INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

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CHAPTER 14: PPI Index Number Computations Using an Artificial Data Set

1. Introduction

In order to give the reader some idea of how the various PPI indexes defined in chapter 13 might perform in practice, in section 2 below, we compute all of the major indexes defined in the previous chapters using an artificial "industrial" data set consisting of prices and quantities for 8 commodities over 5 periods. The period can be thought of as somewhere between a year and 5 years. The trends in the data are generally more pronounced than one would see in the course of a year. The 8 commodities can be thought of as the net deliveries to the final demand sector of all industries in the economy. The first 6 commodities are outputs and correspond to the usual private consumption plus government consumption plus investment plus export deliveries to final demand while the last 2 commodities are imports (and hence are indexed with a negative sign).

In section 3, the same final demand data set is used in order to compute the midyear indexes that were described in section 5 of chapter 7. Recall that these indexes have an important practical advantage over superlative indexes because they can be computed using current data on prices and lagged data on quantities (or equivalently, using lagged data on expenditures).

In section 4, the additive percentage change decompositions for the Fisher ideal price index that were discussed in section 9 of chapter 3 are illustrated using the final demand data set on 8 commodities.

In section 5, price and quantity data for three industrial sectors of the economy are presented. This industrial data set is consistent with the final demand data set listed in section 2 below. We construct value added deflators for these 3 industries in sections 6, 7 and 8. Only the Laspeyres, Paasche, Fisher and Törnqvist formulae are considered in section 5 and subsequent sections since these are the formulae that are likely to be used in practice.

In section 9, 10 and 11, the industry data are used in order to construct national output price indexes, national intermediate input price deflators and national value added deflators respectively. The construction of a national value added deflator by aggregating the national output and intermediate input price indexes is undertaken in section 12. This two stage national value added deflator is then compared with its single stage counterpart and also with the final demand deflator constructed in section 2.

2. Price Indexes for Final Demand Components

The price and quantity data for net deliveries to final demand are listed in Tables 1 and 2 below. For convenience, we have also listed the period t nominal expenditures, $p^t \bullet q^t \equiv \sum_{i=1}^{8} p_i^t q_i^t$, along with the corresponding period t expenditure shares, $s_i^t \equiv p_i^t q_i^t / p^t \bullet q^t$, in Table 3.

Table 1. Prices for Eight Commodities

Period t	p_1^t	p_2^t	p_3^t	p_4^t	p_5^t	p_6^t	p_7^t	p_8^t
1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
2	1.3	2.0	1.3	0.7	1.4	0.8	2.1	0.7
3	1.0	1.0	1.5	0.5	1.7	0.6	1.0	0.5
4	0.7	0.5	1.6	0.3	1.9	0.4	0.6	0.3
5	1.0	1.0	1.7	0.2	2.0	0.2	1.0	0.2

Table 2. Quantities for Eight Commodities

Period t	q_1^t	q_2^t	q_3^t	q_4^t	q_5^t	q_6^t	q_7^t	q_8^t
1	30	10	40	10	45	5	-28	-7
2	28	8	39	13	47	6	-20	-9
3	30	11	38	30	50	8	-29	-21
4	32	14	39	60	56	13	-35	-42
5	29	12	40	100	65	25	-30	-70

Table 3. Net Expenditures and Net Expenditure Shares for Eight Commodities

Period t	$p^t \bullet q^t$	s_1^t	s_2^t	s_3^t	s_4^t	s_5^t	s_6^t	s_7^t	s_8^t
1	105.0	0.2857	0.0952	0.3810	0.0952	0.4286	0.0476	-0.2667	-0.0667
2	134.5	0.2706	0.1190	0.3770	0.0677	0.4892	0.0357	-0.3123	-0.0468
3	163.3	0.1837	0.0674	0.3491	0.0919	0.5205	0.0294	-0.1776	-0.0643
4	187.8	0.1193	0.0373	0.3323	0.0958	0.5666	0.0277	-0.1118	-0.0671
5	220.0	0.1318	0.0545	0.3091	0.0909	0.5909	0.0227	-0.1364	-0.0636

Typically, the statistical agency will not have quantity data available; only price and expenditure data will be collected. However, given the information in Table 3, the period t net expenditure shares s_n^t may be multiplied by period t total net expenditures $p^t \bullet q^t$ in order to obtain final demand expenditures by commodity and then these commodity expenditures may be divided by the corresponding prices in Table 1 in order to obtain the implicit quantities listed in Table 2.[1]

The trends that are built into the above tables can be explained as follows. Think of the first 4 commodities as the final demand consumption of various classes of *goods* in some economy while the next two commodities are the consumption of two classes of *services*. Think of the first good as *agricultural consumption and exports*. The final demand

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[1] Typically, the prices will be price relatives or averages of price relatives but if we make the base period equal to period 1, then these relative prices will all be unity in period 1.

quantity for this good mildly fluctuates around 30 units of output while its price fluctuates more violently around 1. However, as the rest of the economy grows, the share of agricultural output declines to about one half of its initial share. The second good is *energy consumption* in final demand. The quantity of this good trends up gently during the five periods with some fluctuations. However, note that the price of energy fluctuates wildly from period to period.[2] The third good is *traditional manufactures*. There are rather high inflation rates for this commodity for periods 2 and 3 which diminish to a very low inflation rate by the end of our sample period.[3] The final demand consumption of traditional manufactured goods is more or less static in our data set. The fourth commodity is *high technology manufactured goods*; e.g., computers, video cameras, compact disks, etc. The demand for these high tech commodities grows tenfold times over our sample period while the final period price is only one fifth of the first period price. The fifth commodity is *traditional services*. The price trends for this commodity are similar to traditional manufactures, except that the period to period inflation rates are a bit higher. However, we have the demand for traditional services growing much more strongly than for traditional manufactures. Our sixth commodity is *high technology services*; e.g., telecommunications, wireless phones, internet services, stock market trading, etc. For this final commodity, we have the price trending downward very strongly to end up at 20% of the starting level while demand increases fivefold. The final two commodities are *energy imports* and *imports of high technology manufactured goods*. Since imports are intermediate inputs to the economy as a whole, the quantities for these last two commodities are indexed with minus signs. The prices and quantities for the two imported commodities are more or less proportional to the corresponding final consumption demand prices and quantities. The movements of prices and quantities in this artificial data set are more pronounced than the year to year movements that would be encountered in a typical country but they do illustrate the problem that is facing compilers of the Producer Price Index; namely, *year to year price and quantity movements are far from being proportional across commodities so the choice of index number formula will matter.*

Every price statistician is familiar with the *Laspeyres index* P_L and the *Paasche index* P_P defined by (20) and (21) in chapter 1 above. These indexes are listed in Table 4 along with the two unweighted indexes that were considered in chapter 10: the *Carli index* defined by (2) and the *Jevons index* defined by (3) in chapter 10. The indexes in Table 4 compare the prices in period t with the prices in period 1; i.e., they are *fixed base indexes*. Thus the period t entry for the Carli index, P_C, is simply the arithmetic mean of the 8 price relatives, $\sum_{i=1}^{8} (1/8)(p_i^t/p_i^1)$, while the period t entry for the Jevons index, P_J, is the geometric mean of the 8 price relatives, $\prod_{i=1}^{8} (p_i^t/p_i^1)^{1/8}$.

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[2] This is an example of the price bouncing phenomenon noted by Szulc (1983). Note that the fluctuations in the price of energy that we have built into our data set are not that unrealistic: in the past 3 years, the price of a barrel of crude oil has fluctuated in the range $10 to $37 U.S. Note that agricultural prices also bounce but not as violently.

[3] This corresponds roughly to the experience of most industrialized countries over the period starting in 1973 to the mid 1990's. Thus we are compressing roughly 5 years of price movement into one of our periods.

