solution to (74) is

[33] Theil's approach is also pursued by Rao (2002), who considered a generalization of (70) to cover the case of many time periods.
[34] Use the techniques in Diewert (1978).

(75) $\gamma^{***} = \sum_{m=1}^{M} (1/2)(s_m^0+s_m^1) \ln(p_m^1/p_m^0)$

which is the logarithm of the Törnqvist elementary index. Thus the exponential of γ^{***} is precisely the Törnqvist price index.

The results in this section provide some weak support for the use of the Jevons elementary index but they provide much stronger support for the use of weighted elementary indexes of the type defined in section 2 above.

The results in this section also provide support for the use of value based weights in hedonic regressions.

Problems

5. Show that γ^{**} defined by (72) solves (70).

6. Show that γ^{***} defined by (75) solves (74).

7. Show that γ^{**} defined by (72) approximates γ^{***} defined by (75) to the second order around an equal price and quantity point.

I. Conclusion

The main results in this chapter can be summarized as follows:

- In order to define a "best" elementary index number formula, it is necessary to have a target index number concept. In section 2, it is suggested that normal bilateral index number theory applies at the elementary level as well as at higher levels and hence the target concept should be one of the Fisher, Törnqvist or Walsh formulae.
- When aggregating the prices of the same narrowly defined item within a period, the narrowly defined unit value is a reasonable target price concept.
- The axiomatic approach to traditional elementary indexes (i.e., no quantity or value weights are available) supports the use of the Jevons formula under all circumstances. If the items in the elementary aggregate are very homogeneous (i.e., they have the same unit of measurement), then the Dutot formula can be used. In the case of a heterogeneous elementary aggregate (the usual case), the Carruthers, Sellwood and Ward formula can be used as an alternative to the Jevons formula but both will give much the same numerical answers.
- The Carli index has an upward bias and the Harmonic index has a downward bias.
- The economic approach to elementary indexes weakly supports the use of the Jevons formula.

- All five unweighted elementary indexes are not really satisfactory. A much more satisfactory approach would be to collect quantity or value information along with price information and form sample superlative indexes as the preferred elementary indices.
- A simple hedonic regression approach to elementary indexes supports the use of the Jevons formula. However, a more satisfactory approach is to use a weighted hedonic regression approach. The resulting index will closely approximate the ideal indexes defined in section 2.

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