INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

By W.E. Diewert, April 7, 2006.

CHAPTER 8: Fixed Base Versus Chained Indexes

1. Introduction

In this chapter[1], the merits of using the chain system for constructing price indexes in the time series context versus using the fixed base system are discussed.

The chain system[2] measures the change in prices going from one period to a subsequent period using a bilateral index number formula involving the prices and quantities pertaining to the two adjacent periods. These one period rates of change (the links in the chain) are then cumulated to yield the relative levels of prices over the entire period under consideration. Thus if the bilateral price index is P, the chain system generates the following pattern of price levels for the first three periods:

(1) $1, P(p^0,p^1,q^0,q^1), P(p^0,p^1,q^0,q^1)\, P(p^1,p^2,q^1,q^2)$.

On the other hand, the fixed base system of price levels using the same bilateral index number formula P simply computes the level of prices in period t relative to the base period 0 as $P(p^0,p^t,q^0,q^t)$. Thus the fixed base pattern of price levels for periods 0,1 and 2 is:

(2) $1, P(p^0,p^1,q^0,q^1), P(p^0,p^2,q^0,q^2)$.

Note that in both the chain system and the fixed base system of price levels defined by (1) and (2) above, the base period price level is set equal to 1. The usual practice in statistical agencies is to set the base period price level equal to 100. If this is done, then it is necessary to multiply each of the numbers in (1) and (2) by 100.

Due to the difficulties involved in obtaining current period information on quantities (or equivalently, on expenditures), many statistical agencies loosely base[3] their Consumer Price Index on the use of the Laspeyres formula[4] and the fixed base system. Therefore, it is of some interest to look at some of the possible problems associated with the use of fixed base Laspeyres indexes.

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[1] This section is largely based on the work of Hill (1988) (1993; 385-390).

[2] The chain principle was introduced independently into the economics literature by Lehr (1885; 45-46) and Marshall (1887; 373). Both authors observed that the chain system would mitigate the difficulties due to the introduction of new commodities into the economy, a point also mentioned by Hill (1993; 388). Fisher (1911; 203) introduced the term "chain system".

[3] As we saw in chapter 7, Consumer Price Indexes are usually taken to be Lowe (1823) indexes, which have the formula $P_{Lo}(p^0,p^t,q^b) \equiv p^t\bullet q^b/p^0\bullet q^b$, where q^b is the quantity vector of a base year and p^0 and p^t are monthly price vectors pertaining to months 0 and t.

[4] The Laspeyres formula between months 0 and t is $P_L(p^0,p^t,q^0) \equiv p^t\bullet q^0/p^0\bullet q^0$, where q^0 is the quantity vector of the base month 0.

The main problem with the use of the fixed base Laspeyres index is that the period 0 fixed basket of commodities that is being priced out in period t can often be quite different from the period t basket.[5] Thus if there are systematic *trends* in at least some of the prices and quantities[6] in the index basket, the fixed base Laspeyres price index $P_L(p^0,p^t,q^0,q^t)$ can be quite different from the corresponding fixed base Paasche price index, $P_P(p^0,p^t,q^0,q^t)$.[7] This means that both indexes are likely to be an inadequate representation of the movement in average prices over the time period under consideration.

The fixed base Laspeyres quantity index cannot be used forever: eventually, the base period quantities q^0 are so far removed from the current period quantities q^t that the base must be changed. Chaining is merely the limiting case where the base is changed each period.[8]

2. Discussion of the Advantages and Disadvantages of Chaining

The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indexes.[9] These two indexes each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of the aggregate price change should lie between these two estimates. Thus the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth".[10]

Hill (1993; 388), drawing on the earlier research of Szulc (1983) and Hill (1988; 136-137), noted that it is not appropriate to use the chain system when prices oscillate (or "bounce" to use Szulc's (1983; 548) term). This phenomenon can occur in the context of regular seasonal fluctuations or in the context of price wars. However, in the context of roughly monotonically changing prices and quantities, Hill (1993; 389) recommended the use of chained symmetrically weighted indexes.[11] The Fisher and Walsh indexes are examples of symmetrically weighted indices.

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[5] The Lowe index suffers from a similar problem: if the base year b is fairly distant from the base month 0, the base year quantity vector q^b can be unrepresentative for both months 0 and t.

[6] Examples of rapidly downward trending prices and upward trending quantities are computers, electronic equipment of all types, internet access and telecommunication charges.

[7] Note that $P_L(p^0,p^t,q^0,q^t)$ will equal $P_P(p^0,p^t,q^0,q^t)$ if *either* the two quantity vectors q^0 and q^t are proportional *or* the two price vectors p^0 and p^t are proportional. Thus in order to obtain a difference between the Paasche and Laspeyres indexes, nonproportionality in *both* prices and quantities is required.

[8] Regular seasonal fluctuations can cause monthly or quarterly data to "bounce" using the term due to Szulc (1983) and chaining bouncing data can lead to a considerable amount of index "drift"; i.e., if after 12 months, prices and quantities return to their levels of a year earlier, then a chained monthly index will usually not return to unity. Hence, the use of chained indices for "noisy" monthly or quarterly data is not recommended without careful consideration.

[9] See Diewert (1978; 895) and Hill (1988) (1993; 387-388). Later in this chapter, we will examine more closely under what conditions chaining will reduce the spread between the Paasche and Laspeyres indexes.

[10] This observation will be illustrated with an artificial data set in a later chapter.

[11] Note that all known superlative indexes are symmetrically weighted.

It is possible to be a bit more precise under what conditions one should chain or not chain. Basically, one should chain if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indices at each link.[12] Of course, one needs a measure of how similar are the prices and quantities pertaining to two periods. The similarity measures could be *relative* ones or *absolute* ones. In the case of absolute comparisons, two vectors of the same dimension are similar if they are identical and dissimilar otherwise. In the case of relative comparisons, two vectors are similar if they are proportional and dissimilar if they are nonproportional.[13] Once a similarity measure has been defined, the prices and quantities of each period can be compared to each other using this measure and a "tree" or path that links all of the observations can be constructed where the most similar observations are compared with each other using a bilateral index number formula.[14] Hill (1995) defined the price structures between the two countries to be more dissimilar the bigger is the spread between P_L and P_P; i.e., the bigger is max $\{P_L/P_P, P_P/P_L\}$. The problem with this measure of dissimilarity in the price structures of the two countries is that it could be the case that $P_L = P_P$ (so that the Hill measure would register a maximal degree of similarity) but p^0 could be very different than p^t. Thus there is a need for a more systematic study of similarity (or dissimilarity) measures in order to pick the "best" one that could be used as an input into Hill's (1999a) (1999b) (2001) spanning tree algorithm for linking observations. The Appendix to this chapter provides an introduction to the study of dissimilarity indexes. For a more complete discussion, see Diewert (2002).

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[12] Walsh in discussing whether fixed base or chained index numbers should be constructed, took for granted that the precision of all reasonable bilateral index number formulae would improve, provided that the two periods or situations being compared were more similar and hence, for this reason, favored the use of chained indexes: "The question is really, in which of the two courses [fixed base or chained index numbers] are we likely to gain greater exactness in the comparisons actually made? Here the probability seems to incline in favor of the second course; for the conditions are likely to be less diverse between two contiguous periods than between two periods say fifty years apart." Correa Moylan Walsh (1901; 206). Walsh (1921a; 84-85) later reiterated his preference for chained index numbers. Fisher also made use of the idea that the chain system would usually make bilateral comparisons between price and quantity data that was more similar and hence the resulting comparisons would be more accurate: "The index numbers for 1909 and 1910 (each calculated in terms of 1867-1877) are compared with each other. But direct comparison between 1909 and 1910 would give a different and more valuable result. To use a common base is like comparing the relative heights of two men by measuring the height of each above the floor, instead of putting them back to back and directly measuring the difference of level between the tops of their heads." Irving Fisher (1911; 204). "It seems, therefore, advisable to compare each year with the next, or, in other words, to make each year the base year for the next. Such a procedure has been recommended by Marshall, Edgeworth and Flux. It largely meets the difficulty of non-uniform changes in the Q's, for any inequalities for successive years are relatively small." Irving Fisher (1911; 423-424).

[13] Diewert (2002) takes an axiomatic approach to defining various indexes of absolute and relative dissimilarity.

[14] Fisher (1922; 271-276) hinted at the possibility of using spatial linking; i.e., of linking countries that are similar in structure. However, the modern literature has grown due to the pioneering efforts of Robert Hill (1995) (1999a) (1999b) (2001). Hill (1995) used the spread between the Paasche and Laspeyres price indexes as an indicator of similarity and showed that this criterion gives the same results as a criterion that looks at the spread between the Paasche and Laspeyres quantity indexes.

