APPLIED ECONOMICS

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Chapter 8: The Measurement of Income and the Determinants of Income Growth

1. Introduction

In this chapter, we will consider how to measure income. This would seem to be a very straightforward subject but as we shall see, it is far from being simple, even when we assume that there is only a single homogeneous reproducible capital good. We will also study the determinants of income growth; in particular, we will provide a formal production theoretic framework that will enable us to determine the relative importance to income growth of output price changes (including changes in the terms of trade), capital and labour growth and productivity growth.[1]

Virtually all economic discussions about the economic strength of a country use Gross Domestic or Gross National Product as "the" measure of output. But gross product measures do not account for the capital that is used up during the production period; i.e., the gross measures neglect depreciation. Thus in section 2, we consider some possible reasons why gross measures seem to be more popular than net measures.

Even though it may be difficult empirically to estimate depreciation and hence to estimate net output as opposed to gross output, we nevertheless conclude that for welfare purposes, the net measure is to be preferred. Net measures of output are also known as *income* measures. In section 3, we study in some detail Samuelson's (1961) discussion on alternative income concepts and how they might be implemented empirically. In particular, Samuelson (1961; 46) gives a nice geometric interpretation of Hicks' (1939; 174) Income Number 3.

In section 4, we digress temporarily and generalize Samuelson's (1961; 45-46) index number method for measuring "income" change; i.e., we cover the pure theory of the output quantity index that was developed by Samuelson and Swamy (1974), Sato (1976) and Diewert (1983).

In section 5, we note that Samuelson's measures of income do not capture all of the complexities of the concept. Samuelson worked with a net investment framework but net investment is equal to capital at the end of the period less capital at the beginning of the period. Unfortunately, prices at the beginning of the period are not necessarily equal to prices at the end of the period. Thus Hicks noted that there was a "kind of index number problem" in comparing capital stocks at the beginning and end of the period:[2]

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[1] This chapter draws heavily on Diewert (2006b) and Diewert and Lawrence (2006).
[2] We studied this problem in the previous chapter but we revisit it again in the present chapter.

"At once we run into the difficulty that if Net Investment is interpreted as the difference between the value of the Capital Stock at the beginning and end of the year, the transformation would not be possible. It is only in the special case when the prices of all sorts of capital instruments are the same (if their condition is the same) at the end of the year as at the beginning, that we should be able to measure the money value of Real Net Investment by the increase in the Money value of the Capital stock. In all probability these prices will have changed during the year, so that we have a kind of index number problem, parallel to the index number problem of comparing real income in different years. The characteristics of that other problem are generally appreciated; what is not so generally appreciated is the fact that before we can begin to compare real income in different years, we have to solve a similar problem within the single year—we have to reduce the Capital stock at the beginning and end of the years into comparable real terms." J.R. Hicks (1942; 175-176).

In section 5, we look at various possible alternatives for making the capital stocks at the beginning and end of the year comparable to each other in real terms.

In section 6, we return to the accounting problems associated with the profit maximization problem of a production unit, using the Hicks (1961; 23) and Edwards and Bell (1961; 71-72) Austrian production function framework studied in Appendix 2 of chapter 7. In this section, we show how the traditional gross rentals user cost formula can be decomposed into three terms—one reflecting the reward for "waiting", the second one reflecting anticipated asset price changes and the last term reflecting depreciation—and then we show how the depreciation term can be transferred from the list of inputs and regarded as a negative output, which leads to an income concept that was studied in section 5. This transfer is equivalent to treating depreciation as an intermediate input. Another income concept studied in section 5 emerges if we also regard the anticipated price change term as an intermediate input.

In section 7 we present various approximations to the theoretical target income concept—approximations that can be implemented empirically. Sections 6 and 7 also touches on the obsolescence and depreciation controversy that dates back to Hayek (1941) and Pigou (1941). Section 8 summarizes our discussion on income concepts.

The final sections in the chapter develop a production theory framework that tries to explain the various factors behind the growth in real income that the market sector of an economy can generate. The main factors that explain real income growth are:

- Changes in the prices of the outputs that the market sector produces and changes in the prices of the intermediate inputs that it uses. These price changes include changes in the economy's terms of trade.
- Changes in the amounts of primary inputs that the market sector uses.
- Changes in the productivity of the economy.

Section 9 develops the production theory framework in general terms while section 10 uses the assumption that the technology of the market sector can be described by a translog variable profit function. In the latter case, an exact decomposition of real income growth generated by the market sector into explanatory factors can be obtained.[3]

[3] This decomposition is due to Diewert and Lawrence (2006).

Section 11 extends the analysis of section 10 to deal with the contribution of changes in the terms of trade to real income growth.

2. Measuring National Product: Gross versus Net

Real Gross Domestic Product, per capita real GDP and labour productivity (real GDP divided by hours worked in the economy) are routinely used to compare "welfare" levels between countries (and between time periods in the same country). Gross Domestic Product is the familiar C + G + I + X − M or, in a closed economy with no government, it is simply C + I, consumption plus gross investment that takes place during an accounting period. However, economists have argued for a long time that GDP is not the "right" measure of output for welfare purposes; rather NDP (Net National Product), equal to consumption plus net investment accruing to nationals, is a much better measure, where net investment equals gross investment less depreciation.[4] Why has GDP remained so much more popular than NDP, given that NDP seems to be the better measure for "welfare" comparison purposes?[5]

Samuelson (1961) had a good discussion of the arguments that have been put forth to justify the use of GDP over NDP:

> Within the framework of a purely theoretical model such as this one, I believe that we should certainly prefer net national product, NNP, to gross national product, GNP, if we were forced to choose between them. This is somewhat the reverse of the position taken by many official statisticians, and so let me dispose of three arguments used to favour the gross concept." Paul A. Samuelson (1961; 33).

The first argument that Samuelson considered was that our estimates of depreciation are so inaccurate that it is better to measure GDP or GNP rather than NDP or NNP. Samuelson was able to dispose of this argument in his context of a purely theoretical model as follows:

> "Within our simple model, we know precisely what depreciation is and so for our present purpose this argument can be provisionally ruled out of order." Paul A. Samuelson (1961; 33).

However, in our practical measurement context, we cannot dismiss this argument so easily and we have to concede that the fact that our empirical estimates of depreciation are so shaky, is indeed an argument to focus on measuring GDP rather than NDP.[6]

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[4] See Marshall (1890) and Pigou (1924; 46) (1935; 240-241) (1941; 271) for example. A more recent paper that argues for the net product framework is Diewert and Fox (2005).

[5] In the present chapter, we will assume that the economy is closed so that the distinction between domestic product and national product (e.g., NDP versus NNP) vanishes. Hence our focus is on justifying either a gross product or a net product concept.

[6] Hicks (1973; 155) conceded that this argument for GDP or GNP has some validity: "There are items, of which Depreciation and Stock Appreciation are the most important, which do not reflect actual transactions, but are estimates of the changes in the value of assets which have not yet been sold. These are estimates in a different sense from that previously mentioned. They are not estimates of a statistician's true figure, which happens to be unavailable; there is no true figure to which they correspond. They are estimates relative to a purpose; for different purposes they may be made in different ways. This is of course the basic reason why it has become customary to express the National Accounts in terms of Gross National Product (before deduction of Depreciation) so as to clear them of contamination with the

The second argument that Samuelson considered was the argument that GNP reflects the productive potential of the economy:

"Second, there is the argument that GNP gives a better measure than does NNP of the maximum consumption sprint that an economy could make by consuming its capital in time of future war or emergency." Paul A. Samuelson (1961; 33).

Samuelson (1961; 34) is able to dismiss this argument by noting that NNP is not the maximum short run production that could be squeezed out of an economy: by running down capital to an extraordinary degree, we could increase present period output to a level well beyond current GNP.

The third argument that Samuelson considered had to do with the difficulties involved in determining obsolescence:

"A third argument favouring a gross rather than net product figure proceeds as follows: new capital is progressively of better quality than old, so that net product calculated by the subtraction of all depreciation and obsolescence does not yield an ideal measure 'based on the principle of keeping intact the physical productivity of the capital goods in some kind of welfare sense'." Paul A. Samuelson (1961; 35).

Again Samuelson dismisses this argument in the context of his theoretical model (where all is known) but in the practical measurement context, we have to concede that this argument has some validity, just as did the first argument.

From our point of view, the problem with the gross concept is that it gives us a measure of output that is not sustainable. By deducting even an imperfect measure of depreciation (and obsolescence) from gross investment, we will come closer to a measure of output that could be consumed in the present period without impairing production possibilities in future periods. Hence, for welfare purposes, measures of net product seem to be much preferred to gross measures, even if our estimates of depreciation and obsolescence are imperfect.[7]

'arbitrary' depreciation item; though it should be noticed that even with GNP another arbitrary element remains, in stock accumulation."

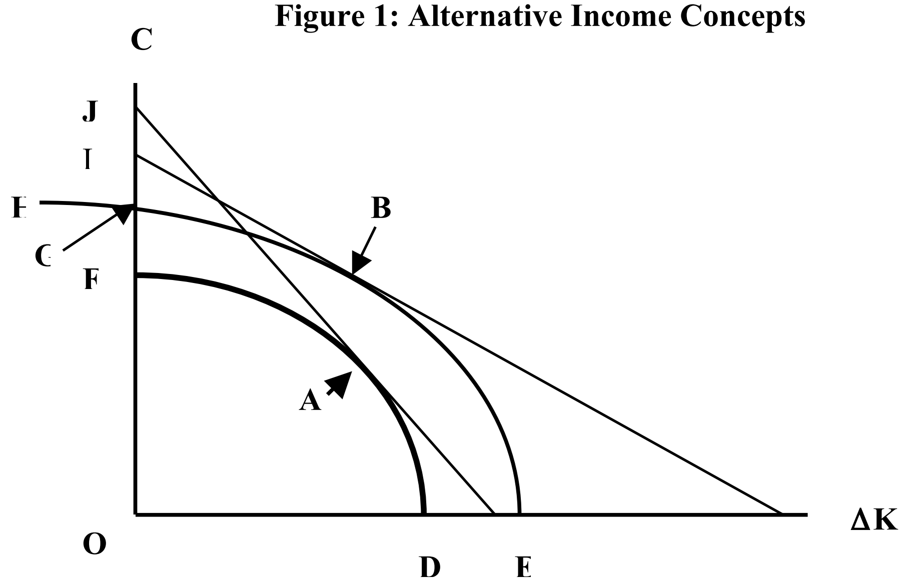
[7] This point of view is also expressed in the *System of National Accounts 1993*: "As value added is intended to measure the additional value created by a process of production, it ought to be measured net, since consumption of fixed capital is a cost of production. However, as explained later, consumption of fixed capital can be difficult to measure in practice and it may not always be possible to make a satisfactory estimate of its value and hence of net value added." Eurostat (1993; 121). "The consumption of fixed capital is one of the most important elements in the System. ... Moreover, consumption of fixed capital does not represent the aggregate value of a set of transactions. It is an imputed value whose economic significance is different from entries in the accounts based only on market transactions. For these reasons, the major balancing items in national accounts have always tended to be recorded both gross and net of consumption of fixed capital. This tradition is continued in the System where provision is also made for balancing items from value added through to saving to be recorded both ways. In general, the gross figure is obviously the easier to estimate and may, therefore, be more reliable, but the net figure is usually the one that is conceptually more appropriate and relevant for analytical purposes." Eurostat (1993; 150).

In the following section, we will look at some alternative definitions of net product. Given a specific definition for net product and given an accounting system that distributes the value of outputs produced to inputs utilized, each definition of net product gives rise to a corresponding definition of "income". In the economic literature, most of the discussion of alternative measures of net output has occurred in the context of alternative "income" measures and so in the following section, we will follow the literature and discuss alternative "income" measures rather than alternative measures of "net product".

3. Measuring Income: Hicks versus Samuelson

Samuelson (1961; 45-46) constructed a nice diagram which illustrated alternative income concepts in a very simple model where the economy produces only two goods: consumption C and a durable capital input K. *Net investment* during period t is defined as $\Delta K^t \equiv K^t - K^{t-1}$, the end of the period capital stock, K^t, less the beginning of the period capital stock, K^{t-1}. In Figure 1 below, let the economy's period 2 production possibilities set for producing combinations of consumption C and net investment ΔK be represented by the curve HGBE[8] and let the economy's period 1 production possibilities set for producing consumption and nonnegative net investment be represented by the curve FAD. Assume that the actual period 2 production point is represented by the point B and the actual period 1 production point is represented by the point A.