Table 4. The Fixed Base Laspeyres, Paasche, Carli and Jevons Indexes

Period t	P_L	P_P	P_C	P_J
1	1.0000	1.0000	1.0000	1.0000
2	1.1552	1.2009	1.2875	1.1853
3	1.4571	1.3957	0.9750	0.8868
4	1.5390	1.3708	0.7875	0.6240
5	1.6343	1.2865	0.9125	0.6373

Note that by period 5, the spread between the fixed base Laspeyres and Paasche price indexes is fairly large: P_L is equal to 1.6343 while P_P is 1.2865*, a spread of about 27%.* Since both of these indices have exactly the same *theoretical* justification, it can be seen that the choice of index number formula matters a lot. There is also a substantial spread between the two unweighted indexes by period 5: the fixed base Carli index is equal to 0.9125, while the fixed base Jevons index is 0.6373, *a spread of about 43%.* However, more troublesome than this spread is the fact that *the unweighted indexes are far below both the Paasche and Laspeyres indices* by period 5.[4] Thus when there are divergent trends in both prices and quantities, it will usually be the case that unweighted price indexes will give very different answers than their weighted counterparts. Since none of the index number theories considered in previous chapters supported the use of unweighted indexes, their use is not recommended. Finally, note that the Jevons index is always considerably below the corresponding Carli index. This will always be the case (unless prices are proportional in the two periods under consideration) because a geometric mean is always equal to or less than the corresponding arithmetic mean.[5]

It is of interest to recalculate the 4 indexes listed in Table 4 above using *the chain principle* rather than the *fixed base principle*. Our expectation is that the spread between the Paasche and Laspeyres indexes will be reduced by using the chain principle. These chain indexes are listed in Table 5.

Table 5. Chained Laspeyres, Paasche, Carli and Jevons Indexes

Period t	P_L	P_P	P_C	P_J
1	1.0000	1.0000	1.0000	1.0000
2	1.1552	1.2009	1.2875	1.1853
3	1.3743	1.4834	1.0126	0.8868
4	1.4374	1.5349	0.7406	0.6240

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[4] The reason for this is that when using weighted indexes, the imports of high technology goods are offset by the final demand expenditures on high technology goods to a large extent; i.e., commodities 6 and 8 have the same dramatic downward price trends but their quantity trends are opposite in sign and cancel each other out to a large extent. However, when calculating the unweighted indexes, this cancellation does not occur and the downward trends in the prices of commodities 6 and 8 get a much higher implicit weight in the unweighted indexes.

[5] This is the Theorem of the Arithmetic and Geometric Mean; see Hardy, Littlewood and Polyá (1934).

5	1.4963	1.5720	0.8372	0.6373

It can be seen comparing Tables 4 and 5 that chaining eliminated about 3/4 of the spread between the Paasche and Laspeyres indexes for period 5. However, even the chained Paasche and Laspeyres indices differ by about 8% in period 3 so the choice of index number formula still matters. Note that chaining did not affect the Jevons index. This is an advantage of the index but the lack of weighting is a fatal flaw. We would expect the "truth" to lie between the Paasche and Laspeyres indexes and from Table 5, we see that the unweighted Jevons index is far below this acceptable range. Note that chaining did not affect the Carli index in a systematic way for our particular data set: in period 3, the chained Carli is above the corresponding fixed base Carli but in periods 4 and 5, the chained Carli is below the fixed base Carli.

We turn now to a systematic comparison of all of *the asymmetrically weighted price indexes*. The *fixed base indexes* are listed in Table 6. The fixed base *Laspeyres* and *Paasche indexes*, P_L and P_P, are the same as those indexes listed in Table 4 above. The *Palgrave index*, P_{PAL}, is defined by equation (53) in chapter 1. The indexes denoted by P_{GL} and P_{GP} are *the geometric Laspeyres and geometric Paasche indexes*[6] which are special cases of the fixed weight geometric indexes defined by Konüs and Byushgens (1926). For *the geometric Laspeyres index*, P_{GL}, we let the weights α_i be the *base period expenditure shares*, s_i^1. This index should be considered an alternative to the fixed base Laspeyres index since each of these indexes makes use of the same information set. For *the geometric Paasche index*, P_{GP}, we let the weights α_i be the *current period expenditure shares*, s_i^t. Finally, the index P_{HL} *is the harmonic Laspeyres index* that was defined by (57) in chapter 1.

Table 6. Asymmetrically Weighted Fixed Base Indexes

Period t	P_{PAL}	P_{GP}	P_L	P_{GL}	P_P	P_{HL}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1520	1.1852	1.1552	1.1811	1.2009	1.1906
3	1.5133	1.4676	1.4571	1.4018	1.3957	1.3212
4	1.6628	1.5661	1.5390	1.4111	1.3708	1.2017
5	1.7673	1.6374	1.6343	1.4573	1.2865	1.0711

By looking at the period 5 entries in Table 6, it can be seen that the spread between all of these fixed base asymmetrically weighted indexes has increased to be even larger than our earlier spread of 27% between the fixed base Paasche and Laspeyres indexes. In Table 6, the period 5 Palgrave index is about 1.65 times as big as the period 5 harmonic Laspeyres index, P_{HL}. Again, *this illustrates the point that due to the nonproportional growth of prices and quantities in most economies today, the choice of index number formula is very important.*

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[6] Vartia (1978; 272) used the terms *logarithmic Laspeyres* and *logarithmic Paasche* respectively.

If there were no negative quantities in the final demand vectors, then it is possible to explain why certain of the indexes in Table 6 are bigger than others. If all weights are positive, it can be shown that a *weighted arithmetic mean* of N numbers is equal to or greater than the corresponding *weighted geometric mean* of the same N numbers which in turn is equal to or greater than the corresponding *weighted harmonic mean* of the same N numbers.[7] It can be seen that the three indexes P_{PAL}, P_{GP} and P_P all use the current period expenditure shares s_i^t to weight the price relatives (p_i^t/p_i^1) but P_{PAL} is a weighted *arithmetic* mean of these price relatives, P_{GP} is a weighted *geometric* mean of these price relatives and P_P is a weighted *harmonic* mean of these price relatives. Thus if there are no negative components in final demand, by Schlömilch's inequality, we would have:[8]

$$(1)\quad P_{PAL} \geq P_{GP} \geq P_P .$$

However, due to the existence of imports in each period (which leads to negative quantities for these components of the final demand vector), the inequalities (1) are not necessarily true. Viewing Table 6, it can be seen that the inequalities (1) hold for periods 3, 4 and 5 but not for period 2. It can also be verified that the three indexes P_L, P_{GL} and P_{HL} all use the base period expenditure shares s_i^1 to weight the price relatives (p_i^t/p_i^1) but P_L is a weighted *arithmetic* mean of these price relatives, P_{GL} is a weighted *geometric* mean of these price relatives and P_{HL} is a weighted *harmonic* mean of these price relatives. If all of these shares were nonnegative, then by Schlömilch's inequality, we would have:[9]

$$(2)\quad P_L \geq P_{GL} \geq P_{HL} .$$

However, due to the existence of imports in each period, the inequalities (2) are not necessarily true. Viewing Table 6, it can be seen that the inequalities (2) hold for periods 3, 4 and 5 but not for period 2.

We continue with our systematic comparison of all of *the asymmetrically weighted price indexes*. These indexes using the *chain principle* are listed in Table 7.

Table 7. Asymmetrically Weighted Indexes Using the Chain Principle

Period t	P_{PAL}	P_{GP}	P_L	P_{GL}	P_P	P_{HL}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1520	1.1852	1.1552	1.1811	1.2009	1.1906
3	1.3444	1.4050	1.3743	1.4569	1.4834	1.6083
4	1.4229	1.4730	1.4374	1.5057	1.5349	1.6342
5	1.4942	1.5292	1.4963	1.5510	1.5720	1.6599

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[7] This follows from Schlömilch's (1858) inequality; see Hardy, Littlewood and Polyá (1934; chapter 11).

[8] These inequalities were noted by Fisher (1922; 92) and Vartia (1978; 278).

[9] These inequalities were also noted by Fisher (1922; 92) and Vartia (1978; 278).

Viewing Table 7, it can be seen that the use of the chain principle dramatically reduced the spread between all of the asymmetrically weighted indexes compared to their fixed base counterparts in Table 6. For period 5, the spread between the smallest and largest asymmetrically weighted fixed base index was 65% but for the period 5 chained indexes, this spread was reduced to 11%.