The method of linking observations explained in the previous paragraph based on the similarity of the price and quantity structures of any two observations may not be practical in a statistical agency context since the addition of a new period may lead to a reordering of the previous links. However, the above "scientific" method for linking observations may be useful in deciding whether chaining is preferable or whether fixed base indexes should be used while making month to month comparisons within a year.

Some index number theorists have objected to the chain principle on the grounds that it has no counterpart in the spatial context:

"They [chain indexes] only apply to intertemporal comparisons, and in contrast to direct indices they are not applicable to cases in which no natural order or sequence exists. Thus the idea of a chain index for example has no counterpart in interregional or international price comparisons, because countries cannot be sequenced in a 'logical' or 'natural' way (there is no k+1 nor k−1country to be compared with country k)." Peter von der Lippe (2001; 12).[15]

This is of course correct but the approach of Robert Hill does lead to a "natural" set of spatial links. Applying the same approach to the time series context will lead to a set of links between periods which may not be month to month but it will in many cases justify year over year linking of the data pertaining to the same month.

3. When Will Chaining Give the Same Answer as Using a Fixed Base Index?

It is of some interest to determine if there are index number formulae that give the same answer when either the fixed base or chain system is used. Comparing the sequence of chain indexes defined by (1) above to the corresponding fixed base indexes defined by (2), it can be seen that we will obtain the same answer in all three periods if the index number formula P satisfies the following functional equation for all price and quantity vectors:

(3) $P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)\ P(p^1,p^2,q^1,q^2)$.

If an index number formula P satisfies (3), then P satisfies the *circularity test*.[16]

If it is assumed that the index number formula P satisfies certain properties or tests in addition to the circularity test above[17], then Funke, Hacker and Voeller (1979) showed

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[15] It should be noted that von der Lippe (2001; 56-58) is a vigorous critic of all index number tests based on symmetry in the time series context although he is willing to accept symmetry in the context of making international comparisons. "But there are good reasons *not* to insist on such criteria in the *intertemporal* case. When no symmetry exists between 0 and t, there is no point in interchanging 0 and t." Peter von der Lippe (2001; 58).

[16] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).

[17] The additional tests are: (i) positivity and continuity of $P(p^0,p^1,q^0,q^1)$ for all strictly positive price and quantity vectors p^0,p^1,q^0,q^1; (ii) the identity test; (iii) the commensurability test; (iv) $P(p^0,p^1,q^0,q^1)$ is

that P must have the following functional form due originally to Konüs and Byushgens[18] (1926; 163-166):[19]

$$(4)\ P_{KB}(p^0,p^1,q^0,q^1) \equiv \prod_{i=1}^{N} \left(\frac{p_i^1}{p_i^0} \right)^{\alpha_i}$$

where the N constants α_i satisfy the following restrictions:

$$(5)\ \sum_{i=1}^{N} \alpha_i = 1 \text{ and } \alpha_i > 0 \text{ for } i = 1,\ldots,N.$$

Thus under very weak regularity conditions, the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios, the weights being constant through time.

An interesting special case of the family of indexes defined by (4) occurs when the weights α_i are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

$$(6)\ P_J(p^0,p^1,q^0,q^1) \equiv \prod_{n=1}^{N} (p_n^1/p_n^0)^{1/N} .$$

The problem with the indexes defined by Konüs and Byushgens and Jevons is that the individual price ratios, p_n^1/p_n^0, have weights (either α_n or 1/n) that are *independent* of the economic importance of commodity n in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity n consumed or the expenditures on commodity n during the two periods. Hence, these indexes are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.

The above results indicate that it is not useful to ask that the price index P satisfy the circularity test *exactly*. However, it is of some interest to find index number formulae that satisfy the circularity test to some degree of *approximation* since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed base systems. Irving Fisher (1922; 284) found that deviations from circularity using his data set and the Fisher ideal

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positively homogeneous of degree one in the components of p^1 and (v) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree zero in the components of q^1.

[18] Konüs and Byushgens show that the index defined by (4) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1983; 119-120). The concept of an exact index number formula was explained in an earlier chapter.

[19] This result can be derived using results in Eichhorn (1978; 167-168) and Vogt and Barta (1997; 47). A simple proof can be found in Balk (1995). This result vindicates Irving Fisher's (1922; 274) intuition who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*…" Fisher (1922; 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. … Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

price index P_F were quite small. This relatively high degree of correspondence between fixed base and chain indexes has been found to hold for other symmetrically weighted formulae like the Walsh index P_W defined in earlier chapters.[20] Thus in most time series applications of index number theory where the base year in fixed base indexes is changed every 5 years or so, it will not matter very much whether the statistical agency uses a fixed base price index or a chain index, provided that a symmetrically weighted formula is used.[21] This of course depends on the length of the time series considered and the degree of variation in the prices and quantities as we go from period to period. The more prices and quantities are subject to large fluctuations (rather than smooth trends), the less the correspondence.[22]

It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for symmetrically weighted index number formulae. Another symmetrically weighted formula is the Törnqvist index P_T.[23] The natural logarithm of this index is defined as follows:

$$(7)\ \ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0+s_n^1)\ln (p_n^1/p_n^0)$$

where the period t expenditure shares s_n^t are defined in the usual way. Alterman, Diewert and Feenstra (1999; 61) show that if the logarithmic price ratios $\ln (p_n^t/p_n^{t-1})$ trend linearly with time t and the expenditure shares s_n^t also trend linearly with time, then the Törnqvist index P_T will satisfy the circularity test exactly.[24] Since many economic time series on prices and quantities satisfy these assumptions approximately, then the Törnqvist index P_T will satisfy the circularity test approximately. As was seen in an earlier chapter, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indexes, so that for many economic time series (with smooth trends), all three of these symmetrically weighted indexes will satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle.

Walsh (1901; 401) (1921a; 98) (1921b; 540) introduced the following useful variant of the circularity test:

[20] See for example Diewert (1978; 894). Walsh (1901; 424 and 429) found that his 3 preferred formulae all approximated each other very well as did the Fisher ideal for his artificial data set.

[21] More specifically, most superlative indexes (which are symmetrically weighted) will satisfy the circularity test to a high degree of approximation in the time series context. See chapter 4 for the definition of a superlative index. It is worth stressing that fixed base Paasche and Laspeyres indices are very likely to diverge considerably over a 5 year period if computers (or any other commodity which has price and quantity trends that are quite different from the trends in the other commodities) are included in the value aggregate under consideration.

[22] Again, see Szulc (1983) and Hill (1988).

[23] This formula was implicitly introduced in Törnqvist (1936) and explicitly defined in Törnqvist and Törnqvist (1937).

[24] This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999; 65).

(8) $1 = P(p^0,p^1,q^0,q^1)\ P(p^1,p^2,q^1,q^2)\ldots P(p^{T-1},p^T,q^{T-1},q^T)\ P(p^T,p^0,q^T,q^0)$.

The motivation for this test is the following one. Use the bilateral index formula $P(p^0,p^1,q^0,q^1)$ to calculate the change in prices going from period 0 to 1, use the same formula evaluated at the data corresponding to periods 1 and 2, $P(p^1,p^2,q^1,q^2)$, to calculate the change in prices going from period 1 to 2, … , use $P(p^{T-1},p^T,q^{T-1},q^T)$ to calculate the change in prices going from period T−1 to T, introduce an artificial period T+1 that has exactly the price and quantity of the initial period 0 and use $P(p^T,p^0,q^T,q^0)$ to calculate the change in prices going from period T to 0. Finally, multiply all of these indexes together and since we end up where we started, then the product of all of these indexes should ideally be one. Diewert (1993a; 40) called this test a *multiperiod identity test*.[25] Note that if T = 2 (so that the number of periods is 3 in total), then Walsh's test reduces to Fisher's (1921; 534) (1922; 64) *time reversal test*.[26]

Walsh (1901; 423-433) showed how his circularity test could be used in order to evaluate how "good" any bilateral index number formula was. What he did was invent artificial price and quantity data for 5 periods and he added a sixth period that had the data of the first period. He then evaluated the right hand side of (8) for various formula, $P(p^0,p^1,q^0,q^1)$, and determined how far from unity the results were. His "best" formulae had products that were close to one.[27]

This same framework is often used to evaluate the efficacy of chained indexes versus their direct counterparts. Thus if the right hand side of (8) turns out to be different than unity, the chained indexes are said to suffer from "chain drift". If a formula does suffer from chain drift, it is sometimes recommended that fixed base indexes be used in place of chained ones. However, this advice, if accepted would *always* lead to the adoption of fixed base indexes, provided that the bilateral index formula satisfies the identity test, $P(p^0,p^0,q^0,q^0) = 1$. Thus it is not recommended that Walsh's circularity test be used to decide whether fixed base or chained indexes should be calculated. However, it is fair to use Walsh's circularity test as he originally used it i.e., as an approximate method for deciding how "good" a particular index number formula is. In order to decide whether to chain or use fixed base indexes, one should decide on the basis of how similar are the observations being compared and choose the method which will best link up the most similar observations.