Figure 1: Alternative Income Concepts



[8] The point H on the period 2 production possibilities set would represent a consumption net investment point where the end of the period capital stock is less than the beginning of the period stock so that consumption is increased at the cost of running down the capital stock. The period 1 production possibilities set could similarly be extended to the left of the point F.

Samuelson used the definition of income that was due to Marshall (1890) and Haig (1921), who (roughly speaking) defined income as consumption plus the consumption equivalent of the increase in net wealth over the period:

"The Haig-Marshall definition of income can be defended by one who admits that consumption is the ultimate end of economic activity. In our simple model, the Haig-Marshall definition measures the economy's *current power to consume* if it wishes to do so." Paul A. Samuelson (1961; 45).

Samuelson went on to describe a number of methods by which the Haig-Marshall definition of income or net product could be implemented. Three of his suggested methods will be of particular interest to us.

Method 1: The Market Prices Method

If producers are maximizing the value of consumption plus net investment subject to available labour and initial capital resources in each period, then in each period, there will be a market revenue line that is tangent to the production possibilities set. Thus the revenue line BI is tangent to the period 2 set and the line JA is tangent to the period 1 production possibilities set. Each of these revenue lines can be used to convert the period's net investment into consumption equivalents at the market prices prevailing in each period. Thus in period 1, the consumption equivalent of the observed production point A is the point J while in period 2, the consumption equivalent of the observed production point B is the point I and so using this method, Haig-Marshall income is higher in period 1 than in 2, since J is above I.[9]

Method 2: Samuelson's Index Number Method

Let the point A be the period 1 consumption, net investment point C^1,I^1 with corresponding market prices P_C^1,P_I^1 and let the point B be the period 2 consumption, net investment point C^2,I^2 with corresponding market prices P_C^2,P_I^2. Samuelson suggested computing the Laspeyres and Paasche quantity indexes for net output, Q_L and Q_P:

(1) $Q_L \equiv [P_C^1C^2 + P_I^1I^2]/[P_C^1C^1 + P_I^1I^1]$;
(2) $Q_P \equiv [P_C^2C^2 + P_I^2I^2]/[P_C^2C^1 + P_I^2I^1]$.

If Q_L and Q_P are both greater than one, then Samuelson would say that income in period 2 is greater than in period 1; if Q_L and Q_P are both less than one, then Samuelson would say that income in period 2 is less than in period 1; if Q_L and Q_P are both equal to one, then Samuelson would say that income in period 1 is equal to period 2 income. If Q_L and Q_P are such that one is less than one and the other greater than one, then Samuelson would term the situation inconclusive.[10]

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9 Some statisticians would, I think, tend to measure incomes by the vertical intercepts of the tangent lines through A and B. On their definition, A would involve more income than B." Paul A. Samuelson (1961; 45).

10 "Neither Haig nor Marshall have told us exactly how they would evaluate and compare A and B in Fig. 3. Certainly some economic statisticians would interpret them as follows: Money national income is meaningless; you must deflate the money figures and reduce things to constant dollars. To deflate, apply

We will indicate in the following section how Samuelson's analysis can be generalized to deal with the indeterminate case, using modern index number theory. Note that this method for measuring the growth in real income boils down to a method which somehow summarizes the shift in the production possibilities frontier going from period 1 to period 2.

Method 3: Hicksian Income

Hicks (1939) made a number of definitions of income. The one that Samuelson chose to model is Hicks' income Number 3:[11]

"Income No. 3 must be defined as the maximum amount of money which the individual can spend this week, and still expect to be able to spend the same amount *in real terms* in each ensuing week." J.R. Hicks (1939; 174).

Referring back to Figure 1 above, Samuelson (1961; 46) interpreted Hicksian income in period 1 as the point F (which is where the period 1 production frontier intersects the consumption axis so that net investment would be 0 at this point) and Hicksian income in period 2 as the point G (which is where the period 2 production frontier intersects the consumption axis so that net investment would be 0 at this point). However, Samuelson also noted that this definition of income is less useful to the economic statistician than the above two definitions because the economic statistician will not be able to determine where the production frontier will intersect the consumption axis:

"Others (e.g. Hicks of the earlier footnote) want to measure income by comparing the vertical intercepts of the curved production possibility schedules passing respectively through A and B. This is certainly one attractive interpretation of the spirit behind Haig and Marshall. The practical statistician might despair of so defining income: using market prices and quantities, he could conceivably apply any of the other definitions; but this one would be non-observable to him." Paul A. Samuelson (1961; 46).

All three of the above definitions of income have some appeal. At this stage, we will not commit to any single definition since we have not yet explored the full complexities of the income concept.[12] We conclude this section with another astute observation made by Samuelson:[13]

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the price ratios of B to the A situation and compare with B; alternatively, apply the price ratios of A to the B situation and compare with A. If both tests give the same answer—and in Fig. 3 they will, because B lies outside A on straight lines parallel to the tangent at either A or at B—then you can be sure that one situation has 'more income' that the other. If these Laspeyres and Paasche tests disagree, reserve judgment or split the difference depending upon your temperament." Paul A. Samuelson (1961; 45-46).

[11] This is the "best" Hicksian definition in my opinion but it has some ambiguity associated with it: how exactly do we interpret the word "real"?

[12] Samuelson's model did not have the added complexities of the Edwards and Bell (1961; 71-72) and Hicks (1961; 23) Austrian production model that distinguished the beginning of the period and end of the period capital stocks as separate inputs and outputs. Also Samuelson had only a single consumption good and a single capital input in his model and we need to also consider the problems involved in aggregating over consumption and capital stock components.

[13] Samuelson's quotation succinctly states the index number problem! For additional material on output indexes, see Balk (1998).

"Our dilemma is now well depicted. The simplest economic model involves two current variables, consumption and investment. A measure of national income is one variable. How can we fully summarize a doublet of numbers by a single number?" Paul A. Samuelson (1961; 47).

4. The Theory of the Output Index

In this section, we will have another look at Samuelson's index number method for measuring income growth; i.e., his second income or net product concept studied in the previous section. We consider a more general model where there are M consumption goods and net investment goods and N primary inputs. We also consider more general indexes than the Laspeyres and Paasche output quantity indexes considered by Samuelson.

We assume that the market sector of the economy produces quantities of M (net) outputs, $y \equiv [y_1,...,y_M]$, which are sold at the positive producer prices $P \equiv [P_1,...,P_M]$. We further assume that the market sector of the economy uses positive quantities of N primary inputs, $x \equiv [x_1,...,x_N]$ which are purchased at the positive primary input prices $W \equiv [W_1,...,W_N]$. In period t, we assume that there is a feasible set of output vectors y that can be produced by the market sector if the vector of primary inputs x is utilized by the market sector of the economy; denote this period t production possibilities set by S^t. We assume that S^t is a closed convex cone that exhibits a free disposal property.[14]

Given a vector of output prices P and a vector of available primary inputs x, we define *the period t market sector net product function*, $g^t(P,x)$, as follows:[15]

$$(3)\ g^t(P,x) \equiv \max_y \{P\cdot y : (y,x) \text{ belongs to } S^t\} ; \qquad t = 0,1,2, ...$$

Thus market sector NDP depends on t (which represents the period t technology set S^t), on the vector of output prices P that the market sector faces and on x, the vector of primary inputs that is available to the market sector.

If P^t is the period t output price vector and x^t is the vector of inputs used by the market sector during period t and if the NDP function is differentiable with respect to the

[14] For more explanation of the meaning of these properties, see Chapter 3 or Diewert (1973) (1974; 134) or Woodland (1982) or Kohli (1978) (1991). The assumption that S^t is a cone means that the technology is subject to constant returns to scale. This is an important assumption since it implies that the value of outputs should equal the value of inputs in equilibrium. In empirical work, this property can be imposed upon the data by using an ex post rate of return in the user costs of capital, which forces the value of inputs to equal the value of outputs for each period. The function g^t is known as the *NDP function* or the *net national product function* in the international trade literature (see Kohli (1978)(1991), Woodland (1982) and Feenstra (2004; 76). It was introduced into the economics literature by Samuelson (1953). Alternative terms for this function include: (i) the *gross profit function*; see Gorman (1968); (ii) the *restricted profit function*; see Lau (1976) and McFadden (1978); and (iii) the *variable profit function*; see Diewert (1973) (1974).

[15] The function $g^t(P,x)$ will be linearly homogeneous and convex in the components of P and linearly homogeneous and concave in the components of x; see Chapter 3 or Diewert (1973) (1974; 136). Notation: $P\cdot y \equiv \sum_{m=1}^M P_m y_m$.

components of P at the point P^t,x^t, then the period t vector of market sector outputs y^t will be equal to the vector of first order partial derivatives of $g^t(P^t,x^t)$ with respect to the components of P; i.e., we will have the following equations for each period t:[16]

$$(4)\ y^t = \nabla_P\, g^t(P^t,x^t)\ ; \qquad t = 1,2.$$

Thus the period t market sector (net) supply vector y^t can be obtained by differentiating the period t market sector NDP function with respect to the components of the period t output price vector P^t.

If the NDP function is differentiable with respect to the components of x at the point P^t,x^t, then the period t vector of input prices W^t will be equal to the vector of first order partial derivatives of $g^t(P^t,x^t)$ with respect to the components of x; i.e., we will have the following equations for each period t:[17]

$$(5)\ W^t = \nabla_x\, g^t(P^t,x^t)\ ; \qquad t = 1,2.$$

Thus the period t market sector input prices W^t paid to primary inputs can be obtained by differentiating the period t market sector NDP function with respect to the components of the period t input quantity vector x^t.

The constant returns to scale assumption on the technology sets S^t implies that the value of outputs will equal the value of inputs in period t; i.e., we have the following relationships:

$$(6)\ g^t(P^t,x^t) = P^t \cdot y^t = W^t \cdot x^t\ ; \qquad t = 1,2.$$

With the above preliminaries out of the way, we can now consider a definition of a family of output indexes which will capture the idea behind Samuelson's second definition of income or net output in the previous section. Diewert (1983; 1063) defined a *family of output indexes* between periods 1 and 2 for each reference output price vector P as follows:[18]

$$(7)\ Q(P,x^1,x^2) \equiv g^2(P,x^2)/g^1(P,x^1).$$

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[16] These relationships are due to Hotelling (1932; 594). Note that $\nabla_P\, g^t(P^t,x^t) \equiv [\partial g^t(P^t,x^t)/\partial P_1, \ldots,\partial g^t(P^t,x^t)/\partial P_M]$.

[17] These relationships are due to Samuelson (1953) and Diewert (1974; 140). Note that $\nabla_x\, g^t(P^t,x^t) \equiv [\partial g^t(P^t,x^t)/\partial x_1, \ldots,\partial g^t(P^t,x^t)/\partial x_N]$.

[18] Diewert generalized the definitions used by Samuelson and Swamy (1974) and Sato (1976; 438). Samuelson and Swamy assumed only one input and no technical change while Sato had many inputs and outputs in his model but no technical change. These authors recognized the analogy of the output quantity index with Allen's (1949) definition of a quantity index in the consumer context.

Note that the above definition combines the effects of technical progress and of input growth. A *family of technical progress indexes* between periods 1 and 2 can be defined as follows for each reference input vector x and each reference output price vector P:[19]

(8) $\tau(P,x) \equiv g^2(P,x)/g^1(P,x)$.

Thus in definition (8), the market sector of the economy is asked to produce the maximum output possible given the same reference vector of primary inputs x and given that producers face the same reference net output price vector P but in the numerator of (8), producers have access to the technology of period 2 whereas in the denominator of (8), they only have access to the technology of period 1. Hence, if $\tau(P,x)$ is greater than 1, there has been *technical progress* going from period 1 to 2.

A *family of input growth indexes* $\gamma(P,t,x^1,x^2)$ between periods 1 and 2 can be defined for each reference net output price vector P and each technology indexed by the time period t as follows:[20]

(9) $\gamma(P,t,x^1,x^2) \equiv g^t(P,x^2)/g^t(P,x^1)$.

Thus using the period t technology and the reference net output price vector P, we say that there has been positive input growth going from the period 1 input quantity vector x^1 to the observed period 2 input quantity vector x^2 if $g^t(P,x^2) > g^t(P,x^1)$ or equivalently, if $\gamma(P,t,x^1,x^2) > 1$.

Problems

1. Show that the output quantity index defined by (7) has the following decompositions:

(a) $Q(P,x^1,x^2) = \tau(P,x^2)\ \gamma(P,1,x^1,x^2)$;
(b) $Q(P,x^1,x^2) = \tau(P,x^1)\ \gamma(P,2,x^1,x^2)$.