Symmetrically weighted indexes can be decomposed into two classes: *superlative indexes* and *other symmetrically weighted indexes*. Superlative indexes have a close connection to economic theory; i.e., as we saw in Chapter 13, a superlative index is exact for a representation of the producer's production function or the corresponding unit revenue function that can provide a second order approximation to arbitrary technologies that satisfy certain regularity conditions. We considered 4 primary superlative indexes in Chapter 13:

- the *Fisher ideal price index* P_F defined by (9) in chapter 13;
- the *Walsh price index* P_W defined by (53) in chapter 13[10];
- the *Törnqvist-Theil price index* P_T defined by (10) in chapter 13 and
- the *implicit Walsh price index* P_{IW} that corresponds to the Walsh quantity index Q_W defined by (4.9) (this is also the index P^1 defined by (58) in chapter 13).

These 4 symmetrically weighted superlative price indexes are listed in Table 8 using the fixed base principle. We also list in this table two symmetrically weighted (but not superlative) price indexes:[11]

- the Marshall Edgeworth price index P_{ME} defined by (19) in chapter 3 and
- the Drobisch Sidgwick Bowley price index P_D (the arithmetic average of the Paasche and Laspeyres indexes) defined above (25) in chapter 1.

Table 8. Symmetrically Weighted Fixed Base Indexes

Period t	$\mathbf{P_T}$	$\mathbf{P_{IW}}$	$\mathbf{P_W}$	$\mathbf{P_F}$	$\mathbf{P_D}$	$\mathbf{P_{ME}}$
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1831	1.1827	1.1814	1.1778	1.1781	1.1788
3	1.4343	1.4339	1.4327	1.4261	1.4264	1.4248
4	1.4866	1.4840	1.4820	1.4525	1.4549	1.4438
5	1.5447	1.5320	1.5193	1.4500	1.4604	1.4188

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[10] Since we cannot take square roots of negative quantities, we need to change our sign conventions when calculating this index: change the negative quantities into positive quantities and change the corresponding positive prices into negative prices.

[11] Diewert (1978; 897) showed that the Drobisch Sidgwick Bowley price index approximates any superlative index to the second order around an equal price and quantity point; i.e., P_{SB} is a *pseudo-superlative index*. Straightforward computations show that the Marshall Edgeworth index P_{ME} is also pseudo-superlative.

Note that the Drobisch index P_D is always equal to or greater than the corresponding Fisher index P_F. This follows from the facts that the Fisher index is the geometric mean of the Paasche and Laspeyres indexes while the Drobisch index is the arithmetic mean of the Paasche and Laspeyres indexes and an arithmetic mean is always equal to or greater than the corresponding geometric mean. Comparing the fixed base asymmetrically weighted indexes, Table 6, with the symmetrically weighted indices, Table 8, *it can be seen that the spread between the lowest and highest index in period 5 is much less for the symmetrically weighted indexes*. The spread was 1.7673/1.0711 = 1.65 for the asymmetrically weighted indexes but only 1.5447/1.4188 = 1.09 for the symmetrically weighted indexes. If we restrict ourselves to the superlative indexes listed for period 5 in Table 8, then this spread is further reduced to 1.5447/1.4500 = 1.065; i.e., the spread between the fixed base superlative indexes is "only" 6.5% compared to the fixed base spread between the Paasche and Laspeyres indexes of 27% (1.6343/1.2865 = 1.27). We expect to further reduce the spread between the superlative indexes by using the chain principle.

We recompute the symmetrically weighted indexes using the chain principle. The results may be found in Table 9.

Table 9. Symmetrically Weighted Indexes Using the Chain Principle

Period t	P_T	P_{IW}	P_W	P_F	P_D	P_{ME}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1831	1.1827	1.1814	1.1778	1.1781	1.1788
3	1.4307	1.4257	1.4298	1.4278	1.4288	1.4290
4	1.4893	1.4844	1.4889	1.4853	1.4861	1.4862
5	1.5400	1.5344	1.5387	1.5337	1.5342	1.5338

A quick glance at Table 9 shows that *the combined effect of using both the chain principle as well as symmetrically weighted indexes is to dramatically reduce the spread between all indexes constructed using these two principles*. The spread between all of the symmetrically weighted indexes in period 5 is only 1.5400/1.5337 = 1.004 or 0.4%, which is the same as the spread between the 4 superlative indexes in period 5.[12]

The results listed in Table 9 reinforce the numerical results tabled in Hill (2004) and Diewert (1978; 894): *the most commonly used chained superlative indexes will generally give approximately the same numerical results*.[13] In particular, the chained Fisher, Törnqvist and Walsh indexes will generally approximate each other very closely.

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[12] On average over the last 4 periods, the chain Fisher and the chain Törnqvist indexes differed by 0.0046 percentage points.

[13] More precisely, the superlative quadratic mean of order r price indexes P^r defined by (55) in chapter 13 and the implicit quadratic mean of order r price indexes P^{r*} defined by (52) in chapter 13 will generally closely approximate each other provided that r is in the interval $0 \le r \le 2$. Note that when one or more of the quantities being aggregated is negative (as in the present situation), we must change our sign conventions when calculating Q^r or P^{r*}: change the negative sign on import quantities to positive and make the import prices negative.

We now turn our attention to the differences between superlative indexes and their counterparts that are constructed in two stages of aggregation; see chapter 9 for a discussion of the issues and a listing of the formulae used. In our artificial data set, we will first aggregate the first 4 commodities into a *goods aggregate*, the next two commodities into a *services aggregate* and the last two commodities into an imports aggregate. In the second stage of aggregation, these three price and quantity components will be aggregated into a net final demand price index.

We report the results in Table 10 for our two stage aggregation procedure using period 1 as the *fixed base* for the Fisher index P_F, the Törnqvist index P_T and the Walsh and implicit Walsh indexes, P_W and P_{IW}.

Table 10. Fixed Base Superlative Single Stage and Two Stage Indexes

Period t	P_F	P_{F2S}	P_T	P_{T2S}	P_W	P_{W2S}	P_{IW}	P_{IW2S}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1778	1.1830	1.1831	1.1837	1.1814	1.1835	1.1827	1.1829
3	1.4261	1.4259	1.4343	1.4351	1.4327	1.4341	1.4339	1.4325
4	1.4525	1.4713	1.4866	1.4974	1.4820	1.4990	1.4840	1.4798
5	1.4500	1.4366	1.5447	1.5440	1.5193	1.5208	1.5320	1.5191

Viewing Table 10, it can be seen that the fixed base single stage superlative indexes generally approximate their fixed base two stage counterparts fairly closely. The divergence between the single stage Fisher index P_F and its two stage counterpart P_{F2S} in period 5 is 1.4500/1.4388 = 1.009 or 0.9%. The other divergences are even less.

Using *chain indexes*, we report the results in Table 11 for our two stage aggregation procedure. Again, the single stage and their two stage counterparts are listed for the Fisher index P_F, the Törnqvist index P_T and the Walsh and implicit Walsh indexes, P_W and P_{IW}.

Table 11. Chained Superlative Single Stage and Two Stage Indexes

Period t	P_F	P_{F2S}	P_T	P_{T2S}	P_W	P_{W2S}	P_{IW}	P_{IW2S}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1778	1.1830	1.1831	1.1837	1.1814	1.1835	1.1827	1.1829
3	1.4278	1.4448	1.4307	1.4309	1.4298	1.4378	1.4257	1.4282
4	1.4853	1.5059	1.4893	1.4907	1.4889	1.4991	1.4844	1.4871
5	1.5337	1.5556	1.5400	1.5419	1.5387	1.5499	1.5344	1.5372

Viewing Table 11, it can be seen that the chained single stage superlative indexes generally approximate their fixed base two stage counterparts quite closely. The divergence between the chained single stage Fisher index P_F and its two stage counterpart P_{F2S} in period 5 is 1.5556/1.5337 = 1.014 or 1.4%. The other divergences are all less than this. Given the large dispersion in period to period price movements, these two stage aggregation errors are not large. However, the important point that emerges from Table 11 is that *the use of the chain principle has reduced the spread between all 8 single stage and two stage superlative indexes* compared to their fixed base counterparts in Table 10. The maximum spread for the period 5 chained index values is 1.4% while the maximum spread for the period 5 fixed base index values is 7.5%.