Appendix: An Introduction to Indexes of Absolute Dissimilarity

A.1 Introduction

[25] Walsh (1921a; 98) called his test the *circular test* but since Fisher also used this term to describe his transitivity test defined earlier by (3), it seems best to stick to Fisher's terminology since it is well established in the literature.

[26] Walsh (1921b; 540-541) noted that the time reversal test was a special case of his circularity test.

[27] This is essentially a variant of the methodology that Fisher (1922; 284) used to check how well various formulae corresponded to his version of the circularity test.

An *absolute index of price dissimilarity* regards the vectors p^1 and p^2 as being dissimilar if $p^1 \neq p^2$ whereas a *relative index of price dissimilarity* regards p^1 and p^2 as being dissimilar if $p^1 \neq \lambda p^2$ where $\lambda > 0$ is an arbitrary positive number. Thus the relative index regards the two price vectors as being dissimilar only if *relative prices* differ in the two situations.

The relative index concept seems to be the most useful for judging whether the structure of prices is similar or dissimilar across two countries. However, assuming that the quantity vectors being compared are per capita quantity vectors, then the absolute concept seems to be more appropriate for judging the degree of similarity across countries. If per capita quantity vectors are quite different, then it is quite likely that the rich country is consuming (or producing) a very different bundle of goods and services than the poorer country and hence big disparities in the absolute level of q^1 versus q^2 are likely to indicate that the components of these two vectors are really not very comparable. In any case, it is of some interest to develop the theory for both the absolute and relative concepts.

Relative indexes of price and quantity similarity or dissimilarity are very useful in deciding how to aggregate up a large number of price and quantity series into a smaller number of aggregates.[28] Finally, absolute indexes of dissimilarity can be useful in deciding when an observation in a large cross sectional data set is an outlier.[29]

In this appendix, we provide an introduction to this topic by studying absolute dissimilarity indexes when the number of commodities is only one. We offer what we think are a fairly fundamental set of axioms or properties that such an absolute dissimilarity index should satisfy and characterize the set of indexes which satisfy these axioms.

A.2 A First Approach to Indexes of Absolute Dissimilarity

We denote our *absolute dissimilarity index* as a function of two variables, d(x,y), where x and y are restricted to be positive scalars. The two variables x and y could be the two prices of the first commodity in the two countries, p_1^1 and p_1^2, or they could be the two per capita quantities of the first commodity in the two countries, q_1^1 and q_1^2. It is obvious that d(x,y) could be considered to be a distance function of the type that occurs in the mathematics literature. However, the axioms that we impose on d(x,y) are somewhat unconventional as we shall see.

The 6 fundamental axioms or properties that we think an absolute dissimilarity index should satisfy are the following ones[30] (note that the domain of definition for d(x,y) is $x > 0$ and $y > 0$):

[28] For applications along these lines, see Allen and Diewert (1981).
[29] Robert Hill pointed out this use for a dissimilarity index.
[30] Counterparts to Axioms A2-A6 in the context of relative dissimilarity indexes were proposed by Allen and Diewert (1981; 433). Sergueev (2001; 4) also proposed counterparts to A2, A4 and A6 in the context of similarity indexes (as opposed to dissimilarity indexes).

A1: *Continuity*: d(x,y) is a continuous function.

A2: *Identity*: d(x,x) = 0 for all x > 0.

A3: *Positivity*: d(x,y) > 0 for all x ≠ y.

A4: *Symmetry*: d(x,y) = d(y,x) for all x and y.

A5: *Invariance to Changes in Units of Measurement*: $d(\alpha x,\alpha y) = d(x,y)$ for all $\alpha > 0$, x > 0, y > 0.

A6: *Monotonicity*: d(x,y) is increasing in y if y ≥ x.

Some comments on the axioms are in order. The continuity assumption is generally made in order to rule out indexes that behave erratically. The identity assumption is a standard one in the mathematics literature; i.e., the absolute distance between two points x and y is zero if x equals y. A3 tells us that there is a positive amount of dissimilarity between x and y if x and y are different. The symmetry property is very important: it says that the degree of dissimilarity between x and y is independent of the ordering of x and y. A5 is another important property from the viewpoint of economics: since units of measurement for commodities are essentially arbitrary, we would like our dissimilarity measure to be independent of the units of measurement. Finally, A6 says that as y gets bigger than x, the degree of dissimilarity between x and y grows. This is a very sensible property.

Problem

1. Show that axiom A3 is implied by the other axioms.

It turns out that there is a fairly simple characterization of the class of dissimilarity indexes d(x,y) that satisfy the above axioms; i.e., we have the following Proposition:

Proposition 1: Let d(x,y) be a function of two variables that satisfies the axioms A1-A6. Then d(x,y) has the following representation:

(1) $d(x,y) = f[\max\{x/y, y/x\}]$

where f(u) is a continuous, monotonically increasing function of one variable, defined for u ≥ 1 with the following additional property:

(2) $f(1) = 0$.

Conversely, if f(u) has the above properties, then d(x,y) defined by (1) has the properties A1-A6.

Proof: Using A5 with $\alpha = x^{-1}$, we have:

(3) $d(x,y) = d(1,y/x)$.

Now use A5 with $\alpha = y^{-1}$ and we find:

(4) $d(x,y) = d(x/y,1)$
$\quad = d(1,x/y)$ using A4.

For $u \geq 1$, define the continuous function of one variable, $f(u)$ as

(5) $f(u) \equiv d(1,u)$; $u \geq 1$.

Using A2 and definition (5), we have

(6) $f(1) = d(1,1) = 0$.

Using A6, we deduce that $f(u)$ is an increasing function of u for $u \geq 1$. Now if $x \geq y$, then from (4) and definition (5), we deduce that $d(x,y) = f(x/y)$. If however, $y \geq x$, then from (3) and definition (5), we deduce that $d(x,y) = f(y/x)$. These two results can be combined into the following result:

(7) $d(x,y) = f[\max\{x/y, y/x\}]$

which completes the first part of the Proposition. Going the other way, if $f(u)$ is an increasing, continuous function for $u \geq 1$ with $f(1) = 0$, then if we define $d(x,y)$ using (1), it is easy to verify that $d(x,y)$ satisfies the axioms A1-A6. Q.E.D.

Problem

2. Show that if $f(u)$ is an increasing, continuous function for $u \geq 1$ with $f(1) = 0$, then $d(x,y)$ defined by (1) satisfies the axioms A1-A6.

Example 1: **The asymptotically linear dissimilarity index:**

Let $f(u) \equiv u + u^{-1} - 2$ for $u \geq 1$. Note that $f'(u) = 1 - u^{-2} > 0$ for $u > 1$, which shows that $f(u)$ is increasing for $u \geq 1$. Note that as u tends to infinity, $f(u)$ approaches the linear function $u - 2$. Hence $f(u)$ is asymptotically linear. Since $f(1) = 0$, we see that $f(u)$ satisfies the required regularity conditions and the associated absolute dissimilarity index is[31]

(8) $d(x,y) = (x/y) + (y/x) - 2 = [(x/y) - 1] + [(y/x) - 1]$; $x > 0$; $y > 0$

[31] If $x \geq y$, then $\max\{x/y, y/x\}$ is x/y and $d(x,y) \equiv f[\max\{x/y, y/x\}] = f[x/y] = (x/y) + (y/x) - 2$. If $y \geq x$, then $\max\{x/y, y/x\}$ is y/x and $d(x,y) \equiv f[\max\{x/y, y/x\}] = f[y/x] = (y/x) + (x/y) - 2 = (x/y) + (y/x) - 2$.

and it satisfies the axioms A1-A6.

Example 2: **The asymptotically quadratic dissimilarity index:**

Let $f(u) \equiv [u - 1]^2 + [u^{-1} - 1]^2$ for $u \geq 1$. Note that $f'(u) = 2[u - 1] + 2[u^{-1} - 1](-1)u^{-2} > 0$ for $u > 1$, which shows that f(u) is increasing for $u \geq 1$. Since $f(1) = 0$, we see that f(u) satisfies the required regularity conditions and the associated absolute dissimilarity index is

(9) $d(x,y) = [(x/y) - 1]^2 + [(y/x) - 1]^2$; $\qquad x > 0 ; y > 0$

and it satisfies the axioms A1-A6.

Note that for both of these examples, the resulting d(x,y) is infinitely differentiable.

Problems

3. Show that the d(x,y) defined by (8) satisfies the axioms A1-A6.
4. Show that the d(x,y) defined by (9) satisfies the axioms A1-A6.

In the following section, we show how a large class of one variable dissimilarity indexes can be defined. Then in the following section, we will add some additional axioms in an attempt to narrow down the choice of a particular index to be used in applications.