Thus the output quantity index between periods 1 and 2 does combine the effects of technical progress and input growth between periods 1 and 2.

2. We now specialize definition (7) to the case where the reference net output price vector is chosen to be the period 1 price vector P^1, which leads to the following *Laspeyres type theoretical output quantity index*:

(a) $Q(P^1,x^1,x^2) \equiv g^2(P^1,x^2)/g^1(P^1,x^1)$.

If we choose P to be the period 2 price vector P^2, we obtain the following *Paasche type theoretical output quantity index*:

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19 Definition (8) may be found in Diewert (1983; 1063), Diewert and Morrison (1986; 662) and Kohli (1990).

20 Definition (9) can also be found in Diewert (1983; 1063).

(b) $Q(P^2,x^1,x^2) \equiv g^2(P^2,x^2)/g^1(P^2,x^1)$.

Under assumptions (6) above, show that the theoretical output quantity indexes defined by (a) and (b) above satisfy the following inequalities:

(c) $Q(P^1,x^1,x^2) \geq P^1 \cdot y^2/P^1 \cdot y^1 \equiv Q_L(P^1,P^2,y^1,y^2)$;
(d) $Q(P^2,x^1,x^2) \leq P^2 \cdot y^2/P^2 \cdot y^1 \equiv Q_P(P^1,P^2,y^1,y^2)$

where $Q_L(P^1,P^2,y^1,y^2)$ and $Q_P(P^1,P^2,y^1,y^2)$ are the observable Laspeyres and Paasche net output quantity indexes.

3. Under what conditions will the inequalities (c) and (d) in problem 2 above hold as equalities?

4. Is the constant returns to scale assumption required to derive the results in problems 1 and 2 above?

5. Illustrate the two inequalities in problem 2 above using Figure 1; i.e., specialize M to the case M = 2, and then modify Figure 1 to illustrate the two inequalities in problem 2.

5. Maintaining Capital Again: the Physical versus Real Financial Perspectives

Recalling the material in section 2 of chapter 7 (on aggregation problems within the period; i.e., the beginning, end and middle of the period decomposition of the period) and Appendix 2 of chapter 7 (on the Austrian production function concept), we see that Samuelson's C + I framework for discussing alternative income concepts is not quite adequate to illustrate all of the problems involved in defining income concepts.

Recall that when using Samuelson's second income concept, nominal income in period 1 was defined as $P_C^1C^1 + P_I^1I^1$ where I^1 was defined to be net investment in period 1. Net investment can be redefined in terms of the difference between the beginning and end of period 1 capital stocks, K^0 and K^1, so that I^1 equals $K^1 - K^0$. If we substitute this definition of net into Samuelson's definition of period 1 nominal income, we obtain the following definition for *period 1 nominal income*:

(10) $\text{Income 1} \equiv P_C^1C^1 + P_I^1I^1 = P_C^1C^1 + P_I^1(K^1 - K^0) = P_C^1C^1 + P_I^1K^1 - P_I^1K^0$.

Note that in the above definition, the beginning and end of period capital stocks are valued at the same price, P_I^1. But this same price concept does not quite fit in with our Austrian one period production function framework where the beginning of the period capital stock should be valued at the beginning of the period opportunity cost of capital, P_K^0 say, and the end of the period capital stock should be valued at the end of the period expected opportunity cost of capital, P_K^1.[21] Replacing P_I^1 in (10) by P_K^1 (for K^1) and by P_K^0 (for K^0) leads to the following estimate for period 1 nominal income:

(11) Income 2 $\equiv P_C^1C^1 + P_K^1K^1 - P_K^0K^0$.

But Income 2 is expressed in heterogeneous units: P_C^1 reflects the average level of prices of the consumption good in period 1 whereas P_K^1 reflects the price of capital at the *end* of period 1 while P_K^0 reflects the price of capital at the *beginning* of period 1. The problem is that there could be a considerable amount of price change going from the beginning to the end of period 1. Hence we need to adjust the beginning of the period price of capital, P_K^0, into a comparable end of period price that eliminates the effects of inflation over the duration of period 1. There are two possible price indexes that we could use: a (capital) *specific price index* $1+i^1$ or a *general price index* $1+\rho^1$ that is based on the movement of consumer prices from the beginning of period 1 to the end of period1; i.e., define i^1 and ρ^1 as follows:

(12) $1+i^0 \equiv P_K^1/P_K^0$;
(13) $1+\rho^0 \equiv P_{CE}^1/P_{CE}^0$

where P_{CE}^1 is the level of consumer prices at the *end* of period 1 and P_{CE}^0 is the level of consumer prices at the *beginning* of period 1 or the *end* of period 0.

Now insert either $1+i^0$ or $1+\rho^0$ in front of the term $P_K^0K^0$ in (11) and we obtain the following *two income concepts that measure income from the perspective of the level of prices prevailing at the end of period 1*:

(14) Income 3 $\equiv P_C^1C^1 + P_K^1K^1 - (1+\rho^0)P_K^0K^0$;
(15) Income 4 $\equiv P_C^1C^1 + P_K^1K^1 - (1+i^0)P_K^0K^0$
$= P_C^1C^1 + P_K^1K^1 - P_K^1K^0$ using (12)
= Income 1 using (10) if $P_K^1 = P_I^1$.

Thus if the end of period 1 price of capital P_K^1 is equal to the period 1 investment price P_I^1, then Income 4 coincides with Income 1; i.e., if $P_K^1 = P_I^1$, then Income 4 ends up equaling Samuelson's Income 1. However, in general, P_K^1 (then end of period 1 price of capital) will not be equal to P_I^1(the average price of capital during period 1) but in a low inflation environment, the differences between Income 1 and 4 will usually be small.

The first line in (15) shows that Income 4 can be interpreted as a type of *specific price level adjusted income* and (14) shows that Income 3 is a type of *general price level adjusted accounting income*.[22] The idea behind the Income 3 measure defined by (14) is

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[21] For now, we will assume that expectations are realized in order to save on notational complexity. We will return to the problem of modeling expectations later in the chapter.

[22] These types of balance sheet adjustments for inflation over an accounting period are discussed in the inflation accounting literature; e.g., see Middleditch (1918), Sweeney (1934) (1935) (1964), Edwards and Bell (1961), Baxter (1975), Sterling (1975), Whittington (1980), Carsberg (1982) and Tweedie and Whittington (1984).

this: at the end of the period, the investors who provided financial capital to the firm have access to the end of period 1 value of the firm's capital stock, $P_K^1K^1$, which they could turn into consumption equivalents if they wanted to do this. However, at the beginning of period 1, they provided financial capital to the firm in the amount $P_K^0K^0$. This amount of money could be turned into consumption at the beginning of period 1 and this amount of consumption represents the opportunity cost of their investment at the beginning of the period. To measure the benefit of the investment ($P_K^1K^1$) against the cost of the investment ($P_K^0K^0$) in comparable amounts of consumption gained versus sacrificed, we need to discount $P_K^1K^1$ by the Consumer Price Index inflation rate over period 1, $1+\rho^0$, or alternatively, escalate $P_K^0K^0$ by $1+\rho^0$. We follow accounting conventions and escalate the beginning of the period sacrifice value to make it comparable to the end of period benefit value and so the net consumption benefit of the investment is $P_K^1K^1 - (1+\rho^0)P_K^0K^0$. If this amount is zero or positive, then *the investor's real financial capital has been kept intact* by the firm's actions over period 1.[23]

Turning now to an interpretation of Income 4 defined by (15), we again start with the investor's end of period benefit of their investment in the firm, $P_K^1K^1$, which again could be turned into consumption equivalents at the end of period 1. However, instead of converting the beginning of the period investment in the firm, $P_K^0K^0$, into consumption forgone, we simply convert the beginning of the period price of the capital stock, P_K^0, into the corresponding end of the period price of the capital stock, $(1+i^0)P_K^0$, which is equal to P_K^1. Thus instead of attempting to maintain the investor's real financial capital intact, *we now attempt to maintain the firm's physical stock of capital in use intact* (at end of period prices). This type of accounting adjustment is called the *specific price level method* for constructing current values for an asset held by a business unit. The method was suggested by Daines (1929; 101), Sweeney (1934; 110) and many other accountants.[24]

Since Income 1 does not fit into the Hicks and Edwards and Bell one period production function framework where beginning of the period capital is regarded as an input and end of the period capital is regarded as an output, we will not consider Income 1 any further. Moreover, we also have rejected Income 2 since it does not adjust for general inflation over the course of period 1. Hence we are left with Incomes 3 and 4 and the question is: how do we choose between Income 3 and Income 4? We will address this question in the following section.

Problem

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[23] Keeping financial capital intact does not include interest payments that typically must be made to investors in order for them to postpone consumption. We will see how interest gets into the picture later.

[24] "Inasmuch as the price level is not stable for any great length of time, and since this calculation is contemplated for each fiscal period, the only feasible procedure for a company with thousands of assets is the use of price index numbers." Albert L. Bell (1953; 49). "Where no market exists for new fixed assets of the type used by the firm, two means of measuring current costs are available: (1) appraisal, and (2) the use of price index numbers for like fixed assets to adjust the original cost base to the level which would now have to be paid to purchase the asset in question." Edgar O. Edwards and Philip W. Bell (1961; 186).

6. Refer back to Samuelson's Method 2 in section 3. Use Income 3 in place of Samuelson's income measure and construct the Laspeyres and Paasche measures of income growth going from period 1 to 2. Are there any potential problems due to the fact that not all components of Income 3 have positive signs?

6. Measuring Business Income: the End of the Period Perspective

In order to see if one of the income concepts explained in the previous sections of this chapter can emerge as being the "right" concept, we will return to the one period profit maximization problem of the market sector of the economy using the Austrian one period production function framework explained in Appendix 2 of chapter 7.[25]

Using the notation introduced in the previous section and adding labour L as an input (with price W) and letting the market sector of the economy face the beginning of period 1 nominal interest rate r^0, the period 1 Austrian profit maximization problem can be defined as follows:

$$(16)\ \max_{C^1,L^1,K^1} \{(1+r^0)^{-1}(P_C{}^1C^1 - W^1L^1 + P_K{}^1K^1) - P_K{}^0K^0 : (C^1,L^1,K^0,K^1)\in S^1\}$$

where S^1 is the period 1 Austrian production possibilities set. Note that we have treated the price $P_C{}^1$ of period 1 consumption and the price of period 1 labour W^1 as end of period 1 prices and hence the corresponding value flows are discounted to their beginning of period 1 equivalents using the beginning of period 1 nominal interest rate r^0. From a practical measurement perspective, it is more useful to work with end of the period equivalents and so if we multiply the objective function in (16) through by $(1+r^0)$, we obtain the following *period 1 (end of period perspective) profit maximization problem*:

$$(17)\ \max_{C^1,L^1,K^1} \{P_C{}^1C^1 - W^1L^1 + P_K{}^1K^1 - (1+r^0)P_K{}^0K^0 : (C^1,L^1,K^0,K^1)\in S^1\}.$$

Recall equation (12) above, $1+i^0 \equiv P_K{}^1/P_K{}^0$, which defined the period 1 asset specific inflation rate i^0, and equation (13) above, $1+\rho^0 \equiv P_{CE}{}^1/P_{CE}{}^0$, which defined the period 1 general inflation rate. The period 1 general inflation rate, ρ^0. can be used to define the beginning of period 1 *real interest rate* r^{0*} and the period 1 *real rate of asset price inflation* i^{0*} as follows:

$$(18)\ 1+r^{0*} \equiv (1+r^0)/(1+\rho^0).$$
$$(19)\ 1+i^{0*} \equiv (1+i^0)/(1+\rho^0).$$

Now substitute (18) into the objective function in (17) and we obtain the following expression for period 1 *pure profits*:

$$(20)\ P_C{}^1C^1 - W^1L^1 + P_K{}^1K^1 - (1+r^0)P_K{}^0K^0$$

[25] Recall that this framework is based on Hicks (1961; 23) and Edwards and Bell (1961; 71-72). Their work is related to the earlier work of Böhm-Bawerk (1891), von Neumann (1937), Hicks (1946; 230) and Malinvaud (1953) and the later work of Diewert (1977) (1980; 472-474).

$$= P_C^1C^1 - W^1L^1 + P_K^1K^1 - (1+r^{0*})(1+\rho^0)P_K^0K^0$$
$$= P_C^1C^1 + P_K^1K^1 - (1+\rho^0)P_K^0K^0 - \{W^1L^1 + r^{0*}(1+\rho^0)P_K^0K^0\}$$
$$= \text{Income 3} - \{W^1L^1 + U^1K^0\}$$

where Income 3 was defined by (14) in the previous section and the *period 1 waiting services user cost of the initial capital stock*[26] is defined as

(21) $U^1 \equiv r^{0*}(1+\rho^0)P_K^0$.