3. Midyear Indexes

The next formulae that we illustrate using our artificial data set are the arithmetic and geometric type midyear indexes defined in section 19 of Chapter 13. Recall that these indexes are due to Schultz (1999) and Okamoto (2001). Basically, midyear indexes are fixed basket type indexes, where the basket of quantities being priced is midway between the base period and the current period. If the current period t less the base period 1 is an even integer, then we use the quantity vector $q^{(t-1)/2}$ as the midyear basket. If the current period t less the base period 1 is an odd integer, then the midyear basket is an average of the two midyear quantity vectors, $q^{t/2}$ and $q^{(t/2)+1}$. If we take the arithmetic average of these two midyear baskets, we obtain the sequence of *fixed base arithmetic type midyear indices*, $P_{OSA}{}^{t}$, defined by (160) in chapter 13. If we take the geometric average of these two midyear baskets, we obtain the sequence of *fixed base geometric type midyear indices*, $P_{OSG}{}^{t}$, defined by (161) in Chapter 13.[14] Recall also that going from period 1 to period 2, the period 2 *midyear arithmetic type index number* $P_{OSA}{}^{2}$ is equal to $P_{ME}(p^1,p^2,q^1,q^2)$, the Marshall (1887) Edgeworth (1925) price index for period 2 and , the period 2 *midyear geometric type index number* $P_{OSG}{}^{2}$ is equal to $P_W(p^1,p^2,q^1,q^2)$, the Walsh (1901) price index for period 2.[15]

The two sequences of *fixed base midyear price indexes*, $P_{OSA}{}^{t}$ and $P_{OSG}{}^{t}$, along with the corresponding *fixed base Fisher, Törnqvist and Walsh price indexes*, $P_F{}^t$, $P_T{}^t$ and $P_W{}^t$ respectively, are listed in Table 12.

Table 12. Fixed Base Arithmetic and Geometric Type Midyear Indexes

Period t	P_{OSA}	P_{OSG}	P_F	P_T	P_W
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1788	1.1814	1.1778	1.1831	1.1814
3	1.4286	1.4286	1.4261	1.4343	1.4327
4	1.4747	1.4783	1.4525	1.4866	1.4820
5	1.5385	1.5385	1.4500	1.5447	1.5193

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[14] Since our quantity vectors have two negative components (and thus we cannot take square roots of these negative components), we need to change our sign conventions when evaluating these geometric type midyear indexes; make all quantities positive but change the prices of the import components from positive to negative. Thus when calculating a geometric type midyear index where it is necessary to take the geometric average of two midyear quantity vectors, we make the same sign conventions as we made when calculating Walsh price indexes where the same problem occurred.

[15] As usual, when calculating this Walsh price index, switch the signs of the negative import quantities to positive signs and make the corresponding import prices negative.

Note that for t odd, the arithmetic and geometric type midyear indexes, $P_{OSA}{}^t$ and $P_{OSG}{}^t$, coincide. This is as it should be because when t is odd, both indexes are set equal to the Schultz midyear index, since there is a single unique midyear basket in this case. The two sequences of midyear indexes differ only for t even since in the even case, there are two midyear baskets and we must decide on arithmetic or geometric averaging of these baskets. Note also that the Walsh index for period 2 is equal to the corresponding geometric type midyear index since this is true by construction. Finally, note that with the exception of the Fisher fixed base index, P_F, the other 4 fixed base indexes listed in Table 12 approximate each other surprisingly closely, given the tremendous variability that was built into the underlying data set.

We turn now to the chained counterparts to the indexes listed in Table 12 above. Recall that the chained sequence of arithmetic and geometric type midyear indexes was defined by (162) and (163) in chapter 13 respectively. The two sequences of *chained midyear price indexes*, $P_{OSA}{}^t$ and $P_{OSG}{}^t$, along with the corresponding *chained Fisher, Törnqvist and Walsh price indexes*, $P_F{}^t$, $P_T{}^t$ and $P_W{}^t$ respectively, are listed in Table 13.

Table 13. Chained Arithmetic and Geometric Type Midyear Indexes

Period t	$\mathbf{P_{OSA}}$	$\mathbf{P_{OSG}}$	$\mathbf{P_F}$	$\mathbf{P_T}$	$\mathbf{P_W}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1788	1.1814	1.1778	1.1831	1.1814
3	1.4286	1.4286	1.4278	1.4307	1.4298
4	1.5230	1.5263	1.4853	1.4893	1.4889
5	1.5388	1.5388	1.5337	1.5400	1.5387

Note that for t odd, the chained arithmetic and geometric type midyear indexes, $P_{OSA}{}^t$ and $P_{OSG}{}^t$, coincide. This is as it should be because when t is odd, both indexes are set equal to chained Schultz midyear indexes. What is striking in looking at Table 13 is how close the chained midyear indexes are to their chained superlative counterparts for odd periods. For year 5, the maximum spread between the 5 indexes is the spread between the chained Fisher and Törnqvist indexes, which was only 1.5400/1.5337 = 1.004 or .4%. The explanation for this rather remarkable result is that for odd periods, the underlying price and quantity data have fairly smooth trends and under these circumstances, we would expect the midyear indexes to approximate the superlative Walsh index rather closely as was indicated in Chapter 13. However, for periods 2 and 4, the underlying data "bounce" considerably and so the trends in the data switch abruptly and thus under these conditions, it is expected that the midyear indexes could deviate from their superlative counterparts. This expectation is borne out by looking at the entries for period 4 in Table 13 where the two midyear indexes are about 2 to 3% higher than their chained superlative counterparts.

The conclusion that emerges from Tables 12 and 13 is that midyear indexes approximate their superlative counterparts surprisingly well but not perfectly well. Given the large amount of variability in the underlying price and quantity data, it appears that the *midyear indexes could be used to give very good advanced estimates of superlative indexes*, which necessarily cannot be evaluated on a very timely basis.

4. Additive Percentage Change Decompositions for the Fisher Index

The final formulae that is illustrated using the artificial final expenditures data set are the *additive percentage change decompositions* for the Fisher ideal index that were discussed in section 9 of Chapter 3. We will first decompose the *chain links* for the Fisher price index using the Diewert (2002) decomposition formulae, (47) to (49) in chapter 3. The results of the decomposition are listed in Table 14. Thus $P_F - 1$ is *the percentage change in the Fisher ideal chain link* going from period t − 1 to t and *the decomposition factor* $v_{Fi}\Delta p_i = v_{Fi}\,(p_i^t - p_i^{t-1})$ is the contribution to the total percentage change of the change in the ith price from p_i^{t-1} to p_i^t for i = 1,2,…,8.

Table 14. An Additive Percentage Change Decomposition of the Fisher Index

t	$P_F - 1$	$v_{F1}\Delta p_1$	$v_{F2}\Delta p_2$	$v_{F3}\Delta p_3$	$v_{F4}\Delta p_4$	$v_{F5}\Delta p_5$	$v_{F6}\Delta p_6$	$v_{F7}\Delta p_7$	$v_{F8}\Delta p_8$
2	0.1778	0.0791	0.0816	0.1079	-0.0316	0.1678	-0.0101	-0.2389	0.0220
3	0.2122	-0.0648	-0.0716	0.0571	-0.0331	0.1084	-0.0105	0.2037	0.0231
4	0.0403	-0.0541	-0.0363	0.0224	-0.0519	0.0616	-0.0121	0.0744	0.0363
5	0.0326	0.0459	0.0326	0.0198	-0.0396	0.0302	-0.0187	-0.0653	0.0277

Viewing Table 14, it can be seen that the price index going from period 1 to 2 grew 17.78% and the major contributors to this change were the increases in the price of commodity 1, finally demanded agricultural products (7.91 percentage points); commodity 2, finally demanded energy (8.16 percentage points); commodity 3, finally demanded traditional manufactures (10.79 percentage points); commodity 5, traditional services (16.78 percentage points); and commodity 7, energy imports (−23.89 percentage points). The sum of the last eight entries for period 2 in Table 14 is equal to .1778, the percentage increase in the Fisher price index going from period 1 to 2. Note that although the price of energy imports *increased* dramatically in period 2, the contribution to the overall price change is *negative* due to the fact that the quantity of energy imports is indexed with a negative sign. Similarly, although the price of high technology imports *decreased* dramatically in period 2, the contribution to the overall price change is *positive* due to the fact that the quantity of high technology imports is indexed with a negative sign.[16] Thus care must be taken in interpreting the last two columns of Table 14 due to the fact that there are negative quantities for some components of the aggregate.[17] It can

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[16] Since the expenditure share of high technology imports is small, the large decrease in price does not translate into a large change in the overall Fisher price index for final demand expenditures.

