The Le Chatelier Principle in Data Envelopment Analysis

by

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Abstract

The paper gives a brief review of the nonparametric approach to efficiency measurement or Data Envelopment Analysis as it is known in the operations research literature. Inequalities are derived between the efficiency measures when different assumptions are made on the technology sets or on the behavior of managers. Of particular interest is the derivation of a Le Chatelier Principle for measures of allocative inefficiency. Finally, DEA methods for measuring the relative efficiency of production units are compared with the more traditional index number and econometric methods.

Key words: efficiency measurement, data envelopment analysis, Le Chatelier Principle, productivity, nonparametric measurement of technology, index numbers.

Classification code: C14, C43, C61, D61

1. Introduction

Data Envelopment Analysis or (DEA) is the term used by Charnes and Cooper (1985) and their co-workers to denote an area of analysis which is called the nonparametric approach to production theory[1] or the measurement of the efficiency of production[2] by economists.

The basic idea in the case of similar firms producing one output and using 2 inputs is due to Farrell (1957; 254). Let there be j firms, denote the output of firm j by $y^j \geq 0$ and denote the amounts of input 1 and 2 used by firm j by $x_1^j \geq 0$ and $x_2^j \geq 0$ respectively, for $j = 1, 2, \ldots, J$. Calculate the input-output coefficients for each firm defined by x_1^j/y^j and x_2^j/y^j for $j = 1, 2, \ldots, J$. Now plot these pairs of input output coefficients in a two dimensional diagram as in Figure 1 where we have labeled these pairs as the points $P^1, P^2, \ldots, P^5$ (so that $j = 5$).

Figure 1:

The convex hull of the 5 data points $P^1, \ldots, P^5$ in Figure 1 is the shaded set: it is the set of all non-negative weighted averages of the 5 points where the weights sum up to 1. The convex free disposal hull of the original 5 points is the shaded set plus all of the points that lie to the north and east of the shaded set. Farrell took the boundary or frontier of this set as an approximation to the unit output isoquant of the underlying production function[3]. In Figure 1, this frontier set is the piecewise linear curve AP^4P^3B. The *Farrell* *technical efficiency* of the point P^1 was defined to be the ratio of distances OC/OP^1, since this is the fraction (of both inputs) that an efficient firm could use to produce the same output as that produced by Firm 1. A point P^i is regarded as being *technically efficient* if its technical efficiency is unity.

Farrell (1957; 254) noted the formal similarity of his definition of technical efficiency to Debreu's (1951) coefficient of resource utilization.

Farrell (1957; 255) also defined two further efficiency concepts using a diagram similar to Figure 1. Suppose Firm 1 faced the fixed input prices w_1 and w_2 for the two inputs. Then we could form a family of isocost lines with slope - w_1/w_2 and find the lowest such isocost line that is just tangent to the free disposal convex hull of the 5 points. In Figure 1, this is the line DE which is tangent to the point P^3. Farrell noted that even if the point P^1 were shrunk in towards the origin to end up at the technically efficient point C, the resulting point would still not be the cost minimizing input combination (which is at P^3). Thus Farrell defined the *price efficiency* of P^1 as the ratio of distances OD/OC. Finally, Farrell (1957; 255) defined the *overall efficiency* of Firm 1 as the ratio of distances OD/OP^1. This measure incorporates both technical and allocative inefficiency. A point P^i is *overall efficient* if its overall efficiency is unity.

There is a problem with Farrell's measure of technical efficiency: Farrell's definition makes the points P^2 and P^5 in Figure 1 technically efficient when it seems clear that they are not: P^2 is dominated by P^3 which uses less of input 1 to produce the same output and P^5 is dominated by P^4 which uses less of input 2 to produce the same output. Charnes, Cooper and Rhodes (1978; 437) and Färe and Lovell (1978; 151) both noticed this problem with Farrell's definition of technical efficiency and suggested remedies. However, in the remainder of this paper we will stick with Farrell's original definition of technical efficiency (with a few modifications due to Mendoza (1989)).