With a constant returns to scale technology, competitive pricing on the part of market sector producers and correct expectations, pure profits will be zero and hence (20) set equal to zero will give us the following equations:[27]

(22) $\text{Income 3} = P_C^1C^1 + P_K^1K^1 - (1+\rho^0)P_K^0K^0$
$= W^1L^1 + U^1K^0$

where W^1L^1 represents period 1 payments to labour and U^1K^0 represents interest payments to holders of the initial capital stock in terms of end of period 1 dollars. Note that all prices in (22) are expressed in end of period 1 equivalents.

What is the significance of equation (22)? This equation seems to suggest that Income 3 is the "right" concept of net output for period 1![28] However, we shall see later see that Income 4 is also consistent with the Austrian production function framework.

At this point, the reader may be slightly confused and may well ask: what happened to our usual user cost formula? The user cost U^1 defined by (21) does not look very familiar and so there might be a suspicion that something might be wrong with the above algebra. In order to address this issue, we will specialize the Austrian model to the usual production function model, defined as follows:

(23) $C^1 = F(I_G^1,L^1,K^0)$; $K^1 = (1-\delta)K^0 + I_G^1$

[26] Rymes (1968) (1983) stressed waiting services as a primary input.

[27] To form a net investment aggregate in this framework, we aggregate over the value difference $P_K^1K^1 - (1+\rho^0)P_K^0K^0$ using normal index number theory provided that this value aggregate is bounded away from 0 over all periods; i.e., we use normal index number theory, with K^1 a positive quantity with the corresponding price P_K^1 and $-K^0$ as a negative quantity with price $(1+\rho^0)P_K^0$. If the value aggregate approaches or passes through 0 during any period, then we cannot form a net investment aggregate for this capital component; i.e., we would have to combine the value aggregate $P_K^1K^1 - (1+\rho^0)P_K^0K^0$ with an additional substantially positive value aggregate.

[28] Note that our Income 3 follows the adjustments to cash flows recommended by the accountant Sterling: "It follows that the appropriate procedure is to (1) adjust the present statement to current values and (2) adjust the previous statement by a price index. It is important to recognize that *both* adjustments are necessary and that neither is a substitute for the other. Confusion on this point is widespread." Robert R. Sterling (1975; 51). Sterling (1975; 50) termed his income concept *Price Level Adjusted Current Value Income*. Unfortunately, Sterling's income concept has not been widely endorsed in accounting circles (but it should be).

where I_G^1 is gross investment in period 1, C^1 is period 1 consumption output, L^1 is period 1 labour input, K^0 is the start of period 1 capital stock, K^1 is the end of period 1 finishing capital stock, $0 < \delta < 1$ is the constant (geometric) physical depreciation rate and F is the production function, which is decreasing in I_G and increasing in L and K.

If we substitute (23) into the objective function in (17) and solve the resulting period 1 profit maximization problem, we find that the optimal objective function can be written as follows:

$$(24)\ P_C^1C^1 - W^1L^1 + P_K^1K^1 - (1+r^0)P_K^0K^0$$
$$= P_C^1C^1 - W^1L^1 + P_K^1[(1-\delta)K^0 + I_G^1] - (1+r^{0*})(1+\rho^0)P_K^0K^0 \qquad \text{using (18)}$$
$$= P_C^1C^1 + P_K^1I_G^1 - W^1L^1 - (1+r^{0*})(1+\rho^0)P_K^0K^0 + (1-\delta)P_K^1K^0$$
$$= P_C^1C^1 + P_K^1I_G^1 - W^1L^1 - (1+r^{0*})(1+\rho^0)P_K^0K^0 + (1-\delta)(1+i^0)P_K^0K^0 \qquad \text{using (12)}$$
$$= P_C^1C^1 + P_K^1I_G^1 - W^1L^1 - (1+r^{0*})(1+\rho^0)P_K^0K^0 + (1-\delta)(1+\rho^0)(1+i^{0*})P_K^0K^0 \qquad \text{using (19)}$$
$$(25)\quad = P_C^1C^1 + P_K^1I_G^1 - W^1L^1 - [(1+r^{0*})(1+\rho^0) - (1-\delta)(1+\rho^0)(1+i^{0*})]P_K^0K^0.$$

The term in square brackets in (25) times P_K^0 represents the usual (end of period) gross rental user cost of capital u^1; i.e., we have[29]

$$(26)\ u^1 \equiv [(1+r^{0*})(1+\rho^0) - (1-\delta)(1+\rho^0)(1+i^{0*})]P_K^0 = [(1+r^0) - (1-\delta)(1+i^0)]P_K^0.$$

Thus if pure profits are zero for the period 1 data, expression (25) set equal to 0 translates into the following usual gross output equals labour payments plus gross payments to the starting stock of capital:

$$(27)\ P_C^1C^1 + P_K^1I_G^1 = W^1L^1 + u^1K^0.$$

We now show that u^1 can be expressed as the sum of three terms where the terms are defined as follows:

$$(28)\ U^1 \equiv r^{0*}(1+\rho^0)P_K^0\ ;$$
$$(29)\ D^1 \equiv \delta(1+i^{0*})(1+\rho^0)P_K^0 = \delta P_K^1\ ;$$
$$(30)\ R^1 \equiv -i^{0*}(1+\rho^0)P_K^0\ .$$

Problem

7. Show that the usual gross rentals user cost formula u^1 defined above by (26) can be written as the sum of the three terms defined by (28)-(30); i.e., show that

$$(31)\ u^1 = U^1 + D^1 + R^1.$$

We now need to provide economic interpretations for the three terms on the right hand side of (31). It can be seen that U^1 defined by (28) is the same definition as the U^1

[29] See section 3 of chapter 7.

defined by (21) and we interpreted this U^1 as a *real waiting services user cost* for the initial beginning of the period capital stock K^0. Obviously, $\delta P_K^0 K^0$ can be interpreted as the amount of wear and tear depreciation that the initial capital stock will undergo during the period. However, this amount of depreciation is expressed in the beginning of the period price of the capital stock, P_K^0. In keeping with our conventions, we convert this beginning of the period price into its consumption equivalent price at the end of the period by multiplying by $1+\rho^0$. Thus D^1K^0 is equal to $\delta(1+i^{0*})(1+\rho^0)P_K^0K^0 = \delta P_K^1 K^0$ and can be interpreted as the *real value of wear and tear depreciation*, expressed in end of period consumption equivalents. Finally, R^1 is a *real revaluation term*; if the real asset inflation rate i^{0*} is negative, then R^1K^0 can be interpreted as an obsolescence charge; i.e., the rate of nominal asset price inflation i^0 is less than the general nominal inflation rate ρ^0 and so an extra charge for the use of the asset in period 0 must be made in addition to normal wear and tear depreciation.[30] On the other hand, if the real asset inflation rate i^{0*} is positive, then $R^1K^0 = -i^{0*}(1+\rho^0)P_K^0K^0 < 0$ can be interpreted as an offset to the wear and tear depreciation charge D^1K^0. This offset is due to the fact that the firm "transports" capital from a time period where it is less valuable in real terms (the beginning of period 0) to a time period where the capital is more highly valued (the end of period 0).

The (real) decomposition of the user cost of capital u^1 defined by (31) and the three definitions (28)-(30) seems a bit awkward compared to the following more straightforward (nominal) decomposition of the user cost:

$$\begin{aligned}(32)\ u^1 &\equiv [(1+r^0) - (1-\delta)(1+i^0)]P_K^0 \\ &= [r^0 - i^0 + \delta(1+i^0)]P_K^0.\end{aligned}$$

The nominal waiting services part of u^1 is obviously $r^0P_K^0$, the nominal revaluation term is $-i^0P_K^0$ and the nominal wear and tear depreciation term is $\delta(1+i^0)P_K^0$. The problem with the user cost decomposition given by (32) is that *it does not readily integrate with the two main income concepts that we defined earlier*.

We now show how the decomposition of the gross user cost u^1 defined by (28)-(31) is related to Income 4 and Income 3 defined earlier. We do this first for Income 3. Substitute (31) into (27) and we obtain the following equation:

$$(33)\ P_C^1C^1 + P_K^1I_G^1 = W^1L^1 + [U^1 + D^1 + R^1]K^0.$$

Now subtract $[D^1 + R^1]K^0$ from both sides of (33) and we obtain the following equation:

$$\begin{aligned}(34)\ P_C^1C^1 + P_K^1I_G^1 - D^1K^0 - R^1K^0 &= W^1L^1 + U^1K^0 \\ &= \text{Income 3} \qquad \text{using (22).}\end{aligned}$$

[30] The sum of the wear and tear depreciation term and the revaluation term is called real time series depreciation by Diewert (2005) (2006) and it formalizes a definition due to Hill (2000).

Equations (34) are simply our earlier equations (22) when we substitute (23) and other equations into (22). Net investment in this model can be viewed as an aggregate of gross investment less real wear and tear depreciation less real revaluations.[31]

As noted earlier, the net output that is generated by the left hand side of (34) can be interpreted as an income concept that maintains real financial capital. Thus at this point, we might tentatively conclude that working with the usual discounted profits model leads to *a preference for a maintenance of real financial capital income concept over a maintenance of a specific inflation adjusted income concept*, at least at the theoretical level. However, this tentative conclusion is not correct:[32] it turns out that we can manipulate the Austrian discounted profits model in a way that will justify the maintenance of real physical capital as opposed to real financial capital. We show this in the following paragraphs.

We establish our result for the general Austrian capital model; i.e., the model that was defined before we specialized the model in equations (23). We first use the definition of R^1 to establish the following identity:

$$(35)\ (1+\rho^0)P_K{}^0K^0 - (1+i^0)P_K{}^0K^0 = (1+\rho^0)P_K{}^0K^0 - (1+i^{0*})(1+\rho^0)P_K{}^0K^0 \quad \text{using (19)}$$
$$= -\, i^{0*}(1+\rho^0)P_K{}^0K^0$$
$$= R^1K^0 \quad \text{using (30).}$$

Using definitions (14) and (15) for Incomes 3 and 1 respectively, we see that if we add the left hand side of (35) to Income 3, we will obtain Income 4. Hence adding the right hand side of (35) to Income 3 will give us Income 4. Thus we have:

$$(36)\ \text{Income } 4 \equiv P_C{}^1C^1 + P_K{}^1K^1 - (1+i^0)P_K{}^0K^0$$
$$= \text{Income } 3 + R^1K^0$$
$$= W^1L^1 + U^1K^0 + R^1K^0 \quad \text{using (22).}$$

Thus if we adopt a physical maintenance of capital point of view to measure income, the matching user cost for the beginning of the period stock of capital K^0 *is* $U^1 + R^1$, *the sum of the real waiting services and revaluation terms.*

If we now specialize the general Austrian model to the geometric depreciation model and substitute (12) and (23) into the first equation in (36), we obtain the following expression for Income 4:

$$(37)\ \text{Income } 4 \equiv P_C{}^1C^1 + P_K{}^1K^1 - (1+i^0)P_K{}^0K^0$$
$$= P_C{}^1C^1 + P_K{}^1[I_G{}^1 + (1-\delta)K^0] - P_K{}^1K^0 \quad \text{using (12) and (23)}$$
$$= P_C{}^1C^1 + P_K{}^1[I_G{}^1 - \delta K^0]$$

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[31] Normal index number theory can be used to aggregate the three terms, provided that gross investment is always larger than the sum of depreciation and revaluation; i.e., treat all three prices as positive, the first quantity as positive and the next two quantities as negative numbers in the index number formula.
[32] I owe this point to Paul Schreyer.

where $I_G^1 - \delta K^0$ is gross investment less wear and tear depreciation so that this difference can clearly be interpreted as net investment, with corresponding price equal to the end of period price of a new investment good, P_K^1.

Thus the Austrian production framework is consistent with both income concepts: the financial maintenance of capital concept (Income 3) and the physical maintenance of capital concept (Income 1 or 4). In fact, (27) shows that the Austrian production framework is also consistent with an "income" concept that is equal to gross product. Table 1 below summarizes the definitions of the different income concepts in the case of geometric depreciation and gives the user cost concept that matches up with the corresponding income concept.