[17] The counterintuitive numbers in the last two columns of Table 14 help to explain why the deflator for final demand expenditures (or the GDP deflator as it is commonly known) is not a satisfactory indicator of inflationary pressures in the economy since a large *increase* in the relative price of imported goods leads to a *decrease* in the index.

be seen that a big price change in a particular component i combined with a big expenditure share in the two periods under consideration will lead to a big decomposition factor, v_{Fi}.

Our final set of computations we illustrate using our artificial data set is the *additive percentage change decomposition* for the Fisher ideal index that is due to Van IJzeren (1987; 6) that was mentioned in section 9 of Chapter 3.[18] The *price* counterpart to the *additive decomposition* for a quantity index is:

$$(3)\ \ P_F(p^0,p^1,q^0,q^1) = \sum_{i=1}^{8} q_{Fi}{*}p_i^1 \,/\, \sum_{i=1}^{8} q_{Fi}{*}p_i^0$$

where the reference quantities need to be defined somehow. Van IJzeren (1987; 6) showed that the following reference weights provided an *exact additive representation for the Fisher ideal price index*:

$$(4)\ \ q_{Fi}{*} \equiv (1/2)q_i^0 + (1/2)q_i^1/Q_F(p^0,p^1,q^0,q^1)\ ; \qquad i = 1,2,\ldots,8$$

where Q_F is the overall Fisher quantity index. Thus using the Van IJzeren quantity weights $q_{Fi}*$, we obtain the *following Van IJzeren additive percentage change decomposition for the Fisher price index*:

$$\begin{aligned}(5)\ \ P_F(p^0,p^1,q^0,q^1) - 1 &= \{\textstyle\sum_{i=1}^{8} q_{Fi}{*}p_i^1 \,/\, \sum_{m=1}^{8} q_{Fi}{*}p_m^0\} - 1\\ &= \{\textstyle\sum_{i=1}^{8} q_{Fi}{*}p_i^1 - \sum_{m=1}^{8} q_{Fi}{*}p_i^0\} \,/\, \sum_{m=1}^{8} q_{Fi}{*}p_m^0\\ &= \textstyle\sum_{i=1}^{8} v_{Fi}{*}\{p_i^1 - p_i^0\}\end{aligned}$$

where the *Van IJzeren weight* for commodity i, $v_{Fi}*$, is defined as

$$(6)\ \ v_{Fi}{*} \equiv q_{Fi}{*} \,/\, \sum_{m=1}^{8} q_{Fi}{*}p_m^0 \qquad ; i = 1,\ldots,8.$$

We will again decompose the *chain links* for the Fisher price index using the formulae (5) listed above. The results of the decomposition are listed in Table 15. Thus $P_F - 1$ is *the percentage change in the Fisher ideal chain link* going from period t − 1 to t and *the Van IJzeren decomposition factor* $v_{Fi}{*}\Delta p_i$ is the contribution to the total percentage change of the change in the ith price from p_i^{t-1} to p_i^t for i = 1,2,…,8.

Table 15. Van IJzeren's Decomposition of the Fisher Price Index

t	$P_F - 1$	$v_{F1}{*}\Delta p_1$	$v_{F2}{*}\Delta p_2$	$v_{F3}{*}\Delta p_3$	$v_{F4}{*}\Delta p_4$	$v_{F5}{*}\Delta p_5$	$v_{F6}{*}\Delta p_6$	$v_{F7}{*}\Delta p_7$	$v_{F8}{*}\Delta p_8$
2	0.1778	0.0804	0.0834	0.1094	-0.0317	0.1697	-0.0101	-0.2454	0.0220
3	0.2122	-0.0652	-0.0712	0.0577	-0.0322	0.1091	-0.0105	0.2021	0.0225
4	0.0403	-0.0540	-0.0361	0.0224	-0.0515	0.0615	-0.0121	0.0741	0.0360
5	0.0326	0.0458	0.0326	0.0197	-0.0393	0.0300	-0.0186	-0.0652	0.0275

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[18] It was also independently derived by Dikhanov (1997) and used by Ehemann, Katz and Moulton (2002).

Comparing the entries in Tables 14 and 15, it can be seen that the differences between the Diewert and Van IJzeren decompositions of the Fisher price index are *very small*. This is somewhat surprising given the very different nature of the two decompositions.[19] As was mentioned in section 9 of Chapter 3, the Van IJzeren decomposition of the chain Fisher *quantity* index is used by the Bureau of Economic Analysis in the U.S.[20]

5. Industry Price Indexes

A highly simplified economy consisting of three industrial sectors is considered. The three sectors are: *the agricultural sector* (or primary sector), *the manufacturing sector* (or secondary sector) and *the services sector* (or tertiary sector). It is assumed that all transactions go through the services sector, which, at first sight, appears to be a bit unusual. However, recall that transportation services reside in the services sector. Hence imported goods are delivered as intermediate inputs to the agricultural and manufacturing sectors using service transportation inputs or they are delivered directly to the final demand sector again using service sector transportation, storage, retailing and wholesaling services. Similarly, the agricultural sector produces unprocessed food which is delivered by the services sector to the manufacturing sector for further processing and packaging and that manufactured food output is then again delivered by the services sector to the final demand sector.[21]

We distinguish three outputs and intermediate inputs for the agricultural sector. The first commodity is agricultural output delivered to the services sector. This is the only output of this sector. There are two intermediate inputs used in the agricultural sector: commodity 2 is deliveries of nonenergy materials (fertilizer, etc.) to agriculture from the services sector and commodity 3 is deliveries of energy from the services sector to agriculture. These prices and quantities are denoted by p_n^{At} and q_n^{At} for n = 1,2,3 and t = 1,...,5. Note that q_1^{At} is positive (because commodity 1 is an output) and q_2^{At} and q_3^{At} are negative (since commodities 2 and 3 in the agriculture sector are intermediate inputs). The data for the agriculture sector for 5 periods are listed in Table 16.

Table 16. Price and Quantity Data for the Agriculture Sector

Period	p_1^A	p_2^A	p_3^A	q_1^A	q_2^A	q_3^A

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[19] The terms in Diewert's decomposition can be given economic interpretations whereas the terms in the other decomposition are more difficult to interpret from the economic perspective. However, Reinsdorf, Diewert and Ehemann (2002) show that the terms in the two decompositions approximate each other to the second order around any point where the two price vectors are equal and where the two quantity vectors are equal.

[20] See Ehemann, Katz and Moulton (2002).

[21] Our treatment of industrial transactions is an extension of Kohli's (1978) approach to modeling the treatment of imports as flowing first through the production sector of the economy rather than being directly delivered to final demand or other industrial sectors.

1	1.0	1.0	1.0	20	-3	-6
2	1.5	1.4	2.2	16	-2	-4
3	1.1	1.6	1.1	20	-3	-5
4	0.6	1.4	0.7	23	-3	-6
5	1.0	1.7	1.1	19	-3	-5

We distinguish two outputs and three intermediate inputs for the manufacturing sector or 5 commodities in all. The first commodity is processed agricultural output delivered to the services sector; the second commodity is traditional manufactures delivered to the services sector; the third commodity is deliveries of transported agricultural intermediate inputs delivered from the services sector; the fourth commodity is deliveries of energy from services to manufacturing and the final commodity is inputs of business services. . These prices and quantities are denoted by p_n^{Mt} and q_n^{Mt} for n = 1,...,5 and t = 1,...,5. Note that q_1^{Mt} and q_2^{Mt} are positive (because these commodities are outputs) and q_3^{Mt}, q_4^{Mt} and q_5^{Mt} are negative (since commodities 3,4 and 5 in the manufacturing sector are intermediate inputs). The data for the manufacturing sector for 5 periods are listed in Table 17.