A.3 An Alternative Approach for Generating Absolute Dissimilarity Indexes.

Let g and h be continuous monotonically increasing functions of one variable with $g(0) = 0$ and consider the following class of dissimilarity indexes:

(10) $d_{g,h}(x,y) \equiv g\{|h(y/x) - h(1)|\}$.

Thus we first transform y/x and 1 by the function of one variable h, calculate the difference, $h(y/x) - h(1)$, take the absolute value of this difference and then transform this difference by g.

It is easy to verify that the d defined by (10) satisfies all of the axioms A1-A6 with the exception of A4, the symmetry axiom, $d(x,y) = d(y,x)$. However, this defect can be readily overcome. Note that $d_{g,h}(y,x) \equiv g\{|h(x/y) - h(1)|\}$ also satisfies A1-A6 with the exception of A4. Thus, if we take a *symmetric mean*[32] of these two indexes[33], we will obtain a new index which satisfies axiom A4. Hence, let m be a symmetric mean

[32] Diewert (1993b; 361) defined a *symmetric mean* of a and b as a function m(a,b) that has the following properties: (1) $m(a,a) = a$ for all $a > 0$ (mean property); (2) $m(a,b) = m(b,a)$ for all $a > 0$, $b > 0$ (symmetry property); (3) m(a,b) is a continuous function for $a > 0$, $b > 0$ (continuity property); (4) m(a,b) is a strictly increasing function in each of its variables (increasingness property).

[33] Our method for converting a measure that is not symmetric into a symmetric method is the counterpart to Irving Fisher's (1922) *rectification* procedure, which is actually due to Walsh (1921).

function of two variables and let g and h be continuous monotonically increasing functions of one variable with g(0) = 0 and consider the following class of *symmetric monotonic transformation dissimilarity indexes*:

(11) $d_{g,h,m}(x,y) \equiv m[g\{|h(y/x) - h(1)|\}, g\{|h(x/y) - h(1)|\}]$.

Proposition 2: Let g and h be continuous monotonically increasing functions of one variable with g(0) = 0 and let m(a,b) be a symmetric mean. Then each member of the class of symmetric monotonic transformation indexes $d_{g,h,m}(x,y)$ defined by (11) satisfies the axioms A1-A6.

Proof: The proofs of A1-A5 are left to a problem. We verify axiom A6. Let $y'' > y' \geq x > 0$. Then

$$
\begin{aligned}
(12)\quad d_{g,h,m}(x,y'') &\equiv m[g\{|h(y''/x) - h(1)|\}, g\{|h(x/y'') - h(1)|\}] \\
&= m[g\{h(y''/x) - h(1)\}, g\{h(1) - h(x/y'')\}] \\
&\qquad \text{using } y'' > x \text{ and the monotonicity of h} \\
&> m[g\{h(y'/x) - h(1)\}, g\{h(1) - h(x/y'')\}] \\
&\qquad \text{using } y'' > y', x > 0 \text{ and the monotonicity of h, g and m} \\
&> m[g\{h(y'/x) - h(1)\}, g\{h(1) - h(x/y')\}] \\
&\qquad \text{using } y'' > y', x > 0 \text{ and the monotonicity of h, g and m} \\
&= m[g\{|h(y'/x) - h(1)|\}, g\{|h(x/y') - h(1)|\}] \\
&\qquad \text{using } y' > x \text{ and the monotonicity of h} \\
&\equiv d_{g,h,m}(x,y').
\end{aligned}
$$

Q.E.D.

Problem

5. Show that the index $d_{g,h,m}(x,y)$ defined by (11) satisfies the axioms A1-A5.

Let us try and specialize the class of functional forms defined by (11). The simplest symmetric mean m of two numbers is the arithmetic mean and so let us set m(a,b) = (1/2)a + (1/2)b. It is also convenient to get rid of the absolute value function in (11) (so that the resulting dissimilarity index will be differentiable) and this can be done in the most simple fashion by setting $g(u) = u^2$.[34] This leads us to following class of *simple symmetric transformation dissimilarity indexes*, which depends only on the continuous monotonic function h:

(13) $d_h(x,y) \equiv (1/2)[h(y/x) - h(1)]^2 + (1/2)[h(x/y) - h(1)]^2$.

[34] There is another good reason for this choice of g. In most applications, we want the slope of g(u) to be zero at u = 0 and then increase as u increases. This means the amount of dissimilarity between x and y will be close to zero in a neighborhood of points where x is close to y but the degree of dissimilarity will grow at an increasing rate as x diverges from y. We will formalize these properties as axioms A7 and A8 in the next section. Hence if we want the slope of g(u) to increase at a constant rate as u increases, then $g(u) = u^2$ is the simplest function which will accomplish this task.

The two simplest choices for h are $h(u) \equiv u$ and $h(u) \equiv \ln u$.[35] These two choices for h lead to the following concrete dissimilarity indexes:

Example 3: The linear quadratic dissimilarity index:

(14) $d(x,y) \equiv (1/2)[(y/x) - 1]^2 + (1/2)[(x/y) - 1]^2$.

Note that this example is essentially the same as Example 2.

Example 4: The log quadratic dissimilarity index:

$$\begin{aligned}(15)\ d(x,y) &\equiv (1/2)[\ln(y/x) - \ln(1)]^2 + (1/2)[\ln(x/y) - \ln(1)]^2 \\ &= (1/2)[\ln y - \ln x]^2 + (1/2)[\ln x - \ln y]^2 \\ &= [\ln y - \ln x]^2 \\ &= [\ln(y/x)]^2.\end{aligned}$$

Our conclusion at this point is that even in the one variable case, there are a large number of possible measures of absolute dissimilarity that could be chosen. Hence, in the following section, we add some additional axioms to our list of axioms, A1-A6, in an attempt to narrow down this large number of possible choices.

A.4 Additional Axioms for One Variable Absolute Dissimilarity Indexes

Consider the following axiom:

A7: *Convexity*: $d(x,y)$ is a convex function of y for $y \geq x > 0$.

The meaning of this axiom is that we want the amount of dissimilarity between x and y to grow at a constant or increasing rate as y grows bigger than x. Put another way, we do not want the rate of increase in dissimilarity to *decrease* as y grows bigger than x. Although this property seems to be a reasonable one for many purposes, it must be conceded that this property is not as fundamental as the previous 6 properties.

Proposition 3: The asymptotically linear dissimilarity index defined by (8) and the linear quadratic dissimilarity index defined by (14) satisfy the convexity axiom A7 but the log quadratic dissimilarity index defined by (15) does *not* satisfy A7.

Proof: Let $y \geq x > 0$. For the $d(x,y)$ defined by (8), we find that $\partial^2 d(x,y)/\partial y^2 = 2x/y^3 > 0$ and so the asymptotically linear dissimilarity index defined by (8) is convex in y.

For the $d(x,y)$ defined by (15), we find that $\partial^2 d(x,y)/\partial y^2 = 2(x/y)^2[1 - x^{-1}\ln(y/x)]$ which is negative for y large enough and hence the log quadratic dissimilarity index defined by (15) does not satisfy A7.

[35] Bert Balk suggested the following choice for h: $h(u) \equiv u^{1/2}$.

For the d(x,y) defined by (14), we find that:

(16) $\partial^2 d(x,y)/\partial y^2 = x^{-2} + 3x^2y^{-4} - 2xy^{-3} \equiv g(y)$.

Let us attempt to minimize g(y) defined in (16) over $y \geq x$. We have:

(17) $g'(y) = -\,12x^2y^{-5} + 6xy^{-4} = 0$.

The positive roots of (17) are $y^* = 2x$ and $y^{**} = +\infty$. We find that g(y) attains a strict local minimum at $y = 2x$ and this turns out to be the global minimum of g(y) for $y \geq x$. Thus we have for $y \geq x$:

(18) $f''(y) \geq f''(2x) = x^{-2} + 3x^2\,(2x)^{-4} - 2x\,(2x)^{-3} > 0$

and hence the linear quadratic dissimilarity index defined by (14) satisfies A7. Q.E.D.

How can we choose between the asymptotically linear dissimilarity index defined by (8) and the asymptotically quadratic dissimilarity index defined by (9) or (14)? Both indexes behave similarly for x close to y but as y diverges from x, the amount of dissimilarity between x and y will grow roughly quadratically in y for the index defined by (14) whereas for the index defined by (8), the amount of dissimilarity will tend towards a linear in y rate. Hence the choice between the two indexes depends on how fast one wants the amount of dissimilarity between x and y to grow as y grows bigger than x. It should be noted that the index defined by (14) will be much more sensitive to outliers in the data so perhaps for this reason, the index defined by (8) should be used when there is the possibility of errors in the data.