Farrell's basic ideas outlined above for the case of a one output, constant returns to scale technology can be generalized in several ways: (i) we can relax the assumption of constant returns to scale; (ii) we can extend the analysis to the multiple output, multiple input case; (iii) we can generalize the analysis to cover situations where it is reasonable to assume profit maximizing behaviour (or partial profit maximizing behaviour) rather than cost minimizing behaviour and (iv) we can measure inefficiency in different metrics (i.e., instead of measuring technical inefficiency in terms of a proportional shrinkage of the input vector, we could choose to measure the inefficiency in terms of a basket of outputs or a basket of outputs and inputs). Drawing on the work of Mendoza (1989) and others, we shall indicate how the above generalizations can be implemented for the case of technologies that produce only 2 outputs and utilize only 2 inputs. The generalization to many outputs and inputs is straightforward. Section 2 below covers approaches that use only quantity data while section 3 describes approaches that utilize both price and quantity data. Section 3 also derives some interesting general relationships between various measures of efficiency loss. Of particular interest is a Le Chatelier Principle for measures of allocative inefficiency.

Mendoza (1989) also undertook an empirical comparison of 3 different methods for measuring productivity change in the context of time series data. The 3 different methods of comparison she considered were: (i) a nonparametric or DEA method; (ii) traditional index number methods and (iii) an econometric method based on the estimation of a unit profit function[4]. In section 4 we give a summary of her results.

Drawing on the empirical and theoretical results reviewed in the previous sections, in section 5 we compare the advantages and disadvantages of DEA methods for measuring the relative efficiency of production units with the more traditional index number and econometric methods.

2. Efficiency tests using quantity data

2.1 The case of a convex technology

Suppose that we have quantity data on J production units that are producing 2 outputs using 2 inputs. Let $y_m^j \geq 0$ denote the amount of output m produced by "firm" j for $m = 1, 2$, and let $x_k^j \geq 0$ denote the amount of input be used by firm j for $k = 1, 2$ and $j = 1, 2, \ldots, J$.

We assume that each firm has the same basic technology except for efficiency differences. An approximation to the basic technology is defined to be the convex free disposal hull of the observed quantity data $\{(y_1^j, y_2^j, x_1^j, x_2^j) : j = 1, \ldots, J\}$. This technology assumption is consistent with decreasing returns to scale (and constant returns to scale) but *not* increasing returns to scale.

It is necessary to specify a *direction* in which possible inefficiencies are measured; i.e., do we measure the inefficiency of observation i in terms of output m or input k or some combination of outputs and inputs? Mendoza's (1989) methodology allows for an arbitrary efficiency direction[5], but for simplicity, we will restrict ourselves to the Debreu (1951) - Farrell (1957) direction; i.e., we shall measure the inefficiency of observation i by the number of multiples δ_i^* of (x_1^i, x_2^i) that could be thrown away by the *ith* firm if it were on the efficient frontier spanned by the convex free disposal hull of the J observations. If the *ith* observation is efficient relative to this frontier, then $\delta_i^* = 0$. The number δ_i^* can be determined as the optimal objective function of the following linear programming problem:[6]

$$\begin{aligned}
\delta_i^* \equiv max_{\delta_i \geq 0, \lambda_1 \geq 0, \ldots, \lambda_J \geq 0}\ \delta_i \text{ subject to} :& \\
\Sigma_{j=1}^J y_1^j \lambda_j \geq y_1^i \ ;& \\
\Sigma_{j=1}^J y_2^j \lambda_j \geq y_2^i \ ;& \\
\Sigma_{j=1}^J x_1^j \lambda_j \leq x_1^i - \delta_i x_1^i \ ;& \\
\Sigma_{j=1}^J x_2^j \lambda_j \leq x_2^i - \delta_i x_2^i \ ;& \\
\Sigma_{j=1}^J \lambda_j = 1.&
\end{aligned} \tag{1}$$

Thus we look for a convex combination of the J data points that can produce at least the observation i combination of outputs (y_1^i, y_2^i) and use at most $(1-\delta_i)$ times the observation i combination of inputs (x_1^i, x_2^i). The largest such δ_i is δ_i^*.

The linear programming problems (1) are run for each observation i and the resulting $\delta_i^* \geq 0$, serves to measure the relative efficiency of observation i; if $\delta_i^* = 0$, then observation i is efficient. At least one of the J observations will be efficient.

We then turn now to the corresponding linear program that tests for efficiency under the maintained hypothesis that the underlying technology is subject to constant returns to scale.