Table 17. Price and Quantity data for the Manufacturing Sector

Period	p_1^M	p_2^M	p_3^M	p_4^M	p_5^M	q_1^M	q_2^M	q_3^M	q_4^M	q_5^M
1	1.0	1.0	1.0	1.0	1.0	26	36	-22	-6	-8
2	1.3	1.2	1.4	2.0	1.2	23	35	-19	-5	-9
3	1.1	1.4	1.1	1.1	1.6	26	34	-22	-5	-10
4	0.8	1.5	0.7	0.8	1.8	27	35	-23	-5	-11
5	1.0	1.6	1.0	1.1	1.9	25	36	-21	-5	-11

We distinguish 11 service sector outputs and 5 service sector intermediate inputs or 16 commodities in all. The 11 *outputs* are listed as follows:

- Commodity 1 is food deliveries to final demand;
- Commodity 2 is energy deliveries to final demand;
- Commodity 3 is traditional manufacturing deliveries to final demand;
- Commodity 4 is deliveries of high technology manufactured goods to final demand;
- Commodity 5 is delivery of personal services to final demand;
- Commodity 6 is deliveries of high technology services to final demand;
- Commodity 7 is deliveries of materials to agriculture;
- Commodity 8 is deliveries of energy to agriculture;
- Commodity 9 is delivery of materials to manufacturing;
- Commodity 10 is deliveries of energy to manufacturing; and
- Commodity 11 is deliveries of business services to manufacturing.

The 5 *intermediate inputs* into the services sector are listed as follows:

- Commodity 12 is imports of energy into the economy;
- Commodity 13 is imports of high technology manufactures into the economy;
- Commodity 14 is deliveries of agricultural output to services;

- Commodity 15 is deliveries of processed food from manufacturing to services; and
- Commodity 16 is deliveries of traditional manufacturing to services.

These prices and quantities are denoted by p_n^{St} and q_n^{St} for n = 1,...,16 and t = 1,...,5. Note that q_1^{St} to q_{11}^{St} are positive (because these commodities are outputs) and q_{12}^{St} to q_{16}^{St} are negative (since these commodities in the services sector are intermediate inputs). The service sector price and quantity data for the 16 commodities are listed in Tables 18 and 19 respectively.

Table 18. Price Data for the Service Sector

t	p_1^S	p_2^S	p_3^S	p_4^S	p_5^S	p_6^S	p_7^S	p_8^S	p_9^S	p_{10}^S	p_{11}^S	p_{12}^S	p_{13}^S	p_{14}^S	p_{15}^S	p_{16}^S
1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
2	1.3	2.0	1.3	0.7	1.4	0.8	1.4	2.2	1.4	2.0	1.2	2.1	0.7	1.5	1.3	1.2
3	1.0	1.0	1.5	0.5	1.7	0.6	1.6	1.1	1.1	1.1	1.6	1.0	0.5	1.1	1.1	1.4
4	0.7	0.5	1.6	0.3	1.9	0.4	1.4	0.7	0.7	0.8	1.8	0.6	0.3	0.6	0.8	1.5
5	1.0	1.0	1.7	0.2	2.0	0.2	1.7	1.1	1.0	1.1	1.9	1.0	0.2	1.0	1.0	1.6

Table 19. Quantity Data for the Service Sector

t	q_1^S	q_2^S	q_3^S	q_4^S	q_5^S	q_6^S	q_7^S	q_8^S	q_9^S	q_{10}^S	q_{11}^S	q_{12}^S	q_{13}^S	q_{14}^S	q_{15}^S	q_{16}^S
1	30	10	40	10	45	5	3	6	22	6	8	-28	-7	-20	-26	-36
2	28	8	39	13	47	6	2	4	19	5	9	-20	-9	-16	-23	-35
3	30	11	38	30	50	8	3	5	22	5	10	-29	-21	-20	-26	-34
4	32	14	39	60	56	13	3	6	23	5	11	-35	-42	-23	-27	-35
5	29	12	40	100	65	25	3	5	21	5	11	-30	-70	-19	-25	-36

The above sectoral data satisfy the conventions of national income accounting in that every value transaction (which is of the form $p_n^{et}q_n^{et}$ where e denotes a sector and n denotes a commodity) in each sector has a *matching transaction* in another sector for each period and each sector. It should be noted that no attempt has been made to balance the supply and demand for each commodity across sectors; put another way, no attempt has been made to produce *balanced input output tables in real terms* commodity by commodity across sectors. In order to produce such constant dollar input output tables, it is necessary to make assumptions about margins in each sector as say a primary commodity is transformed as it progresses from the agriculture sector to the various downstream sectors. However, these margins are not constant from period to period, which makes it difficult to interpret constant dollar input output tables. Moreover, as goods are transformed through the manufacturing process, they often lose their initial identities, which again makes it difficult to interpret a constant dollar input output table. Our approach avoids all of these problems by focusing on transactions between each pair of sectors in the industrial classification. For each pair of sectors, these intersector transactions can be further classified using a commodity classification, which is what has been done in the above data set, but there is no attempt to have a uniform commodity classification across all sectors.

In the following 3 sections, value added deflators for each of the 3 industrial sectors are calculated. Only fixed base and chained Laspeyres, Paasche, Fisher and Törnqvist indexes will be computed since these are the ones most likely to be used in practice.

6. Value Added Deflators for the Agriculture Sector

The data listed in Table 16 for the agriculture sector are used to calculate fixed base Laspeyres, Paasche, Fisher and Törnqvist price indices for periods t equal 1 to 5, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 20.

Table 20. Agriculture Sector Fixed Base Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.1455	1.2400	1.1918	1.2000
3	0.9636	0.9750	0.9693	0.9679
4	0.3273	0.3857	0.3553	0.3472
5	0.7545	0.7636	0.7591	0.7478

From Table 20, it can be seen that all 4 value added deflators are very close to each other for the odd periods but for the even periods (when agricultural and energy prices "bounce" or are quite different from their longer term "normal" values), the Paasche and Laspeyres indexes differ considerably. However, even for the even periods, the two superlative indexes are quite close to each other.

The data listed in Table 16 for the agriculture sector are used to calculate chained Laspeyres, Paasche, Fisher and Törnqvist price value added deflators for periods t equal 1 to 5, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 21.

Table 21. Agriculture Sector Chained Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.1455	1.2400	1.1918	1.2000
3	0.9238	0.9803	0.9516	0.9579
4	0.3395	0.3808	0.3596	0.3584
5	0.7104	0.8646	0.7837	0.7758

It can be seen, comparing Tables 20 and 21, that the chained indexes show considerably *more* variation than their fixed base counterparts. Hence, here is an example of a sector where chaining does *not* reduce the spread between the Paasche and Laspeyres value added deflators. The reason why chaining does not reduce the spread is that agriculture is an example of a sector where price bouncing is much more important than divergent trends in relative prices. The commodities that have divergent prices are high technology goods and services and the agriculture sector does not use or produce these commodities. Even though chaining did not reduce the spread between the Paasche and Laspeyres indexes for the agriculture sector, it can be seen that the chained Fisher and Törnqvist price indexes are still very close to each other although they are somewhat higher than their fixed base counterparts for the later periods.

7. Value Added Deflators for the Manufacturing Sector

The data listed in Table 17 for the manufacturing sector are used to calculate fixed base Laspeyres, Paasche, Fisher and Törnqvist value added deflators for periods t equal 1 to 5, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 22.

Table 22. Manufacturing Sector Fixed Base Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	0.9462	0.9800	0.9629	0.9599
3	1.3615	1.3261	1.3437	1.3425
4	1.5462	1.4870	1.5163	1.5265
5	1.5308	1.4667	1.4984	1.4951

From Table 22, it can be seen that the divergence between the fixed base Laspeyres and Paasche deflators for the value added of the manufacturing sector grows steadily from period 3 when it is 3.6% to period 5 when it is 4.4%. However the divergence between the two superlative value added deflators is quite small for all periods.

The data listed in Table 17 for the manufacturing sector are used to calculate chained Laspeyres, Paasche, Fisher and Törnqvist value added deflators for periods t equal 1 to 5, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 23.

Table 23. Manufacturing Sector Chained Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	0.9462	0.9800	0.9629	0.9599
3	1.2937	1.3711	1.3318	1.3430
4	1.4591	1.5476	1.5027	1.5217
5	1.4335	1.5345	1.4832	1.5013

Comparing Tables 22 and 23, it can be seen that chaining did *not* reduce the spread between the Paasche and Laspeyres value added deflators for the manufacturing sector: the spread between these two chained indexes in period 5 is 7.0% whereas it was only 4.4% for the corresponding fixed base indices. The explanation for this result is the same as it was for agriculture: (traditional) manufacturing is an example of a sector where the bouncing behavior of energy prices is much more important than divergent trends in relative prices. The commodities that have divergent prices are high technology goods and services and the traditional manufacturing sector does not use or produce these commodities. Comparing Tables 22 and 23, it can also be seen that chaining did *not* reduce the spread between the Fisher and Törnqvist value added deflators for the manufacturing sector. Again, bouncing energy prices explain this result. However the chained Fisher and Törnqvist price indexes are still quite close to each other.