Another axiom which is also not fundamental but does seem reasonable is the following one:

A8: *Differentiability*: d(x,y) is a once differentiable function of two variables.

The real impact of the axiom A8 is along the ray where $x = y$. If we look at the proof of Proposition 1, we see that if we add A8 to the list of axioms, the effect of the differentiability axiom is to force the derivative of f(u) at $u = 1$ to be 0; i.e., under A8, we must have $f'(1) = 0$. In many applications, this will be a very reasonable restriction on f since it implies that the amount of dissimilarity between x and y will be very small when x is very close to y. All of our examples 1 to 4 above satisfy the differentiability axiom.

We now consider another axiom for d(x,y), which is perhaps more difficult to justify, but it does determine the functional form for d:

A9: *Additivity*: $d(x,x + y + z) = d(x,x + y) + d(x,x + z)$ for all $x > 0$, $y \geq 0$ and $z \geq 0$.

Proposition 4: Suppose d(x,y) satisfies the axioms A1-A6 and A9. Then d has the following functional form:[36]

(19) $d(x,y) = \alpha[\max\{x/y, y/x\} - 1]$ where $\alpha > 0$.

Proof: If d(x,y) satisfies A1-A6, then by Proposition 1, $d(x,y) = f[\max\{x/y, y/x\}]$ where f(u) is continuous, increasing for $u \geq 1$ with $f(1) = 0$. Substitute this representation for d(x,y) into A9 and letting $x > 0$, $y \geq 0$ and $z \geq 0$, we find that f satisfies the following functional equation:

(20) $f[1 + (y/x) + (z/x)] = f[1 + (y/x)] + f[1 + (z/x)]$; $x > 0, y \geq 0$ and $z \geq 0$.

Define the variables u and v as follows:

(21) $u \equiv y/x$; $v \equiv z/x$.

Substituting (21) into (20), we find that f satisfies the following functional equation:

(22) $f(1 + u + v) = f(1 + u) + f(1 + v)$; $u \geq 0, v \geq 0$.

Define the function g as follows:

(23) $g(u) \equiv f(1 + u)$.

Using (23), (22) can be rewritten as follows:

(24) $g(u + v) = g(u) + g(v)$; $u \geq 0, v \geq 0$.

But (24) is Cauchy's first functional equation or a special case of Pexider's (1903) first functional equation[37] and has the following solution:

(25) $g(x) = \alpha x$; $x \geq 0$

where α is a constant.

Using (23) and (25),

(26) $f(u) = \alpha(u - 1)$; $u \geq 1$.

Equation (26) implies that d is equal to the right hand side of (19). However, in order that f(u) be increasing for $u \geq 1$, we require that $\alpha > 0$, which completes the proof. Q.E.D.

---

[36] The f(u) that corresponds to this functional form is $f(u) \equiv \alpha[u - 1]$ where $\alpha > 0$. The d(x,y) defined by (12) also satisfies the convexity axiom A7 but it does not satisfy the differentiability axiom A8.
[37] See chapter 2 or Eichhorn (1978; 49) for a more accessible reference.

Let us set $\alpha = 1$ in (19) and call the resulting d(x,y), *example 5, the linear dissimilarity index*. It can be seen that for large y, the dissimilarity indexes defined by examples 1 and 5 will approach each other. The big difference between the two indexes is along the ray where x = y: the linear dissimilarity index will not be differentiable along this ray, whereas the asymptotically linear dissimilarity index will be differentiable everywhere. Also for x close to y, the linear dissimilarity index will be greater than the corresponding asymptotically linear dissimilarity measure.

We conclude this section by indicating a simple way for determining the exact functional form for d(x,y): we need only consider the behavior of d(1,y) for $y \geq 1$. This behavior of the function d determines the underlying generator function f(u) that appeared in the Proposition 1. Hence consider the following "axioms" for d:

A10: $d(1,y) = (y - 1)^{\beta}$ $\quad y \geq 1$, where $\beta > 0$;

A11: $d(1,y) = \ln y$; $\quad y \geq 1$;

A12: $d(1,y) = e^{y} - e$; $\quad y \geq 1$.

It is straightforward to show that if d(x,y) satisfies A1-A6 and A10, then d is equal to the following function: (*example 6*):

(27) $d(x,y) = [\max\{x/y, y/x\} - 1]^{\beta}$; $\quad \beta > 0$.

Of course, if $\beta = 1$, then Example 6 reduces to Example 5.[38]

Similarly, it is straightforward to show that if d(x,y) satisfies A1-A6 and A11, then d is equal to the following function: (*example 7*):[39]

(28) $d(x,y) = \ln [\max\{x/y, y/x\}]$.

Finally, if d(x,y) satisfies A1-A6 and A12, then d is equal to the following function: (*example 8*):[40]

(29) $d(x,y) = e^{\max\{x/y,y/x\}} - e$.

The functional forms for the dissimilarity indexes defined by (27)-(29) are all relatively simple but they all have a disadvantage: namely, *they are not differentiable along the ray where x = y*. Hence, they are probably not suitable for many economic applications.

We turn now to N variable measures of absolute dissimilarity.

---

[38] The d(x,y) defined by (27) satisfies the convexity axiom A7 if and only if $\beta \geq 1$.

[39] This d(x,y) does not satisfy A7.

[40] This d(x,y) does satisfy the convexity axiom A7.

A.5 Axioms for Absolute Dissimilarity Indexes in the N Variable Case

We now let $x \equiv [x_1,...,x_N]$ and $y \equiv [y_1,...,y_N]$ be strictly positive vectors (either price or quantity) that are to be compared in an absolute sense. Let D(x,y) be the absolute dissimilarity index, defined for all strictly positive vectors x and y. The following 6 axioms or properties are fairly direct counterparts to the 6 fundamental axioms that were introduced in section A.2 above.

B1: *Continuity*: D(x,y) is a continuous function defined for all $x >> 0_N$ and $y >> 0_N$.

B2: *Identity*: $D(x,x) = 0$ for all $x >> 0_N$.

B3: *Positivity*: $D(x,y) > 0$ for all $x \neq y$.

B4: *Symmetry*: $D(x,y) = D(y,x)$ for all $x >> 0_N$ and $y >> 0_N$.

B5: *Invariance to Changes in Units of Measurement*: $D(\alpha_1x_1,...,\alpha_Nx_N;\alpha_1y_1,...,\alpha_Ny_N) = D(x_1,...,x_N;y_1,...,y_N) = D(x,y)$ for all $\alpha_n > 0$, $x_n > 0$, $y_n > 0$ for $n = 1,...,N$.[41]

B6: *Monotonicity*: D(x,y) is increasing in the components of y if $y \geq x$.

The above axioms or properties can be regarded as fundamental. However, they are not sufficient to give a nice characterization Proposition like Proposition 1 in section A.2. Hence we need to add additional properties to determine D.

Possible additional properties are the following ones:

B7: *Invariance to the ordering of commodities*: D(Px,Py) = D(x,y) where Px denotes a permutation of the components of the x vector and Py denotes the same permutation of the components of the y vector.

B8: *Additive Separability*: $D(x,y) = \sum_{n=1}^{N} d_n(x_n,y_n)$.

The N functions of two variables, $d_n(x_n,y_n)$, are obviously absolute dissimilarity measures that give us the degree of dissimilarity between the components of the vectors x and y.

Proposition 5: Suppose D(x,y) satisfies B1-B8. Then there exists a continuous, increasing function of one variable, f(u), such that $f(1) = 0$ and D(x,y) has the following representation in terms of f:

(30) $D(x,y) = \sum_{n=1}^{N} f[\max\{x_n/y_n, y_n/x_n\}]$.

---

[41] Note that this axiom implies that D has the homogeneity property $D(\lambda x,\lambda y) = D(x,y)$. To see this, let each $\alpha_n = \lambda$.

Conversely, if D(x,y) is defined by (30) where f is a continuous, increasing function of one variable with f(1) = 0, then D satisfies B1-B8.

Proof: Using B2 and B8, we have

$$(31)\ D(1_N,1_N) = \sum_{n=1}^{N} d_n(1,1) = 0.$$

Thus

$$\begin{aligned}(32)\ D(x,y) &= D(x,y) - D(1_N,1_N) && \text{using (31)}\\ &= \sum_{n=1}^{N} d_n(x_n,y_n) - \sum_{n=1}^{N} d_n(1,1) && \text{using B8}\\ &= \sum_{n=1}^{N} d_n^*(x_n,y_n)\end{aligned}$$

where the $d_n^*(x_n,y_n)$ are defined as:

$$(33)\ d_n^*(x_n,y_n) \equiv d_n(x_n,y_n) - d_n(1,1)\ ; \qquad n = 1,2,\ldots,N.$$

It is easy to check that the d_n^* functions satisfy the following restrictions:

$$(34)\ d_n^*(1,1) = 0\ ; \qquad n = 1,2,\ldots,N.$$

Using (32) and (33), we have:

$$\begin{aligned}(35)\ D(x_1,1_{N-1},y_1,1_{N-1}) &= d_1^*(x_1,y_1) + \sum_{n=2}^{N} d_n^*(1,1)\\ &= d_1^*(x_1,y_1) && \text{using (34).}\end{aligned}$$

Properties B1-B6 on D imply that $d_1^*(x_1,y_1)$ will satisfy properties A1-A6 listed in section A.2 of the Appendix above. Hence, we may apply Proposition 1 and conclude that $d_1^*(x_1,y_1)$ has the following representation:

$$(36)\ d_1^*(x_1,y_1) = f[\max\{x_1/y_1,\ y_1/x_1\}]$$

for some continuous, increasing function of one variable f(u) defined for $u \geq 1$ with f(1) = 0.