2.2 The case of a convex, constant returns to scale technology

In this case, the approximation to the underlying technology set is taken to be the free disposal hull of the convex cone spanned by the J data points. The inefficiency of observation i is measured by the number of multiples δ_i^{**} of the ith input vector (x_1^i, x_2^i) that could be thrown away by the ith firm if it were on the above frontier and can be calculated by solving the following linear program:[7]

$$\begin{aligned}
\delta_i^{**} = max_{\delta_i \geq 0, \lambda_1 \geq 0, \ldots, \lambda_J \geq 0}\ \delta_i \text{ subject to} :& \\
\Sigma_{j=1}^J y_1^j \lambda_j \geq y_1^i \ ;& \\
\Sigma_{j=1}^J y_2^j \lambda_j \geq y_2^i \ ;& \\
\Sigma_{j=1}^J x_1^j \lambda_j \leq x_1^i - \delta_i x_1^i \ ;& \\
\Sigma_{j=1}^J x_2^j \lambda_j \leq x_2^i - \delta_i x_2^i \ .&
\end{aligned} \tag{2}$$

Note that the LP (2) is the same as (1) except that the constraint $\sum_{j=1}^J \lambda_j = 1$ has been dropped. Thus the optimal solution for (1) is feasible for (2) and thus $\delta_i^{**} \geq \delta_i^*$; i.e., the constant returns to scale measure of inefficiency for observation i will equal to or greater than the convex technology measure of inefficiency for observation i.

We turn now to models that are consistent with increasing returns to scale.

2.3 Quasiconcave technologies

We first need to define what we mean by a production possibilities set $L(y_1)$ that is conditional on an amount y_1 of output 1. Let S be the set of feasible outputs and inputs. Then $L(y_1)$ is defined to be the set of (y_2, x_1, x_2) such that (y_1, y_2, x_1, x_2) belongs to S; i.e., $L(y_1)$ is the set of other outputs y_2 and inputs x_1 and x_2 that are consistent with the production of y_1 units of output 1. We assume that the family of production possibilities sets $L(y_2)$ has the following three properties: (i) for each $y_1 \geq 0, L(y_1)$ is a closed, convex set[8]; (ii) if $y_1' \leq y_1''$, then $L(y_1'')$ is a subset of $L(y_1)'$ and (iii) the sets $L(y_1)$ exhibit free disposal.

For each observation i, define the following set of indexes:

$$I_i^1 \equiv \{j : y_1^j \geq y_1^i,\ j = 1, 2, \ldots, J\}; \tag{3}$$

i.e., I_i^1 is the set of observations j such that the amount of output 1 produced by observations j is equal to or greater than the amount of output 1 produced by observation i. Note that observation i must belong to I_i^1.

Given our assumptions on the underlying technology, it can be seen that the free disposal convex hull of the points $(y_2^j, x_1^j, x_2^j), j \in I_i^1$, form an approximation to the set $L(y_1^i)$. The frontier of this set is taken to be the efficient set. As usual, we measure the inefficiency of observation i by the number of multiples δ_j^{***} of the ith input vector (x_1^i, x_2^i) that could be thrown away by the ith firm if it were on the above frontier. The number can be calculated by solving the following linear program:[9]

$$\begin{aligned} \delta_i^{***} = max_{\delta_i \geq 0, \lambda_1 \geq 0, \ldots, \lambda_J \geq 0}\ \delta_i \text{ subject to :} \\ \Sigma_{j \in I_i^1} y_2^j \lambda_j \geq y_2^i\ ; \\ \Sigma_{j \in I_i^1} x_1^j \lambda_j \leq x_1^i - \delta_i x_1^i\ ; \\ \Sigma_{j \in I_i^1} x_2^j \lambda_j \leq x_2^i - \delta_i x_2^i\ ; \\ \Sigma_{j \in I_i^1} \lambda_j = 1. \end{aligned} \tag{4}$$

On the left hand side of each constraint in (4), the indexes j must belong to the index set I_i^1 defined by (3) above.

Denote the optimal λ_j for (4) above by λ_j^{***} for $j \in I_i^1$. By the last constraint in (4), we have

$$\Sigma_{j \in I_i^1} \lambda_j^{***} = 1. \tag{5}$$

Using the definition (3), $\lambda_j^{***} \geq 0$ and (5), we can show that

$$\Sigma_{j \in I_i^1} y_1^i \lambda_j^{***} \geq y_1^i. \tag{6}$$

Using (1), (4) and (6), we see that the optimal solution for (4) is feasible for (1) and thus we must have $\delta_i^* \geq \delta_i^{***}$. Recall that we showed that $\delta_i^{**} \geq \delta_i^*$ and so we have

$$0 \leq \delta_i^{***} \leq \delta_i^* \leq \delta_i^{**}. \tag{7}$$

Thus the efficiency loss measures generally increase (or remain constant) as we make stronger assumptions on the underlying technology: the smallest loss measure δ_i^{***} corresponds to a quasiconcave (in output 1) technology, the next measure δ_i^* corresponds to a convex technology, and the largest measure δ_i^{**} corresponds to a constant returns to scale convex technology.