8. Value Added Deflators for the Services sector

The data listed in Tables 18 and 19 for the services sector are used to calculate fixed base Laspeyres, Paasche, Fisher and Törnqvist value added deflators for periods t equal 1 to 5, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 24.

Table 24. Services Sector Fixed Base Laspeyres, Paasche, Fisher and Törnqvist Value Added deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.2368	1.2675	1.2521	1.2561
3	1.5735	1.4768	1.5244	1.5344
4	1.7324	1.4820	1.6023	1.6555
5	1.8162	1.2971	1.5348	1.6547

From Table 24, it can be seen that the divergence between the fixed base Laspeyres and Paasche deflators for the value added of the services sector grows steadily from period 2 when it is 2.5% to period 5 when it is 40.0%. However the divergence between the two superlative value added deflators is much smaller but does grow over time to reach 7.8% in period 5.

The data listed in Tables 18 and 19 for the services sector are used to calculate chained Laspeyres, Paasche, Fisher and Törnqvist value added deflators for periods t equal 1 to 5, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 25.

Table 25. Services Sector Chained Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.2368	1.2675	1.2521	1.2561
3	1.4763	1.6056	1.5396	1.5324
4	1.6104	1.7331	1.6706	1.6662
5	1.6364	1.7410	1.6879	1.6870

Comparing Tables 24 and 25, it can be seen that chaining has substantially reduced the spread between the Paasche and Laspeyres value added deflators for the services sector.

In period 5, the divergence between the chained Paasche and Laspeyres is only 6.4% compared to the 40% divergence between their fixed base counterparts. Similarly, chaining has reduced the spread between the two superlative indexes: in period 5, the chained Fisher and Törnqvist value added deflators differ only by .05% compared to the 7.8% divergence between their fixed base counterparts. Chaining reduces divergences between the 4 indexes for the services sector because several outputs and intermediate inputs for this sector have strongly divergent trends in their prices. This divergent prices effect overwhelms the effects of bouncing agricultural and energy prices.

9. The National Output Price Index

In order to construct a national output price index, we need only collect up the outputs from each of our three industrial sectors and apply normal index number theory to these value flows. There is one output in the agriculture sector, two outputs in the manufacturing sector and eleven outputs in the services sector or 14 outputs in all. The price and quantity data pertaining to these 14 commodities are used to calculate *fixed base* Laspeyres, Paasche, Fisher and Törnqvist output price indexes, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 26.

Table 26. Fixed Base National Laspeyres, Paasche, Fisher and Törnqvist Output Producer Price Indexes

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.3551	1.3295	1.3422	1.3424
3	1.2753	1.2226	1.2487	1.2575
4	1.1622	1.0305	1.0944	1.1203
5	1.3487	1.0697	1.2011	1.2880

Since there are divergent trends in the relative prices of outputs in the economy, it should come as no surprise that the Paasche and Laspeyres output price indexes grow farther apart over time, reaching a difference of 25.7% in period 5. The two superlative indexes show a similar diverging trend, reaching a difference of 7.2% in period 5. Our expectation is that chaining will reduce these divergences.

The price and quantity data pertaining to the 14 sectoral outputs in the economy are used again to calculate *chained* Laspeyres, Paasche, Fisher and Törnqvist output price indexes, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 27.

Table 27. Chained National Laspeyres, Paasche, Fisher and Törnqvist Output Producer Price Indexes

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.3551	1.3295	1.3422	1.3424
3	1.3033	1.2477	1.2752	1.2751

4	1.1806	1.1119	1.1457	1.1456
5	1.3404	1.2221	1.2799	1.2813

Comparing Tables 26 and 27, it can be seen that chaining has indeed reduced the differences between the various national output price indexes. The period 5 difference between the chained Paasche and Laspeyres price indexes is only 9.7% compared to a difference of 25.7% for their fixed base counterparts. Similarly, the period 5 difference between the chained Fisher and Törnqvist price indexes is only 0.1% compared to a difference of 7.2% for their fixed base counterparts.

10. The National Intermediate Input Price Index

In order to construct a national intermediate input price index, we need only collect up the intermediate inputs from each of our three industrial sectors and apply normal index number theory to these value flows.[22] There are two intermediate inputs in the agriculture sector, three intermediate inputs in the manufacturing sector and five intermediate inputs in the services sector or 10 intermediate inputs in all. The price and quantity data pertaining to these 10 commodities are used to calculate *fixed base* Laspeyres, Paasche, Fisher and Törnqvist intermediate input price indexes, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 28.

Table 28. Fixed Base National Laspeyres, Paasche, Fisher and Törnqvist Intermediate Input Producer Price Indexes

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.4846	1.4310	1.4575	1.4582
3	1.1574	1.1069	1.1319	1.1397
4	0.9179	0.8086	0.8615	0.8817
5	1.1636	0.9049	1.0261	1.0997

Since there are divergent trends in the relative prices of intermediate inputs in the economy, it should come as no surprise that the Paasche and Laspeyres intermediate input price indexes grow farther apart over time, reaching a difference of 28.6% in period 5. The two superlative indexes show a similar diverging trend, reaching a difference of 7.2% in period 5. Our expectation is that chaining will reduce these divergences.

The price and quantity data pertaining to the 10 sectoral intermediate inputs in the economy are used again to calculate *chained* Laspeyres, Paasche, Fisher and Törnqvist intermediate input price indexes, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 29.

Table 29. Chained National Laspeyres, Paasche, Fisher and Törnqvist Intermediate Input Producer Price Indexes

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[22] In this section, the negative quantities are changed into positive quantities.

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.4846	1.4310	1.4575	1.4582
3	1.2040	1.1168	1.1596	1.1597
4	0.9485	0.8627	0.9046	0.9052
5	1.1759	1.0296	1.1003	1.1030

Comparing Tables 28 and 29, it can be seen that chaining has reduced the differences between the Paasche and Laspeyres intermediate input price indexes. The period 5 difference between the chained Paasche and Laspeyres price indexes is 14.2% compared to a difference of 28.6% for their fixed base counterparts. Similarly, the period 5 difference between the chained Fisher and Törnqvist price indexes is only 0.2% compared to a difference of 7.2% for their fixed base counterparts.

11. The National Value Added Deflator

In order to construct a national value added deflator, we need only collect up all of the outputs and intermediate inputs from each of our three industrial sectors, make sure that the intermediate input quantities are indexed with negative signs and apply normal index number theory to these value flows. There are two intermediate inputs and one output in the agriculture sector, two outputs and three intermediate inputs in the manufacturing sector and eleven outputs and five intermediate inputs in the services sector or 24 commodities in all. The price and quantity data pertaining to these 24 commodities are used to calculate *fixed base* Laspeyres, Paasche, Fisher and Törnqvist value added deflators, P_L^t, P_P^t, P_F^t and P_T^t respectively. The results are listed in Table 30.

Table 30. Fixed Base National Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.1552	1.2009	1.1778	1.1831
3	1.4571	1.3957	1.4261	1.4343
4	1.5390	1.3708	1.4525	1.4866
5	1.6343	1.2865	1.4500	1.5447

Since there are divergent trends in the relative prices of outputs and intermediate inputs in the economy, it should come as no surprise that the fixed base Paasche and Laspeyres value added deflators grow farther apart over time, reaching a difference of 27.0% in period 5. The two superlative indexes show a similar diverging trend, reaching a difference of 6.5% in period 5. As usual, our expectation is that chaining will reduce these divergences.

The price and quantity data pertaining to the 24 sectoral outputs and intermediate inputs in the economy are used again to calculate *chained* Laspeyres, Paasche, Fisher and

Törnqvist national value added deflators, $P_L{}^t$, $P_P{}^t$, $P_F{}^t$ and $P_T{}^t$ respectively. The results are listed in Table 31.