Using B7, we deduce that

$$\begin{aligned}(37)\ d_n^*(x_n,y_n) &= d_1^*(x_n,y_n)\\ &= f[\max\{x_n/y_n,\ y_n/x_n\}]\ ; && \text{for } n = 2,\ldots,N \text{ using (36)}\end{aligned}$$

and this establishes (30). The second half of the Proposition is straightforward. Q.E.D.

Thus adding the axioms B7 and B8 to the earlier axioms B1-B6 essentially reduces the N dimensional case down to the one dimensional case.

In applications, it is sometimes useful to be able to compare the amount of dissimilarity between two N dimensional vectors x and y to the amount of dissimilarity between two M dimensional vectors u and v. If we decide to use the function of one variable f to generate the dissimilarity index defined by (30), then we can achieve comparability across vectors of different dimensionality if we modify (30) and define the following family of dissimilarity indexes (which depend on N, the dimensionality of the vectors x and y):

(38) $D_N(x,y) \equiv \sum_{n=1}^{N} (1/N)f[\max\{x_n/y_n, y_n/x_n\}]$.

Recall examples 1 and 2 in section 2. We use the generating functions f(u) for these examples to construct N variable measures of absolute dissimilarity between the positive vectors x and y. Using the generating function $f(u) \equiv [u - 1]^2 + [u^{-1} - 1]^2$ in (38) gives us the following *N dimensional asymptotically linear quadratic index of absolute dissimilarity*, which is the N dimensional generalization of Example 1 above, which we now label as *example 9*:

(39) $D_{AL}(x,y) \equiv (1/N)\sum_{n=1}^{N} [(y_n/x_n) + (x_n/y_n) - 2]$.

Using the generating function $f(u) \equiv [u - 1]^2 + [u^{-1} - 1]^2$ in (38) gives us the following *N dimensional asymptotically quadratic index of absolute dissimilarity*, which is the N dimensional generalization of Example 2 above, which we now label as *example 10*:

(40) $D_{AQ}(x,y) \equiv (1/N)\sum_{n=1}^{N} [(y_n/x_n) - 1]^2 + (1/N)\sum_{n=1}^{N} [(x_n/y_n) - 1]^2$.

The indexes defined by (39) and (40) are our preferred indexes of absolute dissimilarity. The index defined by (40) is less sensitive to measurement errors and outliers so under most circumstances, it seems to be a preferred choice.[42]

We turn now to a discussion of relative dissimilarity indexes in the case of N commodity prices or quantities that must be compared.[43]

A.6 Axioms for Relative Dissimilarity Indexes in the N Variable Case

In making relative comparisons, we regard x and y as being completely similar if x is proportional to y or if y is proportional to x; i.e., if $y = \lambda x$ for some scalar $\lambda > 0$. We denote the relative dissimilarity index between two vectors x and y by $\Delta(x,y)$. The earlier axioms B1-B7 for absolute dissimilarity indexes are now replaced by the following axioms:

C1: *Continuity*: $\Delta(x,y)$ is a continuous function defined for all $x >> 0_N$ and $y >> 0_N$.

---

[42] Hill (2004) has adopted the corresponding weighted relative index of dissimilarity in his most recent empirical work.

[43] The case N = 1 is not relevant in the case of relative dissimilarity indexes so we must move right away into the $N \geq 2$ dimensional case.

C2: *Identity*: $\Delta(x,\lambda x) = 0$ for all $x >> 0_N$ and scalars $\lambda > 0$.

C3: *Positivity*: $\Delta(x,y) > 0$ if $y \neq \lambda x$ for any $\lambda > 0$.

C4: *Symmetry*: $\Delta(x,y) = \Delta(y,x)$ for all $x >> 0_N$ and $y >> 0_N$.

C5: *Invariance to Changes in Units of Measurement*: $\Delta(\alpha_1 x_1,...,\alpha_N x_N ;\alpha_1 y_1,...,\alpha_N y_N) = \Delta(x_1,...,x_N;y_1,...,y_N) = \Delta(x,y)$ for all $\alpha_n > 0$, $x_n > 0$, $y_n > 0$ for $n = 1,...,N$.

C6: *Invariance to the Ordering of Commodities*: $\Delta(Px,Py) = \Delta(x,y)$ where Px is a permutation or reordering of the components of x and Py is the same permutation of the components of y.

C7: *Proportionality*: $\Delta(x,\lambda y) = \Delta(x,y)$ for all $x >> 0_N$, $y >> 0_N$ and scalars $\lambda > 0$.

The last axiom says that the degree of relative dissimilarity between the vectors x and y remains the same if y is multiplied by the arbitrary positive number λ.

The above axioms all seem to be fairly fundamental in the relative dissimilarity index context.[44] We have not developed a counterpart to the absolute monotonicity axiom B6 for relative indexes of dissimilarity because it is not clear what the appropriate relative axiom should be. This is a topic for further research. Also, we do not have any nice characterization theorems for relative dissimilarity indexes that are analogous to Proposition 5 in the previous section. However, we do have a strategy for adapting the absolute dissimilarity indexes to the relative context.

Our suggested strategy is this. First, find a *scale index* S(x,y) that is essentially a price or quantity index between the vectors x and y and that has the property $S(x,\lambda x) = \lambda$. Second, find a suitable absolute dissimilarity index, D(x,y). Finally, use the scale index S and the absolute dissimilarity index D in order to define the following relative dissimilarity index Δ:

(41) $\Delta(x,y) \equiv D(S(x,y)x, y)$.

Thus in (23), we scale up the base vector x by the index number S(x,y) which makes it comparable in an absolute sense to the vector y. We then apply an absolute index of dissimilarity D to the scaled up x vector, S(x,y)x, and the vector y. Naturally, in order for the Δ defined by (41) to satisfy the axioms C1-C7, it will be necessary for D and S to satisfy certain properties. We will assume that the absolute dissimilarity index D satisfies B1-B5 and B7 in the previous section. We will also impose the following properties on the scale index S(x,y):

D1: *Continuity*: S(x,y) is a continuous function defined for all $x >> 0_N$ and $y >> 0_N$.

[44] Axioms C2-C7 were proposed by Allen and Diewert (1981; 433).

D2: *Identity*: $S(x,x) = 1$ for all $x >> 0_N$.

D3: *Positivity*: $S(x,y) > 0$ for all $x >> 0_N$ and $y >> 0_N$.

D4: *Time or Place Reversal*: $S(x,y) = 1/S(y,x)$ for all $x >> 0_N$ and $y >> 0_N$.

D5: *Invariance to Changes in Units of Measurement*: $S(\alpha_1x_1,...,\alpha_Nx_N\ ;\alpha_1y_1,...,\alpha_Ny_N) = S(x_1,...,x_N;y_1,...,y_N) = S(x,y)$ for all $\alpha_n > 0$, $x_n > 0$, $y_n > 0$ for $n = 1,...,N$.

D6: *Invariance to the Ordering of Commodities*: $S(Px,Py) = S(x,y)$ where Px is a permutation or reordering of the components of x and Py is the same permutation of the components of y.

D7: *Proportionality*: $S(x,\lambda y) = \lambda S(x,y)$ for all $x >> 0_N$, $y >> 0_N$ and scalars $\lambda > 0$.

Proposition 6: If the scale function S(x,y) satisfies D1-D7 and the absolute dissimilarity index D(x,y) satisfies B1-B5 and B7 listed in the previous section, then the relative dissimilarity index $\Delta(x,y)$ defined by (23) satisfies properties C1-C7.