In definition (3) and the LP (4), output 1 was singled out to play a special role. Obviously, analogues to (3) and (4) could be constructed where output 2 played the asymmetric role. In this latter case, the underlying technological assumption is that the $y_2 = f(y_1, x_1, x_2)$ production function is quasiconcave. This is a somewhat different technological assumption than our initial one, but both assumptions can be consistent with an increasing returns to scale technology.[10]

This completes our brief overview of nonparametric efficiency tests that involve the use of quantity data. We now turn to test that involve both price and quantity data so that overall efficiency measures can be constructed in place of the technical efficiency measures of this section.

3. Efficiency tests using price and quantity data

3.1 The convex technology case

We make the same assumptions on the underlying technology as in section 2.1 above. However, we now assume that each producer may be either minimizing cost or maximizing profits. We consider each case in turn.

Case (i): Cost Minimization We assume that producer j faces the input prices (w_1^j, w_2^j) for the two inputs. To determine whether producer i is minimizing cost subject to our convex technology assumptions, we solve the following linear program:[11]

$$\begin{aligned} & min_{\lambda_1 \geq 0, \ldots, \lambda_J \geq 0} \; w_1^i(\Sigma_{j=1}^J x_1^j \lambda_j) + w_2^i(\Sigma_{j=1}^J x_2^j \lambda_j) \\ & \text{subject to:} \quad \Sigma_{j=1}^J y_1^j \lambda_j \geq y_1^i \; ; \\ & \qquad\qquad\qquad \Sigma_{j=1}^J y_2^j \lambda_j \geq y_2^i \; ; \\ & \qquad\qquad\qquad \Sigma_{j=1}^J \lambda_j = 1 \; ; \end{aligned} \tag{8}$$

$$\equiv (1 - \varepsilon_i^*)[w_1^i x_1^i + w_2^i x_2^i]. \tag{9}$$

The meaning of (9) is that we define the overall efficiency loss measure ε_i^* by equating (9) to the optimized objective function in (8). If we set $\lambda_i = 1$ and the other $\lambda_j = 0$, we have a feasible solution for (8) which yields a value of the objective function equal to $w_1^i x_1^i + w_2^i x_2^i$. Thus $\varepsilon_i^* \geq 0$. The number ε_i^* can be interpreted as the number of multiples of (x_1^i, x_2^i) that we could throw away and obtain a shrunken firm i input vector that would be on the minimum cost isocost line; i.e., $1 - \varepsilon_i^*$ is an analogue to the overall efficiency measure OD/OP^1 which occurred in Figure 1.

Comparing (1) and (8), it can be seen that the optimal λ_j^* solution for (1) is a feasible solution for (8) and thus:

$$0 \leq \delta_i^* \leq \varepsilon_i^* \tag{10}$$

The inequality (10) simply reflects the fact that overall inefficiency is equal to or greater than technical efficiency (recall Figure 1 again).

Case (ii): Profit Maximization We now assume that firm i also faces the output prices (p_1^j, p_2^j) for the two outputs. To determine whether producer i is maximizing profits subject to our convex technology assumptions; we solve the following linear program:[12]

$$\begin{aligned} &max_{\lambda_1 \geq 0, \ldots, \lambda_J \geq 0} \ \Sigma_{m=1}^2 p_m^i (\Sigma_{j=1}^J y_m^j \lambda_j) - \Sigma_{k=1}^2 w_k^i (\Sigma_{j=1}^J x_k^j \lambda_j) \\ &\text{subject to:} \qquad \Sigma_{j=1}^J \lambda_j = 1 \end{aligned} \tag{11}$$

$$\equiv p_1^i y_1^i + p_2^i y_2^i - (1 - \alpha_i^*)[w_1^i x_1^i + w_2^i x_2^i]. \tag{12}$$

Equating (11) to (12) defines the efficiency loss measure α_i^* for observation i. If we set $\lambda_i = 1$ in (11) and the other $\lambda_j = 0$, we obtain a feasible value for the objective function equal to $p_1^i y_1^i + p_2^i y_2^i - [w_1^i x_1^i + w_2^i x_2^i]$. Thus $\alpha_i^* \geq 0$. If $\alpha_i^* = 0$, then observation i is efficient relative to our assumptions on the technology and relative to the hypothesis of complete profit maximization. The interpretation of α_i^* is similar to that of ε_i^* defined above by (9).