Table 1: Alternative Income Concepts and the Corresponding User Costs of Capital

Income Concept	Corresponding Net Output Definition	Corresponding User Cost Value
Gross Output	$P_C^1C^1 + P_K^1I_G^1$	$U^1K^0 + D^1K^0 + R^1K^0 = u^1K^0$
Income 4	$P_C^1C^1 + P_K^1I_G^1 - D^1K^0$	$U^1K^0 + R^1K^0$
Income 3	$P_C^1C^1 + P_K^1I_G^1 - D^1K^0 - R^1K^0$	U^1K^0

Looking at Table 1, it can be seen that the usual gross output (or GDP) definition of "income" matches up with the usual gross rentals user cost of capital, u^1. The Income 4 definition, which corresponds to a physical maintenance of capital income concept, takes the physical depreciation term D^1K^0 out of the gross rentals user cost and treats it as a negative contribution to output. Finally, the Income 3 definition, which corresponds to a maintenance of real financial capital concept, takes both the physical depreciation term D^1K^0 and the revaluation or obsolescence term R^1K^0 out of the gross rentals user cost and treats them both as a negative contributions to output. Thus the Austrian model of production is consistent with all three income concepts.

Typically, the price of capital goods will decline relative to the price of consumption goods and services (or will increase at a lower rate) so that the real asset inflation rate, i^{0*} defined by (19), will usually be negative. Under this hypothesis, R^1 will be positive[33] and we will have the following inequalities between the three income concepts:

(38) Gross Income > Income 4 > Income 3.

We conclude this section with a brief discussion on which income concept is "best" from the perspective of describing household consumption possibilities over time. The gross income concept clearly overstates long run consumption for the consumer and so this concept can be dismissed. It is clear that Income 2 is a very defective measure of sustainable consumption prospects, since this measure can be made very large if there is a substantial amount of inflation between the beginning and end of the period. All of the other income measures are invariant to general inflation between the beginning and end

[33] Under these conditions, we can say that the capital good is experiencing a form of obsolescence.

of the accounting period.[34] However, choosing between the physical and real financial maintenance perspectives is more problematical: reasonable economists could differ on this choice. The merits of the two perspectives were discussed by Pigou and Hayek over 60 years ago. Pigou (1941; 273-274) favored the maintenance of physical capital approach (Incomes 1 or 4) while Hayek (1941; 276-277) favored the maintenance of real financial capital approach (Income 3). Our preference is for Income 3, following in the footsteps of Hayek and Hill (2000; 6), who felt that Income 1 or 4 would generally overstate the real value of consumption in any period due to its neglect of (foreseen) obsolescence (due to expected real price decreases in the asset). Conversely, if real price increases in the asset are foreseen, then the revaluation term can be regarded as a positive contribution to the net revenues produced by the production unit under consideration; i.e., the unit "transports" the asset from a time when it is less valued (in real terms) to a time when it is more highly valued. However, it is certainly possible to argue in favour of the physical maintenance of capital concept of income.

To summarize: there are two ways that can be used to justify the Haig Marshall Pigou Samuelson Income 1 or 4 versus the Hayek Sterling Hill Income 3 (or Price level Adjusted Current Value Income to use Sterling's (1975; 50) terminology):

- One can look at income from the output side perspective and think about the relative merits of preserving physical capital versus real financial capital or
- One can look at income from the primary input side perspective and ask whether the (anticipated) revaluation term is a source of primary income or not.

As Hicks (1939; 184) said in his Income chapter: "What a tricky business this all is!"

7. Approximations to the Income Concept

We now relax the perfect foresight assumptions that were made in the previous section. This means that i^0 and ρ^0 defined by (12) and (13) are not known variables; rather ρ^0 is the anticipated (Consumer Price Index) nominal inflation rate for consumption goods and services and i^0 is the anticipated specific asset inflation rate, where the anticipations are formed at the start of period 1. This means that the period 1 real interest rate r^{0*} and real asset inflation rate i^{0*} defined by (18) and (19) are also anticipated variables. Thus we must now address the question as to how exactly these anticipated variables will be estimated in empirical applications of the income concept.[35]

We will consider three alternative methods for approximating these anticipated variables but the reader will be able to construct many additional approximations, depending on the purpose at hand.

Approximation Method 1:

[34] We regard this invariance property as a fundamental property that any sensible income measure should satisfy.
[35] The approximations that we suggest below can be used to implement either Gross Income, Income 1 or Income 3, depending on the user's choice of income concept.

The two assumptions that are made in order to implement this first method are the following ones:

- Approximate general inflation adjusted beginning of the period price of capital, $(1+\rho^0)P_K{}^0$ by the period 1 average price for the corresponding investment good $P_I{}^1$ and
- Set the anticipated specific asset real inflation rate i^{0*} equal to zero.

Thus we make the following assumptions:

(39) $(1+\rho^0)P_K{}^0 = P_I{}^1$;
(40) $i^{0*} = 0$.

With the above two assumptions, we find that the waiting user cost of capital U^1 defined by (21) or (28), the gross rentals user cost of capital u^1 defined by (26) and the wear and tear depreciation term D^1 defined by (29) simplify as follows:

(41) $U^1 \equiv r^{0*}(1+\rho^0)P_K{}^0 = r^{0*}P_I{}^1$;
(42) $u^1 \equiv [(1+r^{0*})(1+\rho^0) - (1-\delta)(1+\rho^0)(1+i^{0*})]P_K{}^0 = [r^{0*} + \delta]P_I{}^1$;
(43) $D^1 \equiv \delta(1+i^{0*})(1+\rho^0)P_K{}^0 = \delta P_K{}^1 = \delta P_I{}^1$.

The only remaining approximation issue is the approximation for the anticipated period 1 real interest rate r^{0*}. There are three obvious possible choices for r^{0*}:

- Calculate the balancing real rate of return that will make the profits of the production unit under consideration equal to zero;[36]
- Smooth past balancing real rates of return and use the smoothed rate as the predicted rate; or
- Simply pick a plausible constant real rate of return, such as 3 or 4 percent (for annual data).

This method of approximation should be appealing to national income accountants.

Approximation Method 2:

One problem with the previous method is that it neglects *foreseen obsolescence* that is due to anticipated (real) asset price decline.[37] For example, for the past 40 years, the real price of computers in constant quality units has steadily declined and these declines are very likely to continue. Hence, the i^{0*} term in our definition of the real depreciation term, D^1 defined by (29) and in the real revaluation term R^1 defined by (30), should be a negative number if the asset under consideration is computers (or any related asset that

[36] Substitute u^1 defined by (44) into equation (28) and solve the resulting equation for the balancing real rate r^{0*}.
[37] Not all foreseen obsolescence is due to expected future real price declines; i.e., some models of obsolescence imply contracting asset lives.

has a substantial computer chip component). Thus again make assumption (39) above but now for assets that are expected to decline in price, estimate i^0* by smoothed past real declines in the asset price (expressed in constant quality units). With these assumptions, we find that the real waiting user cost of capital U^1 defined by (21) or (28), the gross rentals user cost of capital u^1 defined by (26) and R^1 and D^1 simplify as follows:

$$(44)\ U^1 \equiv r^0*(1+\rho^0)P_K{}^0 = r^0*P_I{}^1 ;$$
$$(45)\ u^1 \equiv [(1+r^0*)(1+\rho^0) - (1-\delta)(1+\rho^0)(1+i^0*)]P_K{}^0 = [r^0* + \delta - (1-\delta)i^0*]P_I{}^1 ;$$
$$(46)\ D^1 \equiv \delta(1+i^0*)(1+\rho^0)P_K{}^0 = \delta(1+i^0*)P_I{}^1 ;$$
$$(47)\ R^1 \equiv -i^0*(1+\rho^0)P_K{}^0 = -i^0*P_I{}^1$$

Examining (45)-(47), it can be seen that if i^0* is negative, then the gross rental user cost u^1 and the real revaluation term R^1 term become *larger* compared to the corresponding values when i^0* is set equal to 0, while the real wear and tear depreciation term D^1 becomes *smaller*. Once an estimate for i^0* has been obtained, r^0* can be estimated using any of the three methods outlined under Approximation Method 1 above.

However some assets have a long history of real price appreciation (e.g., urban land) and so the question is: for those assets which we expect to appreciate in real terms (i.e., i^0* is expected to be positive instead of negative), should we insert these positive expected values into the user cost terms defined by (45)-(47)?

Symmetry suggests that the answer to the above question is yes.[38] However, note that if i^0* is large and positive enough, then it could lead to the gross rental price u^1 defined by (47) being *negative*. It is not plausible that expected gross rentals be negative, since under these conditions, we would expect the corresponding asset price to be immediately be bid up to eliminate this negative expected rental price. Alternatively, if we regard the gross rental user cost as an approximation to an actual market rental rate for the asset, then since usually rental rates are positive, it is not plausible to approximate this market rental rate by a negative number. Thus some caution is called for when simply inserting a smoothed value of past real rates of asset price appreciation i^0* into the formulae (45)-(47): we do not want to insert a value of i^0* that is so large that it makes u^1 negative. Hence we want i^0* to satisfy the following inequality:

$$(48)\ i^0* < (1-\delta)^{-1}[r^0* + \delta].$$

In practical applications of this method, we suggest that for most assets, the assumption that the corresponding anticipated real inflation i^0* is zero is appropriate. Only in exceptional cases where we are fairly certain that producers are anticipating real capital gains or losses should we insert a nonzero i^0* into formulae (45)-(47).

Approximation Method 3:

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[38] See the discussion on the basic forms of productive activity in section 4 of Diewert (2006a) where it was concluded that anticipated capital gains were productive.

Approximation Methods 1 and 2 explained above are suitable for applications of the income concept at the national economy or industry levels where current prices for each class of asset can be obtained.[39] However, these methods are not usually suitable for applications at the level of the individual firm or enterprise, because current objective and replicable prices for each asset used by the enterprise will generally not be available. Hence the question arises: how can we approximate the income concept at the firm level?

The simplest and most useful assumptions in this context are the following ones:

$$(49)\ (1+\rho^0) = P_C^1 / P_C^0 ;$$
$$(50)\quad i^{0*} = 0.$$

Thus we set $1+\rho^0$ equal to the ex post amount of Consumer Price Index inflation that occurred from the beginning to the end of the accounting period and assume that specific real asset price inflation is 0.

With these assumptions, we find that the user cost components defined in equations (44)-(47) simplify as follows:

$$(51)\ U^1 \equiv r^{0*}(1+\rho^0)P_K^0 = r^{0*}(1+\rho^0)P_K^0 ;$$
$$(52)\ u^1 \equiv [(1+r^{0*})(1+\rho^0) - (1-\delta)(1+\rho^0)(1+i^{0*})]P_K^0 = [r^{0*} + \delta](1+\rho^0)P_K^0 ;$$
$$(53)\ D^1 \equiv \delta(1+i^{0*})(1+\rho^0)P_K^0 = \delta(1+\rho^0)P_K^0 ;$$
$$(54)\ R^1 \equiv -i^{0*}(1+\rho^0)P_K^0 = 0.$$

The net effect of the above assumptions is that we can basically use historical cost accounting, except that historical cost depreciation allowances should be escalated each accounting period by the amount of CPI inflation that occurred over the period; i.e., our present approximations lead to *purchasing power adjusted historical cost accounting*, see section 3 in Diewert (2005b).

The accounting profession is unlikely to embrace the above very simple and straightforward accounting adjustments to historical cost depreciation[40] but it is important that the tax authorities recognize the importance of indexing depreciation allowances for general inflation. Using the notation developed in section 6 above and taking Income 3 as our desired income concept, *taxable income* should be defined as follows:

$$(55)\ \text{Taxable income} \equiv P_C^1C^1 + P_K^1I_G^1 - D^1K^0 - R^1K^0$$

[39] Typically, we will have to rely on national statistical agency index numbers for estimates of current asset prices.
[40] Recall the discussions in chapters II and III.

where D^1 and R^1 are defined by (29) and (30).[41] With this definition of taxable income, the real return to capital will be taxed and the 'unjust" taxation of inflation inflated "profits" will be prevented.

For additional material on the role of expectations in income measures, the reader is referred to Hill and Hill (2003).