Proof: Properties C1 and C5 are obvious. Now check property C2:

$$
\begin{aligned}
(42)\quad \Delta(x,\lambda x) &\equiv D(S(x,\lambda x)x, \lambda x) && \text{using definition (41)} \\
&= D(\lambda S(x,x)x, \lambda x) && \text{using D7} \\
&= D(\lambda x, \lambda x) && \text{using D2} \\
&= 0 && \text{using B2.}
\end{aligned}
$$

Now check property C3. Given x and y, suppose that $y \neq \lambda x$ for any $\lambda > 0$. Using definition (41), we have:

$$
\begin{aligned}
(43)\quad \Delta(x,y) &\equiv D(S(x,y)x, y) \\
&= D(\mu x, y) && \text{where } \mu = S(x,y) > 0 \text{ using D3} \\
&> 0 && \text{using B3 since } y \neq \mu x.
\end{aligned}
$$

Check property C4:

$$
\begin{aligned}
(44)\quad \Delta(x,y) &\equiv D(S(x,y)x, y) && \text{using definition (41)} \\
&= D(1_N, y_1/x_1S(x,y),...,y_N/x_NS(x,y)) && \text{using B5} \\
&= D(1_N, S(y,x)y_1/x_1,...,S(y,x)y_N/x_N) && \text{using D4} \\
&= D(x, S(y,x)y) && \text{using B5 again} \\
&= D(S(y,x)y,x) && \text{using B4} \\
&\equiv \Delta(y,x) && \text{using definition (23).}
\end{aligned}
$$

Property C6 follows from Properties B7 and D6.

Finally, check Property C7. Let $x >> 0_N$, $y >> 0_N$ and scalars $\lambda > 0$. Then by definition (23),

$$
\begin{aligned}
(45)\ \Delta(x,\lambda y) &\equiv D(S(x,\lambda y)x,\ \lambda y) \\
&= D(\lambda S(x,y)x,\ \lambda y) && \text{using D7} \\
&= D(S(x,y)x,\ y) && \text{using B5 with all } \alpha_n = \lambda \\
&= \Delta(x,y) && \text{using definition (41).}
\end{aligned}
$$

Q.E.D.

The above Proposition can be used in order to generate a wide class of relative dissimilarity indexes.

We conclude this section by giving some examples of how Proposition 7 could be applied in order to define some indexes of relative dissimilarity.

Example 11: Recall the N variable index of dissimilarity $D_{AL}(x,y)$ defined by (39) above. It can be verified that this absolute index of dissimilarity satisfies axioms B1-B9. We need to choose a scale index $S(x,y)$ that satisfies the axioms D1-D7. The simplest choice for such an S is:

$$(46)\quad S_J(x,y) \equiv \Pi_{n=1}^{N} (y_n/x_n)^{1/N}.$$

Thus $S(x,y)$ is the geometric mean of the y_n divided by the geometric mean of the x_n. This functional form (for a price index) is due to Jevons (1865) and it is still used today as a functional form for an elementary price index. It can be verified that S_J satisfies the axioms D1-D7. It should be noted that the following scale indexes do *not* satisfy the time reversal test, D4:

$$(47)\ S_A(x,y) \equiv \sum_{n=1}^{N} (1/N)(y_n/x_n)\ ;$$
$$(48)\ S_H(x,y) \equiv [\sum_{n=1}^{N} (1/N)(y_n/x_n)^{-1}]^{-1}.$$

Thus S_A is the arithmetic mean[45] of the ratios y_n/x_n and S_H is the harmonic mean of the ratios y_n/x_n.

Inserting S_J defined by (46) into formula (41) where D is defined by (39) leads to the following *asymptotically linear index of relative dissimilarity* (which satisfies C1-C7):

$$(49)\ \Delta_{AL}(x,y) \equiv D_{AL}(S_J(x,y)x,\ y) = \sum_{n=1}^{N}(1/N)[(S_J(x,y)x_n/y_n) + (y_n/S_J(x,y)x_n) - 2]\ .$$

[45] S_A is known in the price index literature as the Carli (1764) index. Note that the geometric mean of S_A and S_H does satisfy the axioms D1-D7 and hence could be used in place of the Jevons scale index S_J. $S_{AH}(x,y) \equiv [S_A(x,y)S_H(x,y)]^{1/2}$ has been suggested as the functional form for an elementary price index by Carruthers, Sellwood and Ward (1980).

Example 12: Recall the N dimensional asymptotically quadratic index of absolute dissimilarity, $D_{AQ}(x,y)$ defined by (40) above. It can be verified that this absolute index of dissimilarity satisfies axioms B1-B8. Inserting S_J defined by (46) into formula (41) where D is defined by (40) leads to the following *asymptotically quadratic index of relative dissimilarity* (which also satisfies C1-C7):

$$(50)\ \Delta_{AQ}(x,y) \equiv D_{AQ}(S_J(x,y)x,y) = \sum_{n=1}^{N}(1/N)[(S_J(x,y)x_n/y_n) - 1]^2 + \sum_{n=1}^{N}(1/N)[(y_n/S_J(x,y)x_n) - 1]^2.$$

Example 13: Recall the log quadratic single variable measure of absolute dissimilarity defined by (15) above. The additively separable extension of this measure to the N variable case is the following *log squared index of absolute dissimilarity*:

$$(51)\ D_{LS}(x,y) \equiv \sum_{n=1}^{N} (1/N)[\ln(y_n/x_n)]^2.$$

It can be verified that this absolute index of dissimilarity satisfies axioms B1-B9. Inserting S_J defined by (46) into formula (41) where D is defined by (51) leads to the following *log squared index of relative dissimilarity* (which also satisfies C1-C7):

$$(52)\ \Delta_{LS}(x,y) \equiv D_{LS}(S_J(x,y)x, y) = \sum_{n=1}^{N} (1/N)[\ln(y_n/S_J(x,y)x_n)]^2$$
$$= (1/N)\sum_{n=1}^{N} [\ln(y_n/x_n) - \ln S_J(x,y)]^2$$
$$= (1/N)\sum_{n=1}^{N} [\ln(y_n/x_n) - \ln\{\Pi_{n=1}^{N} (y_n/x_n)^{1/N}\}]^2.$$

The last line of (52) shows that $\Delta_{LQ}(x,y)$ is equal to a constant times the Allen Diewert (1981; 433) measure of nonproportionality between the vectors x and y. Allen and Diewert derived their measure by regressing the N logarithmic ratios, $\ln(y_n/x_n)$, on a constant, obtaining $(1/N)\sum_{n=1}^{N} \ln(y_n/x_n) = \ln\{\Pi_{n=1}^{N} (y_n/x_n)^{1/N}\}$ as the least squares estimator of this constant. They then used the sum of squared residuals from their regression as their measure of nonproportionality, which is N times the last line of (52).

We turn now to *weighted absolute and relative dissimilarity indexes*.

A.7 Weighted Absolute Dissimilarity Indexes

The analysis up to this point has implicitly assumed (using the axioms B7 or C6) that the amount of dissimilarity between each component of the x and y vectors is equally important and hence gets an equal weight in the overall index of dissimilarity. In many applications, this assumption is not justified, which suggests that the individual component measures of dissimilarity should be weighted according to the economic importance of that commodity. However, there are several ways that this economic importance could be measured. If we are constructing an index of price dissimilarity, then it might be natural to weight by either the *quantities transacted* in the two situations or by the *expenditures* pertaining to that component. However, if the prices of a large country are being compared to those of a small country, then using either of these two methods of weighting will perhaps give too much weight to the large country. Hence, we will follow the example of Theil (1967; 136-137) and weight the importance of commodities by their *expenditure shares* in the two countries.[46] Thus define the expenditure share of commodity n in country i as

(53) $s_n^i \equiv p_n^i q_n^i / p^i \cdot q^i$; $\quad$ i = 1,2 ; n = 1,...,N.

Let m(a,b) be a symmetric mean of the positive numbers a and b and let f(u) be an increasing continuous function of one variable, defined for u ≥ 1 with the property that f(1) = 0. Then we can use the functions m and f in order to define the following *weighted absolute indexes of price and quantity dissimilarity*, D_P and D_Q:

(54) $D_P(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N m(s_n^1,s_n^2)\, f[\max\{p_n^1/p_n^2, p_n^2/p_n^1\}]$;
(55) $D_Q(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N m(s_n^1,s_n^2)\, f[\max\{q_n^1/q_n^2, q_n^2/q_n^1\}]$.

It can be seen that we have just used the characterization of D(x,y) in the unweighted case given by Proposition 5 and weighted the commodities according to their economic importance, which is reflected in the weights $m(s_n^1,s_n^2)$.[47]

It will be necessary to make concrete choices for the mean function m and the generator function f in empirical examples. As in the earlier sections, on the grounds of simplicity, we choose the arithmetic mean so that

(56) m(a,b) = (1/2)a + (1/2)b.

Our two preferred choices for f made at the end of section 2 and in examples 9 and 10 in section A.5. With the first preferred choice, (54) and (55) become the *weighted asymptotically linear indexes of absolute dissimilarity*: (*Example 14*):

(57) $D_{PAL}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\, [(p_n^1/p_n^2) + (p_n^2/p_n^1) - 2]$;
(58) $D_{QAL}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\, [[(q_n^1/q_n^2) + (q_n^2/q_n^1) - 2]$.