It can be seen that the optimal λ_j^* solution to (8) is feasible for (11). Using this fact and the inequalities in (8), Mendoza (1989; 76-77) shows that

$$\varepsilon_i^* \leq \alpha_i^*. \tag{13}$$

In fact, Mendoza shows in general as the producer is assumed to optimize over more goods, the corresponding efficiency loss measures cannot decrease; i.e., they will increase or stay constant.

We now turn to the corresponding linear programs that test for the efficiency of observation i under the maintained hypothesis that the underlying technology is subject to constant returns to scale.

The convex conical technology case

Case (ii): Cost Minimization Guided by the results of section 2,2, it can be seen that all we have to do is to drop the constraint $\Sigma_{j=1}^{J}\lambda_j = 1$ from (8). The resulting optimized objective function is set equal to $(1-\varepsilon_i^{**})[w_1^i x_1^i + w_2^i x_2^i]$. Since the new LP has one less constraint than (8), it will generally attain a smaller optimized objective function and so ε_i^{**} will generally be bigger than ε_i^{*}; i.e.,

$$\varepsilon_i^{*} \leq \varepsilon_i^{**}. \tag{14}$$

By comparing the new LP to (2), we can also show

$$\delta_i^{**} \leq \varepsilon_i^{**}. \tag{15}$$

The inequality (14) shows that making stronger assumptions on the underlying technology tends to increase the efficiency loss measure and the inequality (15) shows that assuming cost minimizing behaviour tends to increase the efficiency loss.[13]

Case (ii): Profit Maximization As in section 2.2, we approximate the underlying technology set by the free disposal hull of the convex cone spanned by the J data points. To determine whether observation i is on the frontier of this set, we could attempt to solve the LP problem (11) after dropping the constraint $\Sigma_{j=1}^{J}\lambda_j = 1$. Unfortunately, the resulting optimal objective function is either 0 or plus infinity. Hence a different approach is required.

In order to obtain an operational approach, we consider a conditional profit maximization problem in place of the full profit maximization problem that appears in the objective function of (11); i.e., we allow firm i to maximize profits excluding the cost of one input (which is regarded as being fixed in the short run). Thus if the fixed input is input 1, to determine whether producer i is maximizing (variable) profits subject to our convex, conical technology assumptions, we solve the following linear programming problem:[14]

$$\begin{aligned} &max_{\lambda_1 \geq 0,\ldots,\lambda_J \geq 0}\ \Sigma_{m=1}^{2} p_m^i (\Sigma_{j=1}^{J} y_m^j \lambda_j) - w_2^i(\Sigma_{j=1}^{J} x_2^j \lambda_j) \\ &\text{subject to}: \ \Sigma_{j=1}^{J} x_1^j \lambda_j \leq x_1^i \ ; \end{aligned} \tag{16}$$

$$\equiv [p_1^i y_1^i + p_2^i y_2^i](1+\beta_i^{**}) - w_2^i x_2^i \tag{17}$$

where (17) serves to define the observation i efficiency loss measure β_i^{**}. Note that $\lambda_i = 1$ and the other $\lambda_j = 0$ is a feasible solution for (16).

We have changed the way we measure the loss of efficiency in (16) compared to (11): the efficiency loss of firm i is now the number of extra multiples β_i^{**} of the firm i observed output vector (y_1^i, y_2^i) that an efficient firm could throw away and still be on the same isoprofit surfaces firm i. Thus in (16), we measure the loss of efficiency in terms of proportional output bundles whereas previously we had measured the loss in terms of proportional input bundles. Since we have switched loss metrics, we cannot deduce that $\beta_i^{**} \geq \alpha_i^*$ where α_i^* was defined by (12).