8. Choosing an Income Concept: A Summary

Table 1 in section 6 presented 3 "income" or output concepts:

- Gross output;
- Income 1 or 4 or "wear and tear" adjusted net product[42] and
- Income 3 or "wear and tear" and "anticipated revaluation" adjusted net product.[43]

Table 1 also indicated that the "traditional" user cost of capital (which approximates a market rental rate for the services of a capital input for the accounting period), u^1, consists of three additive terms; i.e., we have:

$$(56)\quad u^1 = U^1 + D^1 + R^1$$

where U^1 is the reward for waiting term (interest rate term), D^1 is the cross sectional depreciation term (or wear and tear depreciation term) and R^1 is the anticipated revaluation term, which can be interpreted as an obsolescence charge if the asset is anticipated to fall in price over the accounting period. The Gross output income concept corresponds to the traditional user cost term u^1. This income measure can be used as an approximate indicator of short run production potential. However, it is not suitable for use as an indicator of sustainable consumption. In order to obtain indicators of sustainable consumption, we turn to Incomes 1 or 4 and 3.

To obtain Income 4, we simply take the wear and tear component of the traditional user cost, D^1, times the beginning of period corresponding capital stock, K^0, out of the primary input category and treat it as a negative offset to the period's gross investment. The resulting Income 4 can be interpreted to be consistent with the position of Pigou (1941), who argued against including any kind of revaluation effect in an income concept. This position can also be interpreted as a *maintenance of physical capital approach* to income measurement. In terms of the Hicks (1961) and Edwards and Bell (1961) Austrian production model, capital at the beginning and end of the period (K^0 and K^1 respectively) are both valued at the end of period stock price for a unit of capital, $P_K^{\,1}$, and the contribution of capital accumulation to period income is simply the difference between

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[41] If we wanted the business income tax to fall on pure profits or rents (rather than on the gross return to capital), we would also subtract U^1K^0 equal to $r^{0*}(1+\rho^0)P_K^{\,0}K^0$ from the right hand side of (55) where r^{0*} would be a "normal" real rate of return to capital that would be chosen by the tax authorities. The resulting system of business income taxation would lead to minimal deadweight loss.

[42] We can associate this income concept with Marshall (1890), Haig (1921), Pigou (1941) and Samuelson (1961).

[43] We can associate this income concept with Hayek (1941), Sterling (1975) and Hill (2000).

the end of period value of the capital stock and the beginning of the period value (valued at end of period prices), $P_K{}^1K^1 - P_K{}^1K^0$.[44] This difference between end and beginning of period values for the capital stock can be converted into consumption equivalents and then can be added to actual period 1 consumption in order to obtain Income 1. This income concept is certainly defensible.

To obtain Income 3, we subtract both wear and tear depreciation from gross output, D^1K^0, as well as the revaluation term, R^1K^0, and treat both of these terms as negative offsets to the period's gross investment. The resulting Income 3 can be interpreted to be consistent with the position of Hayek (1941), Sterling (1975) and Hill (2000). This position can also be interpreted as a *maintenance of real financial capital* approach to income measurement. In terms of the Hicks (1961) and Edwards and Bell (1961) Austrian production model, capital stocks at the beginning of the period and end of the period are valued at the prices prevailing at the beginning and the end of the period,[45] $P_K{}^0$ and $P_K{}^1$ respectively, and then these beginning and end of period values of the capital stock are converted into consumption equivalents and then differenced. Thus the end of the period value of the capital stock is $P_K{}^1K^1$ and this value can be converted into consumption equivalents at the consumption prices prevailing at the end of the period. The beginning of the period value of the capital stock is $P_K{}^0K^0$ but to convert this value into consumption equivalents at end of period prices, we must multiply this value by $(1+\rho^0)$, which is one plus the rate of consumer price inflation over the period. This price level adjusted difference between end and beginning of period values for the capital stock, $P_K{}^1K^1 - (1+\rho^0)P_K{}^0K^0$, can be converted into consumption equivalents and then can be added to actual period 1 consumption in order to obtain Income 3. Thus the difference between Income 1 and Income 3 can be viewed as follows: Income 1 uses the end of period stock price of capital to value both the beginning and end of period capital stocks and then converts the resulting difference in values into consumption equivalents at the prices prevailing at the end of the period whereas Income 3 values beginning and end of period capital stocks at the stock prices prevailing at the beginning and end of the period and *directly* converts these values into consumption equivalents and then adds the difference in these consumption equivalents to actual consumption. Thus Income 3 also seems to be a defensible concept.[46]

In order to highlight the difference between Incomes 4 and 3, use definitions (10), (12) and (14) in order to compute their difference:

$$(57)\ \text{Income 4} - \text{Income 3} = P_C{}^1C^1 + P_I{}^1K^1 - P_I{}^1K^0 - [P_C{}^1C^1 + P_K{}^1K^1 - (1+\rho^0)P_K{}^0K^0]$$

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[44] Using Samuelson's (1961) Figure 1 above, this income can be interpreted as the distance OJ along the consumption axis.

[45] Strictly speaking, the end of period price is an expected end of period price.

[46] Income 1 is much easier to justify to national income accountants because it relies on the standard production function model. On the other hand, Income 3 relies on the Austrian model of production as developed by Hicks (1961) and Edwards and Bell (1961) and this production model is not very familiar. This Austrian model of production has its roots in the work of Böhm-Bawerk (1891), von Neumann (1937) and Malinvaud (1953) but these authors did not develop the user cost implications of the model. On the user cost implications of the Austrian model, see Hicks (1973; 27-35) and Diewert (1977;108-111) (1980; 472-474).

$$= (\rho^0 - i^0)P_K{}^0K^0.$$

If ρ^0 (the general consumer price inflation rate) is greater than i^0 (the asset inflation rate) over the course of the period, then there is a negative real revaluation effect (so that obsolescence effects dominate). In this case, Income 3 is less than Income 4, reflecting the fact that capital stocks have become less valuable (in terms of consumption equivalents) over the course of the period. If ρ^0 is less than i^0 over the course of the period, then the real revaluation effect is positive (so that capital stocks have become more valuable over the period). In this case, Income 3 exceeds Income 4, reflecting the fact that capital stocks have become more valuable over the course of the period and this real increase in value contributes to an increase in the period's income which is not reflected in Income 4.

To summarize: both Income 3 and Income 4 both have reasonable justifications. Choosing between them is not a straightforward matter.[47]

In the following sections of this chapter, we develop a formal economic model of production that can help to explain the factors behind growth in real income in a market economy.

9. Productivity and Real Income Growth: A Theoretical Framework

Recall the notation and assumptions made in section 4 above. The same assumptions will be made in the present section. Recall that in section 4, we assumed that there was a period t market sector technology set S^t that exhibited constant returns to scale, the period t net output and input quantity vectors were y^t and x^t respectively and the corresponding period t price vectors were P^t and W^t. We assume that the components of y are the components of C + G + I + X − M − D − R, which are the usual components of market sector GDP less wear and tear depreciation D and less the revaluation term R. The components of x consist of different types of labour services supplied to the market sector by households and the various types of (waiting) capital services used by the market sector.

The constant returns to scale assumption on the technology sets S^t implies that the value of outputs will equal the value of inputs in period t; i.e., we have the following relationships:

(58) $g^t(P^t,x^t) = P^t\cdot y^t = W^t\cdot x^t$; $\qquad t = 0,1,2, \ldots$.

The above material will be useful in what follows but of course, our focus is not on the outputs produced by the market sector; instead our focus is on the income generated by

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[47] However, we lean towards Income 3 over income 1 for two reasons: (i) It seems to us that (expected) obsolescence charges are entirely similar to normal depreciation charges and Income 3 reflects this similarity and (ii) it seems to us that waiting services (U^1K^0) along with labour services and land rents are natural primary inputs whereas depreciation and revaluation services (D^1K^0 and R^1K^0 respectively) are more naturally regarded as a kind of intertemporal intermediate input charge (or benefit if R^1 is negative).

the market sector or more precisely, on *the real income generated by the market sector*. However, since market sector net output is distributed to the factors of production used by the market sector, nominal market sector NDP will be equal to nominal market sector income; i.e., from (58), we have $g^t(P^t,x^t) = P^t\cdot y^t = W^t\cdot x^t$. As an approximate welfare measure that can be associated with market sector production,[48] we will choose to measure the *real income generated by the market sector in period t*, ρ^t, in terms of the number of consumption bundles that the nominal income could purchase in period t; i.e., define ρ^t as follows:[49]

$$
\begin{aligned}
(59)\ \rho^t &\equiv W^t\cdot x^t/P_C{}^t\,; \qquad\qquad t = 0,1,2, \ldots \\
&= w^t\cdot x^t \\
&= p^t\cdot y^t \\
&= g^t(p^t,x^t)
\end{aligned}
$$

where $P_C{}^t > 0$ is the *period t consumption expenditures deflator* and the market sector period t *real output price* p^t and *real input price* w^t vectors are defined as the corresponding nominal price vectors deflated by the consumption expenditures price index; i.e., we have the following definitions:[50]

$$(60)\ p^t \equiv P^t/P_C{}^t\,;\ w^t \equiv W^t/P_C{}^t\,; \qquad\qquad t = 0,1,2, \ldots .$$

The first and last equality in (59) imply that period t real income, ρ^t, is equal to the period t NDP function, evaluated at the period t real output price vector p^t and the period t input vector x^t, $g^t(p^t,x^t)$. Thus *the growth in real income over time can be explained by three main factors: t (Technical Progress or Total Factor Productivity growth), growth in real output prices and the growth of primary inputs*. We will shortly give formal definitions for these three growth factors.

Using the linear homogeneity properties of the GDP functions $g^t(P,x)$ in P and x separately, we can show that the following counterparts to the relations (4) and (5) hold using the deflated prices p and w:[51]

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[48] Since some of the primary inputs used by the market sector can be owned by foreigners, our measure of *domestic* welfare generated by the market production sector is only an approximate one. Moreover, our suggested welfare measure is not sensitive to the distribution of the income that is generated by the market sector.

[49] Note that our use of the symbol ρ in the present section is different from our use of the symbol in previous sections.

[50] Our approach is similar to the approach advocated by Kohli (2004b; 92), except he essentially deflates nominal GDP by the domestic expenditures deflator rather than just the domestic (household) expenditures deflator; i.e., he deflates by the deflator for C+G+I, whereas we suggest deflating by the deflator for C. Another difference in his approach compared to the present approach is that we restrict our analysis to the market sector GDP, whereas Kohli deflates all of GDP (probably due to data limitations). Our treatment of the balance of trade surplus or deficit is also different.

[51] If producers in the market sector of the economy are solving the profit maximization problem that is associated with $g^t(P,x)$, which uses the original output prices P, then they will also solve the profit maximization problem that uses the normalized output prices $p \equiv P/P_C$; i.e., they will also solve the problem defined by $g^t(p,x)$.

(61) $y^t = \nabla_p\, g^t(p^t,x^t)$; $t = 0,1,2, ...$
(62) $w^t = \nabla_x\, g^t(p^t,x^t)$; $t = 0,1,2, ...$.

Now we are ready to define a family of *period t productivity growth factors or technical progress shift factors* $\tau(p,x,t)$:[52]

(63) $\tau(p,x,t) \equiv g^t(p,x)/g^{t-1}(p,x)$; $t = 1,2, ...$.

Thus $\tau(p,x,t)$ measures the proportional change in the real income produced by the market sector at the reference real output prices p and reference input quantities used by the market sector x where the numerator in (63) uses the period t technology and the denominator in (63) uses the period t−1 technology. Thus each choice of reference vectors p and x will generate a possibly different measure of the shift in technology going from period t−1 to period t. Note that we are using the chain system to measure the shift in technology.

It is natural to choose special reference vectors for the measure of technical progress defined by (63): a *Laspeyres type measure* $\tau_L^{\,t}$ that chooses the period t−1 reference vectors p^{t-1} and x^{t-1} and a *Paasche type measure* $\tau_P^{\,t}$ that chooses the period t reference vectors p^t and x^t:

(64) $\tau_L^{\,t} \equiv \tau(p^{t-1},x^{t-1},t) = g^t(p^{t-1},x^{t-1})/g^{t-1}(p^{t-1},x^{t-1})$; $t = 1,2, ...$;
(65) $\tau_P^{\,t} \equiv \tau(p^t,x^t,t) \quad = g^t(p^t,x^t)/g^{t-1}(p^t,x^t)$; $t = 1,2, ...$.

Since both measures of technical progress are equally valid, it is natural to average them to obtain an overall measure of technical change. If we want to treat the two measures in a symmetric manner and we want the measure to satisfy the time reversal property from index number theory[53] (so that the estimate going backwards is equal to the reciprocal of the estimate going forwards), then the geometric mean will be the best simple average to take.[54] Thus we define the geometric mean of (64) and (65) as follows:[55]

(66) $\tau^t \equiv [\tau_L^{\,t}\,\tau_P^{\,t}]^{1/2}$; $t = 1,2, ...$.