With the second preferred choice, (54) and (55) become the *weighted asymptotically quadratic index of absolute dissimilarity*: (*Example 15*):

(59) $D_{PAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\, [\{(p_n^1/p_n^2) - 1\}^2 + \{(p_n^2/p_n^1) - 1\}^2]$;
(60) $D_{QAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\, [\{(q_n^1/q_n^2) - 1\}^2 + \{(q_n^2/q_n^1) - 1\}^2]$.

We can follow Theil (1967; 138) and give the following statistical interpretation of the right hand side of (57) when m is defined by (56). Define *the absolute dissimilarity of the nth price ratio* between the two countries, r_n, by:

(61) $r_n \equiv f[\max\{p_n^1/p_n^2, p_n^2/p_n^1\}]$ $\quad$ for n = 1,…,N.

---

[46] Recent papers that also pursue this weighted approach are Heston, Summers and Aten (2001), Sergueev (2001b) and Diewert (2002). Our analysis follows that of Diewert.
[47] In (38), we used the normalizing factor (1/N) in place of our present normalizing factor, $m(s_n^1,s_n^2)$. Thus the dissimilarity measures defined by (57) and (58) are comparable for differing N.

Now define the discrete random variable, R say, as the random variable which can take on the values r_n with probabilities $\rho_n \equiv (1/2)[s_n^0 + s_n^1]$ for n = 1,…,N. Note that since each set of expenditure shares, s_n^0 and s_n^1, sums to one, the probabilities ρ_n will also sum to one. It can be seen that the expected value of the discrete random variable R is:

$$(62)\quad E[R] \equiv \sum_{n=1}^{N} \rho_n r_n = \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) f[\max\{p_n^1/p_n^2, p_n^2/p_n^1\}] = D_P(p^1,p^2,q^1,q^2).$$

using (54) and (61). Thus $D_P(p^1,p^2,q^1,q^2)$ can be interpreted as *the expected value of the absolute dissimilarities of the price ratios between the two countries*, where the N discrete price dissimilarities are weighted according to Theil's probability weights, $\rho_n \equiv (1/2)[s_n^0 + s_n^1]$ for n = 1,…,N.

A similar interpretation can be given to $D_Q(p^1,p^2,q^1,q^2)$ defined by (55) when m is defined by (55). Thus $D_Q(p^1,p^2,q^1,q^2)$ can be interpreted as *the expected value of the absolute dissimilarities of the quantity ratios between the two countries*, where the N discrete absolute quantity dissimilarities, $f[\max\{q_n^1/q_n^2, q_n^2/q_n^1\}]$, are weighted according to Theil's probability weights, $\rho_n \equiv (1/2)[s_n^0 + s_n^1]$ for n = 1,…,N.

A.8 Weighted Relative Dissimilarity Indexes

Let $P(p^1,p^2,q^1,q^2)$ and $Q(p^1,p^2,q^1,q^2)$ be the "best" bilateral price and quantity indexes that one could choose.[48] We want the index number formulae P and Q to satisfy counterparts to the axioms D1-D7 listed above.[49] Adapting the strategy outlined in section A.6 above, we again use the functions m and f in order to define the following *weighted relative indexes of price and quantity dissimilarity*, Δ_P and Δ_Q:

$$(63)\quad \Delta_P(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} m(s_n^1,s_n^2)\, f[\max\{P(p^1,p^2,q^1,q^2)p_n^1/p_n^2,\ p_n^2/P(p^1,p^2,q^1,q^2)p_n^1\}]\ ;$$
$$(64)\quad \Delta_Q(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} m(s_n^1,s_n^2)\, f[\max\{Q(p^1,p^2,q^1,q^2)q_n^1/q_n^2,\ q_n^2/Q(p^1,p^2,q^1,q^2)q_n^1\}].$$

As in the previous section, we specialize m to be the arithmetic mean. With this choice, (63) and (64) become the following *weighted relative indexes of price and quantity dissimilarity*:

$$(65)\quad \Delta_P(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1+s_n^2) f[\max\{P(p^1,p^2,q^1,q^2)p_n^1/p_n^2,\ p_n^2/P(p^1,p^2,q^1,q^2)p_n^1\}]\ ;$$
$$(66)\quad \Delta_Q(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2) f[\max\{Q(p^1,p^2,q^1,q^2)q_n^1/q_n^2,\ q_n^2/Q(p^1,p^2,q^1,q^2)q_n^1\}]$$

where f(u) is an increasing continuous function of one variable, defined for $u \geq 1$ with the property that f(1) = 0.

---

[48] Diewert (1992) argues that the Fisher (1922) price and quantity indexes are "best" from the axiomatic point of view but Von Auer (2001) and Balk (1995) argue for some other choices as well.
[49] The Fisher ideal indexes satisfy these properties.

Example 16: Consider the following special case where we choose $f(u) \equiv [\ln u]^2$. The resulting *weighted log quadratic index of relative price dissimilarity* using the bilateral index number formula P is:

$$(67)\ \Delta_{PLQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2)\,[\ln(p_n^2/P(p^1,p^2,q^1,q^2)p_n^1)]^2 .$$

The above formula is a generalization of the Allen Diewert (1981) unweighted formula (52) above. The Törnqvist Theil (1967) bilateral index number formula $P_T(p^1,p^2,q^1,q^2)$ seems to be the appropriate generalization of the unweighted Jevons formula to use in (67) for $P(p^1,p^2,q^1,q^2)$ but any superlative price index formula $P(p^1,p^2,q^1,q^2)$ could be used in (67).

Example 17: Consider the following special case of (65) where $f(u) \equiv [u + u^{-1} - 2]$ for $u \geq 1$. The resulting *weighted asymptotically linear index of relative price dissimilarity* using the bilateral index number formula P is:

$$(68)\ \Delta_{PAL}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2)\{(p_n^2/P(p^1,p^2,q^1,q^2)p_n^1) + P(p^1,p^2,q^1,q^2)(p_n^1/p_n^2) - 2\}.$$

The above formula is the weighted generalization of the unweighted relative formula (49) above.[50]

Example 18: Consider the following special case of (65) where $f(u) \equiv (1/2)[u - 1]^2 + (1/2)[u^{-1} - 1]^2$ for $u \geq 1$. The resulting *weighted asymptotically quadratic index of relative price dissimilarity* using the bilateral index number formula P is:

$$(69)\ \Delta_{PAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2)\{[(p_n^2/P(p^1,p^2,q^1,q^2)p_n^1) - 1]^2 + [(P(p^1,p^2,q^1,q^2)(p_n^1/p_n^2) - 1]^2\}.$$

The above formula is the weighted generalization of the unweighted relative formula (50) above.

Our preferred indexes of relative dissimilarity are those defined by (68) and (69), with a preference for (68) if the underlying data are subject to large measurement errors. Of course, analogous indexes can be defined for quantities rather than prices.

A.9 Conclusion

Our tentative conclusion is that chaining or linking between countries should be based on the sum of a *weighted absolute dissimilarity index of quantities* and a *weighted relative dissimilarity index of prices*.

We have exhibited many different functional forms for these two dissimilarity indexes but until more theoretical and empirical research becomes available, we recommend the

[50] This is the formula used by Hill (2004) in his recent empirical work.

use of the *asymptotically linear* or *asymptotically quadratic* functional forms. Both of these functional forms are differentiable when the price vectors being compared are proportional and when the quantity vectors being compared are equal but the asymptotically quadratic functional form penalizes large deviations between the two vectors much more heavily than does the asymptotically linear functional form.[51] Thus we are specifically recommending either the *weighted asymptotically linear index of relative price dissimilarity* $\Delta_{PAL}(p^1,p^2,q^1,q^2)$ defined by (68) where the price index $P(p^1,p^2,q^1,q^2)$ is the Fisher (1922) ideal formula or the *weighted asymptotically quadratic index of relative price dissimilarity* $\Delta_{PAQ}(p^1,p^2,q^1,q^2)$ defined by (69) as our preferred measures of relative price dissimilarity. Similarly, we are specifically recommending either the *weighted asymptotically linear index of absolute quantity dissimilarity* $D_{QAL}(p^1,p^2,q^1,q^2)$ defined by (58) or the *weighted asymptotically quadratic index of quantity dissimilarity* $D_{QAQ}(p^1,p^2,q^1,q^2)$ defined by (60) as our preferred measures of absolute quantity dissimilarity. These indexes satisfy all of the important axioms that we have discussed.

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[51] Researchers who prefer the sum of absolute deviations as a measure of dispersion will probably be comfortable with the asymptotically linear functional form whereas researchers who prefer the variance as a measure of dispersion will probably be more comfortable using the asymptotically quadratic functional form.

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