In order to obtain a loss measure α_i^{**} for observation i in our old input loss metric, we need to solve the following linear programming problem:[15]

$$\begin{aligned} & max_{\lambda_1 \geq 0, \ldots, \lambda_J \geq 0} \ \Sigma_{m=1}^2 p_m^i (\Sigma_{j=1}^J y_m^j \lambda_j) - w_1^i x_1^i - w_2^i (\Sigma_{j=1}^J x_2^j \lambda_j) \\ & \text{subject to: } \Sigma_{j=1}^J x_1^i \lambda_j \leq x_1^i \ ; \end{aligned} \tag{18}$$

$$\equiv [p_1^i y_1^i + p_2^i y_2^i] - (1 - \alpha_i^{**})[w_1^i x_1^i + w_2^i x_2^i] \tag{19}$$

where (19) serves to define α_i^{**}. As usual, $\lambda_i = 1$ and the other $\lambda_j = 0$ is feasible for (18) and this fact implies that $\alpha_i^{**} \geq 0$. Comparison of (2) and (18) shows that the optimal solution to (2) generates a feasible α_i for (18) and (19) and thus

$$\delta_i^{**} \leq \alpha_i^{**} \ ; \tag{20}$$

i.e., the observation i technical efficiency loss measure δ_i^{**} is always equal to or less than the overall observation i profit maximization loss α_i^{**}.

3.3 The quasiconcave technology case

We consider only the cost minimization case. More general cases are considered in Mendoza's (1989; 83) Test 6.

We make the same technology assumptions as were made in section 2.3. Recall the index set I_i^1 defined by (3). To determine whether producer i is minimizing cost subject to our quasiconcave technology in output 1 assumption, we solve the following linear program:

$$\begin{aligned} min_{\lambda_1 \geq 0, \ldots, \lambda_J \geq 0} \ & w_1^i (\Sigma_{j \in I_i^1} x_1^i \lambda_j) + w_2^i (\Sigma_{j \in I_i^1} x_2^j \lambda_j) \\ \text{subject to: } & \Sigma_{j \in I_i^1} y_1^j \lambda_j \geq y_1^i \ ; \\ & \Sigma_{j \in I_i^1} y_2^j \lambda_j \geq y_2^i \ ; \\ & \Sigma_{j \in I_i^1} \lambda_j = 1 \ ; \end{aligned} \tag{21}$$

$$\equiv (1 - \varepsilon_i^{***})[w_1^i x_1^i + w_2^i x_2^i] \tag{22}$$

As usual, ε_i^{***} is our measure of overall efficiency loss for observation i under our present assumptions on the technology and on the producer's behaviour. Since the index i belongs to the index set I_i^1 (recall (3)), it can be seen that $\lambda_i = 1$ and the other $\lambda_j = 0$ is feasible for the LP(21) and gives rise to the feasible value for the objective function equal to

$w_1^i x_1^i + w_2^i x_2^i$. Thus $\varepsilon_i^{***} \geq 0$. It is also possible to see that the optimal $\delta_i^{***}, \lambda_j^{***}$ solution to (4) is a feasible ε_i, λ_j solution for (21). Thus

$$0 \leq \delta_j^{***} \leq \varepsilon_i^{***} \; ; \tag{23}$$

i.e., the cost minimizing overall efficiency loss for observation i, ε_i^{***}, will be equal to or greater than the technical efficiency loss for observation i, δ_i^{***}.

Comparing (21) with (8) and using the definition of the index set I_i^1 (recall (3)), it can be seen that the optimal $\lambda_j^{***}, \varepsilon_i^{***}$ solution a feasible solution for (8). Thus

$$\varepsilon_i^{***} \leq \varepsilon_i^{*} \; ; \tag{24}$$

i.e., the observation i loss measure assuming a quasiconcave technology and cost minimizing behaviour ε_i^{***} will be equal to or less than the observation i loss measure assuming a convex technology and cost minimizing behaviour ε_i^{*}.

The inequalities derived in this section and the previous section can be summarized by two rules. We assume that all efficiency losses are measured in the same metric. *Rule 1*: The nonparametric efficiency loss measures tend to grow as we make more restrictive technological assumptions; i.e., the quasiconcave technology loss measure will be equal to or less than the corresponding convex technology loss measure which in turn will be equal to or less than the corresponding convex conical technology loss measure.
Rule 2: The nonparametric efficiency loss measures tend to grow as we assume optimizing behaviour over a larger number of goods; i.e., the technical efficiency loss measure will be equal to or less than the corresponding cost minimizing loss measure which will be equal to or less than the corresponding profit maximizing loss measure. This is Mendoza's (1989; 76-77) Le Chatelier Principle for measures of allocative inefficiency.