At this point, it is not clear how we will obtain empirical estimates for the theoretical productivity growth indexes defined by (64)-(65). One obvious way would be to assume a functional form for the NDP function $g^t(p,x)$, collect data on output and input prices and quantities for the market sector for a number of years (and for the consumption expenditures deflator), add error terms to equations (61) and (62) and use econometric

[52] This measure of technical progress is due to Diewert (1983; 1063) and Diewert and Morrison (1986; 662).
[53] See Fisher (1922; 64).
[54] See the discussion in Diewert (1997) on choosing the "best" symmetric average of Laspeyres and Paasche indexes that will lead to the satisfaction of the time reversal test by the resulting average index.
[55] The specific theoretical productivity change indexes defined by (64)-(66) were first defined by Diewert and Morrison (1968; 662-663). See Diewert (1993) for properties of symmetric means.

techniques to estimate the unknown parameters in the assumed functional form. However, econometric techniques are generally not completely straightforward: different econometricians will make different stochastic specifications and will choose different functional forms.[56] Moreover, as the number of outputs and inputs grows, it will be impossible to estimate a flexible functional form. Thus we will suggest methods for implementing measures like (66) in this paper that are based on exact index number techniques.

We turn now to the problem of defining theoretical indexes for the effects on real income due to changes in real output prices. Define a family of *period t real output price growth factors* $\alpha(p^{t-1},p^t,x,s)$:[57]

(67) $\alpha(p^{t-1},p^t,x,s) \equiv g^s(p^t,x)/g^s(p^{t-1},x)$; s = 1,2, ...

Thus $\alpha(p^{t-1},p^t,x,s)$ measures the proportional change in the real income produced by the market sector that is induced by the change in real output prices going from period t−1 to t, using the technology that is available during period s and using the reference input quantities x. Thus each choice of the reference technology s and the reference input vector x will generate a possibly different measure of the effect on real income of a change in real output prices going from period t−1 to period t.

Again, it is natural to choose special reference vectors for the measures defined by (67): a *Laspeyres type measure* α_L^t that chooses the period t−1 reference technology and reference input vector x^{t-1} and a *Paasche type measure* α_P^t that chooses the period t reference technology and reference input vector x^t:

(68) $\alpha_L^t \equiv \alpha(p^{t-1},p^t,x^{t-1},t-1) = g^{t-1}(p^t,x^{t-1})/g^{t-1}(p^{t-1},x^{t-1})$; t = 1,2, ... ;
(69) $\alpha_P^t \equiv \alpha(p^{t-1},p^t,x^t,t) = g^t(p^t,x^t)/g^t(p^{t-1},x^t)$; t = 1,2, ...

Since both measures of real output price change are equally valid, it is natural to average them to obtain an overall measure of the effects on real income of the change in real output prices:[58]

(70) $\alpha^t \equiv [\alpha_L^t \alpha_P^t]^{1/2}$; t = 1,2, ...

[56] "The estimation of GDP functions such as (19) can be controversial, however, since it raises issues such as estimation technique and stochastic specification. ... We therefore prefer to opt for a more straightforward index number approach." Ulrich Kohli (2004a; 344).
[57] This measure of real output price change was essentially defined by Fisher and Shell (1972; 56-58), Samuelson and Swamy (1974; 588-592), Archibald (1977; 60-61), Diewert (1980; 460-461) (1983; 1055) and Balk (1998; 83-89). Readers who are familiar with the theory of the true cost of living index will note that the real output price index defined by (13) is analogous to the Konüs (1924) *true cost of living index* which is a ratio of cost functions, say $C(u,p^t)/C(u,p^{t-1})$ where u is a reference utility level: g^s replaces C and the reference utility level u is replaced by the vector of reference variables x.
[58] The indexes defined by (67)-(70) were defined by Diewert and Morrison (1986; 664) in the nominal GDP function context.

Finally, we look at the problem of defining theoretical indexes for the effects on real income due to changes in real output prices. Define a family of *period t real input quantity growth factors* $\beta(x^{t-1},x^t,p,s)$:[59]

(71) $\beta(x^{t-1},x^t,p,s) \equiv g^s(p,x^t)/g^s(p,x^{t-1})$; $s = 1,2, \ldots$.

Thus $\beta(x^{t-1},x^t,p,s)$ measures the proportional change in the real income produced by the market sector that is induced by the change in input quantities used by the market sector going from period t−1 to t, using the technology that is available during period s and using the reference real output prices p. Thus each choice of the reference technology s and the reference real output price vector p will generate a possibly different measure of the effect on real income of a change in input quantities going from period t−1 to period t.

Again, it is natural to choose special reference vectors for the measures defined by (71): a *Laspeyres type measure* β_L^t that chooses the period t−1 reference technology and reference real output price vector p^{t-1} and a *Paasche type measure* β_P^t that chooses the period t reference technology and reference real output price vector p^t:

(72) $\beta_L^t \equiv \beta(x^{t-1},x^t,p^{t-1},t-1) = g^{t-1}(p^{t-1},x^t)/g^{t-1}(p^{t-1},x^{t-1})$; $t = 1,2, \ldots$;
(73) $\beta_P^t \equiv \beta(x^{t-1},x^t,p^t,t) \quad = g^t(p^t,x^t)/g^t(p^t,x^{t-1})$; $t = 1,2, \ldots$.

Since both measures of real input growth are equally valid, it is natural to average them to obtain an overall measure of the effects of input growth on real income:[60]

(74) $\beta^t \equiv [\beta_L^t \beta_P^t]^{1/2}$; $t = 1,2, \ldots$.

Recall that market sector real income for period t was defined by (59) as ρ^t equal to nominal period t factor payments $W^t \cdot x^t$ deflated by the household consumption price deflator P_C^t. It is convenient to define γ^t as the *period t chain rate of growth factor for real income*:

(75) $\gamma^t \equiv \rho^t/\rho^{t-1}$; $t = 1,2, \ldots$.

It turns out that the definitions for γ^t and the technology, output price and input quantity growth factors $\tau(p,x,t)$, $\alpha(p^{t-1},p^t,x,s)$, $\beta(x^{t-1},x^t,p,s)$ defined by (63), (67) and (71) respectively satisfy some interesting identities, which we will now develop. We have:

(76) $\gamma^t \equiv \rho^t/\rho^{t-1}$; $t = 1,2, \ldots$.
$= g^t(p^t,x^t)/g^{t-1}(p^{t-1},x^{t-1})$ using definitions (59)
$= [g^t(p^t,x^t)/g^{t-1}(p^t,x^t)][g^{t-1}(p^t,x^t)/g^{t-1}(p^{t-1},x^t)][g^{t-1}(p^{t-1},x^t)/g^{t-1}(p^{t-1},x^{t-1})]$
$= \tau_P^t\, \alpha(p^{t-1},p^t,x^t,t-1)\, \beta_L^t$ using definitions (65), (67) and (72).

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[59] This type of index was defined as a true index of value added by Sato (1976; 438) and as a real input index by Diewert (1980; 456).
[60] The theoretical indexes defined by (71)-(74) were defined in Diewert and Morrison (1986; 665) in the nominal GDP context.

In a similar fashion, we can establish the following companion identity:

$$(77)\ \gamma^t \equiv \tau_L{}^t\ \alpha(p^{t-1},p^t,x^{t-1},t)\ \beta_P{}^t \qquad \text{using definitions (64), (68) and (73).}$$

Thus multiplying (76) and (77) together and taking positive square roots of both sides of the resulting identity and using definitions (66) and (74), we obtain the following identity:

$$(78)\ \gamma^t \equiv \tau^t\ [\alpha(p^{t-1},p^t,x^t,t-1)\alpha(p^{t-1},p^t,x^{t-1},t)]^{1/2}\ \beta^t\ ; \qquad t = 1,2, \ldots .$$

In a similar fashion, we can derive the following alternative decomposition for γ^t into growth factors:

$$(79)\ \gamma^t \equiv \tau^t\ \alpha^t\ [\beta(x^{t-1},x^t,p^t,t-1)\beta(x^{t-1},x^t,p^{t-1},t)]^{1/2}\ ; \qquad t = 1,2, \ldots .$$

It is quite likely that the real output price growth factor $[\alpha(p^{t-1},p^t,x^t,t-1)\alpha(p^{t-1},p^t,x^{t-1},t)]^{1/2}$ is fairly close to α^t defined by (70) and it is quite likely that the input growth factor $[\beta(x^{t-1},x^t,p^t,t-1)\beta(x^{t-1},x^t,p^{t-1},t)]^{1/2}$ is quite close to β^t defined by (74); i.e., we have the following approximate equalities:

$$(80)\ [\alpha(p^{t-1},p^t,x^t,t-1)\alpha(p^{t-1},p^t,x^{t-1},t)]^{1/2} \approx \alpha^t\ ; \qquad t = 1,2, \ldots ;$$
$$(81)\ [\beta(x^{t-1},x^t,p^t,t-1)\beta(x^{t-1},x^t,p^{t-1},t)]^{1/2} \approx \beta^t\ ; \qquad t = 1,2, \ldots .$$

Substituting (80) and (81) into (78) and (79) respectively leads to the following approximate decompositions for the growth of real income into explanatory factors:

$$(82)\ \gamma^t \approx \tau^t\ \alpha^t\ \beta^t\ ; \qquad t = 1,2, \ldots$$

where τ^t is a technology growth factor, α^t is a growth in real output prices factor and β^t is a growth in primary inputs factor.

Rather than look at explanatory factors for the growth in real market sector income, it is sometimes convenient to express the level of real income in period t in terms of an *index of the technology level* or of Total Factor Productivity in period t, T^t, of the *level of real output prices* in period t, A^t, and of the *level of primary input quantities* in period t, B^t.[61] Thus we use the growth factors τ^t, α^t and β^t as follows to define the levels T^t, A^t and B^t:

$$(83)\ T^0 \equiv 1\ ;\ T^t \equiv T^{t-1}\ \tau^t\ ;\ t = 1,2, \ldots ;$$
$$(84)\ A^0 \equiv 1\ ;\ A^t \equiv A^{t-1}\alpha^t\ ;\ t = 1,2, \ldots ;$$
$$(85)\ B^0 \equiv 1\ ;\ B^t \equiv B^{t-1}\beta^t\ ;\ t = 1,2, \ldots .$$

[61] This type of levels presentation of the data is quite instructive when presented in graphical form. It was suggested by Kohli (1990) and used extensively by him; see Kohli (1991), (2003) (2004a) (2004b) and Fox and Kohli (1998).

Using the approximate equalities (82) for the chain links that appear in (83)-(85), we can establish the following approximate relationship for the level of real income in period t, ρ^t, and the period t levels for technology, real output prices and input quantities:

$$(86)\ \rho^t/\rho^0 \approx T^t A^t B^t ; \qquad t = 0,1,2, \ldots .$$

In the following section, we note a set of assumptions on the technology sets that will ensure that the approximate real income growth decompositions (82) and (86) hold as exact equalities.

10. The Translog GDP Function Approach

We now follow the example of Diewert and Morrison (1986; 663) and assume that the log of the period t (deflated) NDP function, $g^t(p,x)$, has the following translog functional form:[62]

$$(87)\ \ln g^t(p,x) \equiv a_0^t + \sum_{m=1}^M a_m^t \ln p_m + (1/2) \sum_{m=1}^M\sum_{k=1}^M a_{mk} \ln p_m \ln p_k + \sum_{n=1}^N b_n^t \ln x_n + (1/2)\sum_{n=1}^N\sum_{j=1}^N b_{nj} \ln x_n \ln x_j + \sum_{m=1}^M\sum_{n=1}^M c_{mn} \ln p_m \ln x_n ; \quad t = 0,1,2, \ldots .$$

Note that the coefficients for the quadratic terms are assumed to be constant over time. The coefficients must satisfy the following restrictions in order for g^t to satisfy the linear homogeneity properties that we have assumed in section 4 above:[63]

$(88)\ \sum_{m=1}^M a_m^t = 1$ for $t = 0,1,2, \ldots$;
$(89)\ \sum_{n=1}^N b_n^t = 1$ for $t = 0,1,2, \ldots$;
$(90)\ a_{mk} = a_{km}$ for all k,m ;
$(91)\ b_{nj} = b_{jn}$ for all n,j ;
$(92)\ \sum_{k=1}^M a_{mk} = 0$ for $m = 1,\ldots,M$;
$(93)\ \sum_{j=1}^N b_{nj} = 0$ for $n = 1,\ldots,N$;
$(94)\ \sum_{n=1}^N c_{mn} = 0$ for $m = 1,\ldots,M$;
$(95)\ \sum_{m=1}^M c_{mn} = 0$ for $n = 1,\ldots,N$.