Table 31. Chained National Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	$P_L{}^t$	$P_P{}^t$	$P_F{}^t$	$P_T{}^t$
1	1.0000	1.0000	1.0000	1.0000
2	1.1552	1.2009	1.1778	1.1831
3	1.3743	1.4834	1.4278	1.4307
4	1.4374	1.5349	1.4853	1.4893
5	1.4963	1.5720	1.5337	1.5400

Comparing Tables 30 and 31, it can be seen that chaining has reduced the differences between the Paasche and Laspeyres deflators. The period 5 difference between the chained Paasche and Laspeyres deflators is 5.1% compared to a difference of 27.0% for their fixed base counterparts. Similarly, the period 5 difference between the chained Fisher and Törnqvist deflators is only 0.4% compared to a difference of 6.5% for their fixed base counterparts.

At the beginning of this chapter, we calculated the Laspeyres, Paasche, Fisher and Törnqvist *final demand deflators* using a fixed base principle in Tables 4 and 8 and using the chain principle in Tables 5 and 9. If these final demand deflators are compared with their *national value added deflator* counterparts listed in Tables 30 and 31, the reader will find that *these two types of deflator give exactly the same answer*. This type of exactness result was obtained theoretically in section 18 of chapter 13, but under the stronger assumption that there was a common commodity classification and prices were constant across sectors. In this chapter, we do not assume that there is a common commodity classification but we do assume that *all transactions are classified on a bilateral sectoral basis*; i.e., we keep track of all transactions between each pair of sectors in the economy. Under these conditions, if any of the commonly used index number formulae are used, then it can be shown that the final demand deflator will be *exactly equal* to the national value added deflator.[23]

12. National Two Stage Aggregation

Given that we have constructed the national output price index and the national intermediate input price index, it is natural to use the two stage aggregation procedure explained in section 15 of chapter 13 in order to aggregate these two indexes into a national value added deflator, which then can be compared to the national value added

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[23] The index number formula used must be consistent with either Hicks' (1946; 312-313) or Leontief's (1936) aggregation theorems; i.e., if all prices vary in strict proportion across the two periods under consideration, then the price index is equal to this common factor of proportionality (Hicks) or if all quantities vary in strict proportion across the two periods under consideration, then the quantity index that corresponds to the price index is equal to this common factor of proportionality (Leontief). See Allen and Diewert (1981; 433) for additional material on these aggregation theorems.

deflator that was obtained in the previous section (which was a single stage aggregation procedure). This comparison is undertaken in this section.

Using the computations made in the previous section and the theory outlined in section 15 of chapter 13, *two stage fixed base* Laspeyres, Paasche, Fisher and Törnqvist value added deflators, P_L^t, P_P^t, P_F^t and P_T^t respectively, were constructed.[24] The resulting two stage national value added deflators are listed in Table 32.

Table 32. Two Stage Fixed Base National Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.1552	1.2009	1.1815	1.1830
3	1.4571	1.3957	1.4259	1.4379
4	1.5390	1.3708	1.4510	1.5018
5	1.6343	1.2865	1.4485	1.5653

Comparing the two stage value added deflators listed in Table 32 with the corresponding single stage deflators listed in Table 30, it can be seen that the Paasche and Laspeyres estimates *are exactly the same* but there are some small differences between the single stage and two stage Fisher and Törnqvist value added deflators. For period 5, the difference in the two fixed base Fisher deflators is only 0.1% and the difference in the two fixed base Törnqvist deflators is 1.3%.

Using the computations made in the previous section and the theory outlined in section 15 of chapter 13, *two stage chained* Laspeyres, Paasche, Fisher and Törnqvist value added deflators, P_L^t, P_P^t, P_F^t and P_T^t respectively, were constructed. The resulting two stage national value added deflators are listed in Table 33.

Table 33. Two Stage Chained National Laspeyres, Paasche, Fisher and Törnqvist Value Added Deflators

Period t	P_L^t	P_P^t	P_F^t	P_T^t
1	1.0000	1.0000	1.0000	1.0000
2	1.1552	1.2009	1.1815	1.1830
3	1.3743	1.4834	1.4281	1.4277
4	1.4374	1.5349	1.4853	1.4861
5	1.4963	1.5720	1.5342	1.5368

Comparing the two stage chained value added deflators listed in Table 33 with the corresponding chained single stage deflators listed in Table 31, it can be seen that the Paasche and Laspeyres estimates *are exactly the same* but there are some small

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[24] Stage 1 constructs the national output and national intermediate input price indexes. Stage 2 aggregates these two indexes into a national value added index.

differences between the single stage and two stage Fisher and Törnqvist value added deflators. For period 5, the difference in the chained Fisher deflators is only 0.03% and the difference in the two chained Törnqvist deflators is 0.2%. Thus chaining has led to a closer correspondence between the single stage and two stage national value added deflators.

References

Allen, R.C. and W.E. Diewert (1981), "Direct versus Implicit Superlative Index Number Formulae", *The Review of Economics and Statistics* 63, 430-435.

Diewert, W.E. (1978), "Superlative Index Numbers and Consistency in Aggregation", *Econometrica* 46, 883-900.

Diewert, W. E. (2002), "The Quadratic Approximation Lemma and Decompositions of Superlative Indexes", *Journal of Economic and Social Measurement* 28, 63-88.

Dikhanov, Y. (1997), "The Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures", mimeo, International Economics Department, The World Bank, Washington D.C., January.

Edgeworth, F.Y. (1925), *Papers Relating to Political Economy*, Volume 1, New York: Burt Franklin.

Ehemann, C., A. J. Katz and B. R. Moulton (2002), "The Chain-Additivity Issue and the U.S. National Accounts", *Journal of Economic and Social Measurement* 28, 37-49.

Fisher, I. (1922), *The Making of Index Numbers*, Houghton-Mifflin, Boston.

Hardy, G.H., J.E. Littlewood and G. Polyá (1934), *Inequalities*, Cambridge: Cambridge University Press.

Hicks, J.R. (1946), *Value and Capital*, Second Edition, Oxford: Clarenden Press.

Hill, R.J. (2004), "Superlative Index Numbers: Not All of them Are Super", School of Economics, University of New South Wales, Sydney 2052, Australia, September 10, forthcoming in the *Journal of Econometrics*.

Kohli, U. (1978), "A Gross National Product Function and the Derived Demand for Imports and Supply of Exports", *Canadian Journal of Economics* 11, 167-182.

Konüs, A.A. and S.S. Byushgens (1926), "K probleme pokupatelnoi cili deneg", *Voprosi Konyunkturi* 2, 151-172.

Leontief, W.W. (1936), “Composite Commodities and the Problem of Index Numbers”, *Econometrica* 4, 39-59.

Marshall, A. (1887), “Remedies for Fluctuations of General Prices’, *Contemporary Review* 51, 355-375.

Okamoto, M. (2001), “Mid-Year Basket Index as a Practical Approximation to a Superlative Index”, Paper presented at the Sixth Meeting of the International Working Group on Price Indices, Canberra, April 2-6, 2001.

Reinsdorf, M. B., W. E. Diewert and C. Ehemann (2002), “Additive Decompositions for the Fisher, Törnqvist and Geometric Mean Indexes”, *Journal of Economic and Social Measurement* 28, 51-61.

Schlömilch, O., (1858), “Über Mïttelgrössen verschiedener Ordnungen”, *Zeitschrift für Mathematik und Physik* 3, 308-310.

Schultz, B. (1999), “Effects of Using Various Macro-Index Formulae in Longitudinal Price and Comparisons: Empirical Studies”, pp. 236-249 in *Proceedings of the Fourth Meeting of the International Working Group on Price Indices*, W. Lane (ed.), Bureau of Labor Statistics, U.S. Department of Labor, Washington D.C.

Szulc (Schultz), B.J. (1983), “Linking Price Index Numbers,” pp. 537-566 *in Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Van IJzeren (1987), *Bias in International Index Numbers: A Mathematical Elucidation*, Dissertation for the Hungarian Academy of Sciences, Den Haag: Koninklijke Bibliotheek.

Vartia, Y.O. (1978), “Fisher’s Five-Tined Fork and Other Quantum Theories of Index Numbers”, pp. 271-295 in *Theory and Applications of Economic Indices*, W. Eichhorn, R. Henn, O. Opitz and R.W. Shephard (eds.), Würzburg: Physica-Verlag.

Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: Macmillan and Co.