We turn now to a brief description of some of Mendoza's empirical results.

4. An empirical comparison of alternative efficiency measures

Mendoza (1989) compared three alternative methods for computing annual rates of productivity change for 4 Canadian production sectors over the years 1961 to 1980 using Canadian input-output data. The 3 alternative methods were: (i) a method based on DEA techniques; (ii) a method based on the use of superlative index numbers[16] and (iii) a method based on the statistical estimation of unit profit functions for each of the four sectors.[17] The 4 sectors were: (i) a resources sector; (ii) an export oriented manufacturing sector; (iii) a domestic market oriented manufacturing sector and (iv) a services sector. Each sector produced one output and used the outputs of the other 3 sectors as inputs. In addition, each sector used 5 primary inputs: imports, labor, inventories, machinery and equipment and an aggregate of land and structures. Thus each sector produced one output, used 3 intermediate inputs and used 5 primary inputs. In the index number computations and in the DEA analysis, the intermediate inputs were treated as negative outputs.

For each sector and of each year i, Mendoza calculated nonparametric measures of efficiency loss that are analogous to the profit maximizing, convex conical loss measures β_i^{**} defined by (17) above.[18] For sector 1, the most efficient observation occurred in 1973; for sector 2, the efficient year was 1978; for sector 3, the efficient year was 1979 and for sector 4, the efficient years the last year 1980. As Mendoza (1989; 111) observed, her violation indexes can be interpreted as chained indexes of technical change. Thus she used her violation indexes to construct to year measures of technical change (or productivity change) and these year to year nonparametric productivity change indexes were compared to the corresponding index number measures of productivity change.[20] Somewhat surprisingly, the nonparametric and index number estimates of technical change are very similar except that the magnitudes of the nonparametric changes tended to be substantially less than the corresponding index number changes. These differences in magnitude are due to Mendoza's treatment of intermediate inputs[21]; a more appropriate treatment would greatly diminish these differences in magnitude.

Mendoza's (1989) econometric estimates of sectoral technical change are tabled on page 129 of her thesis and compared graphically with her nonparametric estimates of sectoral technical change on pages 133-134. Her results show that the econometric estimates of efficiency change are simply a highly smoothed version of the corresponding nonparametric estimates.

Mendoza's results are very encouraging: the three methods of computing annual productivity change given roughly the same answers with the econometric estimates being much less variable than the other two sets of estimates.

5. A comparison of the alternative methods for measuring efficiency

We summarize our comparison of alternative methods in point form.

(i) Nonparametric of DEA techniques have an overwhelming advantage over index number and econometric methods when *only* quantity data are available. Index number methods cannot be implemented without a complete set of price and quantity data. Econometric methods (i.e., production function methods) are not likely to be successful if only quantity data are available due to limited degrees of freedom (see Diewert (1992)).

(ii) The relative efficiency of any single observation will tend to decrease as the sample size increases. Actually, all three methods have this problem.

(iii) Nonparametric and econometric efficiency scores will tend to decrease (i.e., relative inefficiencies will tend to increase) as we make stronger assumptions on the underlying technology. Index number estimates of efficiency remain unchanged as we change our assumptions on the technology.

(iv) Nonparametric and economic efficiency scores will tend to decrease as we make stronger assumptions about the optimizing behaviour of producers; recall Rule 2 in section 3. It is not clear what will happen to econometric based efficiency scores under the same conditions. Since index number methods are based on the assumption of complete optimizing behaviour we cannot vary our assumptions on optimizing behaviour when using index number methods.

(v) If we hold the number of observations in our sample constant but disaggregate the data so that the number of inputs or outputs is increased, then nonparametric efficiency scores will tend to increase[22]. However, index number efficiency scores will generally remain unaffected by increasing disaggregation[23]. It is not clear what will happen using econometric methods.

(vi) The cost of computing index number estimates of relative efficiency is extremely low; the cost of the nonparametric estimates is low and the cost of computing econometric estimates can be very high if the number of goods exceeds 10.

(vii) When complete price and quantity data are available, the nonparametric estimates based on a constant returns to scale technology and complete profit maximizing behaviour are approximately equal to the corresponding index number estimates. Econometric estimates based on the same assumptions will tend to be similar to the first two sets of estimates (but much smoother in the time series context).