Recall the approximate decomposition of real income growth going from period t−1 to t given by (82) above, $\gamma^t \approx \tau^t \alpha^t \beta^t$. Diewert and Morrison (1986; 663) showed that[64] if g^{t-1} and g^t are defined by (87)-(95) above and there is competitive profit maximizing behavior

[62] This functional form was first suggested by Diewert (1974; 139) as a generalization of the translog functional form introduced by Christensen, Jorgenson and Lau (1971). Diewert (1974; 139) indicated that this functional form was flexible.
[63] There are additional restrictions on the parameters which are necessary to ensure that $g^t(p,x)$ is convex in p and concave in x. Note that this functional form is a special case of the translog variable profit function defined in the previous chapter. In the present chapter, we assume price taking competitive behavior on the part of producers and constant returns to scale so there are no monopolistic markups.
[64] Diewert and Morrison established their proof using the nominal GDP function $g^t(P,x)$. However, it is easy to rework their proof using the deflated GDP function $g^t(p,x)$ using the fact that $g^t(p,x) = g^t(P/P_C,x) = g^t(P,x)/P_C$ using the linear homogeneity property of $g^t(P,x)$ in P.

on the part of all market sector producers for all periods t, then (82) holds as an exact equality; i.e., we have[65]

(96) $\gamma^t = \tau^t\,\alpha^t\,\beta^t$; $\qquad$ t = 1,2, ...

In addition, Diewert and Morrison (1986; 663-665) showed that τ^t, α^t and β^t could be calculated using empirically observable price and quantity data for periods t−1 and t as follows:

(97) $\ln\alpha^t = \sum_{m=1}^{M} (1/2)[(p_m^{t-1}y_m^{t-1}/p^{t-1}\cdot y^{t-1}) + (p_m^t y_m^t/p^t\cdot y^t)] \ln(p_m^t/p_m^{t-1})$
$\qquad = \ln P_T(p^{t-1},p^t,y^{t-1},y^t)$;
(98) $\ln\beta^t = \sum_{n=1}^{N} (1/2)[(w_n^{t-1}x_n^{t-1}/w^{t-1}\cdot x^{t-1}) + (w_n^t x_n^t/w^t\cdot x^t)] \ln(x_n^t/x_n^{t-1})$
$\qquad = \ln Q_T(w^{t-1},w^t,x^{t-1},x^t)$;
(99) $\quad \tau^t = \gamma^t/\alpha^t\,\beta^t$

where $P_T(p^{t-1},p^t,y^{t-1},y^t)$ is the Törnqvist (1936) and Törnqvist and Törnqvist (1937) output price index and $Q_T(w^{t-1},w^t,x^{t-1},x^t)$ is the Törnqvist input quantity index.

Since equations (96) now hold as exact identities under our present assumptions, equations (86), the cumulated counterparts to equations (82), will also hold as exact decompositions; i.e., under our present assumptions, we have

(100) $\rho^t/\rho^0 = T^t\,A^t\,B^t$; $\qquad$ t = 1,2, ...

We will implement the real income decompositions (96) and (100) in our empirical projects.

11. The Translog GDP Function Approach and Changes in the Terms of Trade

For some purposes, it is convenient to decompose the aggregate period t contribution factor due to changes in all deflated output prices α^t into separate effects for each change in each output price. Similarly, it can sometimes be useful to decompose the aggregate period t contribution factor due to changes in all market sector primary input quantities β^t into separate effects for each change in each input quantity. In this section, we indicate how this can be done, making the same assumptions on the technology that were made in the previous section.

We first model the effects of a change in a single (deflated) output price, say p_m, going from period t−1 to t. Counterparts to the theoretical Laspeyres and Paasche type price indexes defined by (68) and (69) above for changes in all (deflated) output prices are the following *Laspeyres type measure* $\alpha_{Lm}{}^t$ that chooses the period t−1 reference technology and holds constant other output prices at their period t−1 levels and holds inputs constant at their period t−1 levels x^{t-1} and a *Paasche type measure* $\alpha_{Pm}{}^t$ that chooses the period t

[65] We essentially proved a more general version of this result in section 10 of chapter 6 and this more general result can be specialized to give us the exact decomposition (96).

reference technology and reference input vector x^t and holds constant other output prices at their period t levels:

$$(101)\ \alpha_{Lm}{}^t \equiv g^{t-1}(p_1{}^{t-1},...,p_{m-1}{}^{t-1},p_m{}^t,p_{m+1}{}^{t-1},..., p_M{}^{t-1},x^{t-1})/g^{t-1}(p^{t-1},x^{t-1}) ; \qquad m = 1,...,M; \; t = 1,2, ... ;$$

$$(102)\ \alpha_{Pm}{}^t \equiv g^t(p^t,x^t)/g^t(p_1{}^t,...,p_{m-1}{}^t,p_m{}^{t-1},p_{m+1}{}^t,..., p_M{}^t,x^t) ; \qquad m = 1,...,M; \; t = 1,2, ...$$

Since both measures of real output price change are equally valid, it is natural to average them to obtain an *overall measure of the effects on real income of the change in the real price of output m*:[66]

$$(103)\ \alpha_m{}^t \equiv [\alpha_{Lm}{}^t \alpha_{Pm}{}^t]^{1/2} ; \qquad m = 1,...,M ; t = 1,2, ...$$

Under the assumption that the deflated GDP functions $g^t(p,x)$ have the translog functional forms as defined by (87)-(95) in the previous section, the arguments of Diewert and Morrison (1986; 666) can be adapted to give us the following result:

$$(104)\ \ln\alpha_m{}^t = (1/2)[(p_m{}^{t-1}y_m{}^{t-1}/p^{t-1}\cdot y^{t-1}) + (p_m{}^t y_m{}^t/p^t\cdot y^t)] \ln(p_m{}^t/p_m{}^{t-1}) ; \qquad m = 1,...,M ; t = 1,2, ...$$

Note that $\ln\alpha_m{}^t$ is equal to the mth term in the summation of the terms on the right hand side of (97). This observation means that we have the following exact decomposition of the period t aggregate real output price contribution factor α^t into a product of separate price contribution factors; i.e., we have under present assumptions:

$$(105)\ \alpha^t = \alpha_1{}^t\alpha_2{}^t... \alpha_M{}^t ; \qquad t = 1,2, ...$$

The above decomposition is useful for analyzing how real changes in the price of exports (i.e., a change in the price of exports relative to the price of domestic consumption) and in the price of imports impact on the real income generated by the market sector. In your empirical work, we let M equal three. The three net outputs are:

- Domestic sales less depreciation and revaluation (C+I+G−D−R);
- Exports (X) and
- Imports (M).

Since commodities 1 and 2 are outputs, y_1 and y_2 will be positive but since commodity 3 is an input into the market sector, y_3 will be negative. Hence an increase in the real price of exports will *increase* real income but an increase in the real price of imports will *decrease* the real income generated by the market sector, as is evident by looking at the contribution terms defined by (104) for m = 2 (where $y_m{}^t > 0$) and for m = 3 (where $y_m{}^t < 0$).

[66] The indexes defined by (101)-(103) were defined by Diewert and Morrison (1986; 666) in the nominal GDP function context.

As mentioned above, it is also useful to have a decomposition of the aggregate contribution of input growth to the growth of real income into separate contributions for each important class of primary input that is used by the market sector. We now model the effects of a change in a single input quantity, say x_n, going from period t−1 to t. Counterparts to the theoretical Laspeyres and Paasche type quantity indexes defined by (72) and (73) above for changes in input n are the following *Laspeyres type measure* $\beta_{Ln}{}^t$ that chooses the period t−1 reference technology and holds constant other input quantities at their period t−1 levels and holds real output prices at their period t−1 levels p^{t-1} and a *Paasche type measure* $\beta_{Pn}{}^t$ that chooses the period t reference technology and reference real output price vector p^t and holds constant other input quantities at their period t levels:

$$(106)\ \beta_{Ln}{}^t \equiv g^{t-1}(p^{t-1},x_1{}^{t-1},...,x_{n-1}{}^{t-1},x_n{}^t,x_{n+1}{}^{t-1},..., x_N{}^{t-1})/g^{t-1}(p^{t-1},x^{t-1})\ ; \qquad n = 1,...,N;\ t = 1,2, ... ;$$

$$(107)\ \beta_{Pn}{}^t \equiv g^t(p^t,x^t)/g^t(p^t,x_1{}^t,...,x_{n-1}{}^t,x_n{}^{t-1},x_{n+1}{}^t,..., x_N{}^t)\ ; \qquad n = 1,...,N;\ t = 1,2, ...$$

Since both measures of input change are equally valid, as usual, we average them to obtain *an overall measure of the effects on real income of the change in the quantity of input n*:[67]

$$(108)\ \beta_n{}^t \equiv [\beta_{Ln}{}^t\beta_{Pn}{}^t]^{1/2}\ ; \qquad n = 1,...,N\ ;\ t = 1,2, ...$$

Under the assumption that the deflated GDP functions $g^t(p,x)$ have the translog functional forms as defined by (87)-(95) in the previous section, the arguments of Diewert and Morrison (1986; 667) can be adapted to give us the following result:

$$(109)\ \ln\beta_n{}^t = (1/2)[(w_n{}^{t-1}x_n{}^{t-1}/w^{t-1}\cdot x^{t-1}) + (w_n{}^t x_n{}^t/w^t\cdot x^t)]\ \ln(x_n{}^t/x_n{}^{t-1})\ ; \qquad n = 1,...,N\ ;\ t = 1,2, ...$$

Note that $\ln\beta_n{}^t$ is equal to the nth term in the summation of the terms on the right hand side of (98). This observation means that we have the following exact decomposition of the period t aggregate input growth contribution factor β^t into a product of separate input quantity contribution factors; i.e., we have under present assumptions:

$$(110)\ \beta^t = \beta_1{}^t\beta_2{}^t...\ \beta_N{}^t\ ; \qquad t = 1,2, ...$$

For an empirical application of the methodology (to Australia) explained in the last 3 sections of this chapter, see Diewert and Lawrence (2006).

Problems

[67] The indexes defined by (106)-(108) were defined by Diewert and Morrison (1986; 667) in the nominal GDP function context.

8. Let x and y be N and M dimensional vectors respectively and let f^1 and f^2 be two general quadratic functions defined as follows:

(i) $f^1(x,y) \equiv a_0^{\,1} + a^{1T}x + b^{1T}y + (1/2)x^TA^1x + (1/2)\, y^TB^1y + x^TC^1y$; $A^{1T} = A^1$; $B^{1T} = B^1$;
(ii) $f^2(x,y) \equiv a_0^{\,2} + a^{2T}x + b^{2T}y + (1/2)x^TA^2x + (1/2)\, y^TB^2y + x^TC^2y$; $A^{2T} = A^2$; $B^{2T} = B^2$

where the $a_0^{\,i}$ are scalar parameters, the a^i and b^i are parameter vectors and the A^i, B^i and C^i are parameter matrices for i = 1,2. Note that the A^i and B^i are symmetric matrices.

(a) If $A^1 = A^2$, show that the following equation holds for all x^1, x^2, y^1 and y^2:

(iii) $f^1(x^2,y^1) - f^1(x^1,y^1) + f^2(x^2,y^2) - f^2(x^1,y^2) = [\nabla_x f^1(x^1,y^1) + \nabla_x f^2(x^2,y^2)]^T[x^2 - x^1]$.

(b) If $B^1 = B^2$, show that the following equation holds for all x^1, x^2, y^1 and y^2:

(iii) $f^1(x^1,y^2) - f^1(x^1,y^1) + f^2(x^2,y^2) - f^2(x^2,y^1) = [\nabla_y f^1(x^1,y^1) + \nabla_y f^2(x^2,y^2)]^T[y^2 - y^1]$.

Hint: Straightforward substitution into both sides of (a) and (b) will establish the above identities. These identities are a generalization of Diewert's (1976; 118) *quadratic identity*. Logarithmic versions of the above identities correspond to the *translog identity* which was established in the Appendix to Caves, Christensen and Diewert (1982; 1412-1413).

9. Prove (104).

10. Prove (109).

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