(viii) Nonparametric techniques can be adapted to deal with situations where input prices are available but not output prices. Econometric techniques can also deal with this situation but index number methods cannot be used in this situation.

(ix) Nonparametric methods may be severely biased due to measurement errors; i.e., the best or most efficient observation in a DEA study may be best simply because some output was greatly overstated or some important input was greatly understated. Index number methods are also subject to measurement errors but econometric methods may be adapted to deal with gross outliers.

Our overall conclusion is that DEA methods for measuring relative efficiency can be used profitably in a wide variety of situations when other methods are not practical or are impossible to use.

Footnotes

1. See Hanoch and Rothschild (1972), Diewert (1980) (1981), Diewert and Parkan (1983) and Varian (1984).

2. See Farrell (1957), Afriat (1972), Färe and Lovell (1978), and Färe, Grosskopf and Lovell (1985).

3. Farrell (1957; 254) was assuming constant returns to scale in this part of his paper.

4. For material on variable and unit profit functions, see Diewert (1973) (1974) and Diewert and Wales (1992).

5. See Mendoza (1989; 25-30).

6. See Mendoza (1989; 30) for a general version of Test 1. The use of linear programming techniques to calculate nonparametric efficiencies was first suggested by Hoffman (1957; 284) and first used by Farrell and Fieldhouse (1962).

7. See Mendoza (1989; 44) for a general version of Test 2.

8. If we represent the underlying technology by means of the production function $y_1 = f(y_2, x_1, x_2)$, assumption (i) implies that f is a quasiconcave function.

9. See Mendoza (1989; 54) for a general version of (4) which she calls Test 3. The one output quasiconcavity test is due to Hanoch and Rothschild (1972).

10. Mendoza's (1989; 54) Test 3 can also be modified to model quasiconcave technologies of the form $x_1 = g(y_1, y_2, x_2)$, where g is now a factor requirements function.

11. See Mendoza (1989; 67) for a general version of (8) which she calls Test 4.

12. This is Mendoza's (1989; 88) Test 7. It is also a special case of her Test 4.

13. These results and the appropriate general test may be found in Mendoza (1989; 78), which she calls Test 5.

14. This problem is a special case of Mendoza's (1989; 78) Test 5.

15. This problem was not considered in Mendoza's (1989) thesis.

16. Two index number formulae were used: (i) the Fisher ideal and (ii) the Translog or Törnqvist formula. These index number methods of measuring productivity change are described in Diewert (1992).

17. The specific functional form used was Diewert and Wales' (1987) (1988) (1992) normalized quadratic with quadratic splines in the time trend. Constant returns to scale were assumed and the fixed input was the structures and land aggregate.

18. See Mendoza (1989; 111) for a table of these violation indexes. Unfortunately, her violation indexes are not quite analogous to the β_i^{**} defined by (17) due to her treatment of intermediate inputs. Suppose that output 2 were an intermediate input. Then (16) and (17) are still valid where the y_2^j are now negative numbers. The analogous Mendoza loss measure ε_i^i is obtained by replacing (17) by the following expression that serves to define $\varepsilon^i : [p_1^i y_1^i + p_2^i y_2^i] - w_2^i x_2^i + [p_1^i y_1^i + p_2^i \mid y_2^i \mid]\varepsilon_i$. Thus if the loss is positive, Mendoza's loss ε_i will always be less than our β_i^{**}.

19. These nonparemetric year to year indexes are tables in Mendoza (1989; 114) and plotted on pages 115-116.

20. The Translog indexes of productivity change are tables on page 120 and the Fisher indexes are tabled on page 121. These estimates coincide to 4 decimal places (as we expect from theoretical considerations). The index number estimates of productivity change for the 4 sectors are compared graphically with the corresponding nonparametric estimates on pages 122-125.

21. See footnote 18.

22. As we disaggregate, the objective functions of the various linear programming problems will remain unchanged but the feasible regions for the problems become more constrained or smaller and hence the objective function minimums for the linear programming problems will become larger. Hence, the loss measures will decrease or remain constant and thus efficiency will tend to increase as we disaggregate. This point was first made by Nunamaker (1985). The profit maximization problems (11) and (18) are not affected by disaggregation.

23. This follows from the approximate consistency in aggregation property of superlative index number formulae like the Fisher and Translog; see Diewert (1978; 889 and 895).

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