Journal of Economic and Social Measurement 28 (2002) 51–61
IOS Press

Additive decompositions for Fisher, Törnqvist and geometric mean indexes

Marshall B. Reinsdorf[a], W. Erwin Diewert[b] and Christian Ehemann[a]
[a] *US Bureau of Economic Analysis, 1441 L St. NW, Mail Stop BE-40, Washington, DC 21230, USA*
[b] *University of British Columbia, Department of Economics, Vancouver, BC, Canada V6T 1Z1*

Users of a price or quantity index often want to know how much each item in the index contributes to its overall change. Consequently, statistical agencies generally publish items' contributions to changes in the indexes that they publish. For fixed basket index formulas, calculating contributions to index change that add up to the correct total is straightforward, but for Fisher, Törnqvist and geometric mean index formulas – which statistical agencies are beginning to use – it is not. We use economic and axiomatic approaches to derive additive decompositions of the change in a Fisher index, and we use an axiomatic approach to derive an additive decomposition of the change in a Törnqvist or geometric mean index.

1. Introduction

Users of a price or quantity index often want to know how much each item included in the index contributes to its overall change. Consequently, statistical agencies generally publish items' contributions to changes in the indexes that they publish.

With the Laspeyres or Paasche index formulas that agencies have traditionally used, calculation of contributions to index change that add up to the correct total is straightforward. Let the vectors of initial prices and quantities be $\mathbf{p}_0 = (p_{10}, \ldots, p_{n0})'$ and $\mathbf{q}_0 = (q_{10}, \ldots, q_{n0})'$, and let final prices and quantities be $\mathbf{p}_1$ and $\mathbf{q}_1$. Then the contribution of the ith item to the change in the Laspeyeres price index is $(p_{i1} - p_{i0})q_{i0}/\mathbf{p}_0'\mathbf{q}_0$. Its contribution to the change in the Paasche price index is $(p_{i1} - p_{i0})q_{i1}/\mathbf{p}_0'\mathbf{q}_1$. Similarly, the contribution of the ith item to the change in the Laspeyres quantity index is $(q_{i1} - q_{i0})p_{i0}/\mathbf{p}_0'\mathbf{q}_0$ and its contribution to the change in the Paasche quantity index is $(q_{i1} - \mathbf{q}_{i0})p_{i1}/\mathbf{p}_1'\mathbf{q}_0$.

Recently statistical agencies have begun to use additional types of index formulas besides the Laspeyres and Paasche indexes. For example, the US Bureau of Labor Statistics (BLS) adopted a geometric mean formula for the construction of elementary aggregates of the Consumer Price Index (CPI) in 1999, and plans to start publication of a Törnqvist index of consumer prices in 2002. Moreover, in 1996 the US Bureau of Economic Analysis (BEA) began to use a chained Fisher index for the calculation of real GDP.

With these more complicated index formulas, how to calculate additive contributions to the change in the index is no longer obvious. To choose among the many possible additive decompositions that exist for these indexes, we seek solutions that have important desirable properties. We use economic and axiomatic approaches to derive additive decompositions of changes in Fisher indexes. We next use an axiomatic approach to derive an additive decomposition of the change in the T örnqvist and other geometric mean indexes.

2. An economic approach to additive decomposition of the fisher index

The Fisher quantity index $Q_F = [(\mathbf{p}_0'\mathbf{q}_1/\mathbf{p}_0'\mathbf{q}_0)(\mathbf{p}_1'\mathbf{q}_1/\mathbf{p}_1'\mathbf{q}_0)]^{1/2}$ is a geometric mean of the Laspeyres and Paasche indexes. Diewert [3] shows that Q_F measures the change in a flexible utility or production function that has the following functional form:

$$f(\mathbf{q}) = [\Sigma_i \Sigma_j a_{ij} q_i q_j]^{1/2} \tag{1}$$

where $a_{ij} = a_{ji}$. That is, $Q_F = f(\mathbf{q}_1)/f(\mathbf{q}_0)$.

The change in a Fisher quantity index has an additive decomposition that reflects contributions of individual quantity changes to the change in the level of the production or utility function $f(\cdot)$. Moreover, an analogous additive decomposition exists for the change in a Fisher price index.

Let $g(\mathbf{q}) = [f(\mathbf{q})]^2$. Then:

$$g(\mathbf{q}) = \Sigma_i \Sigma_j a_{ij} q_i q_j \tag{2}$$

The derivative of $g(\mathbf{q})$ with respect to $\mathbf{q}$, $g_q(\mathbf{q})$, equals $2f(\mathbf{q})f_q(\mathbf{q})$. Since Eq. (2) is a quadratic, the Quadratic Approximation Lemma [3] implies that the change in $g(\cdot)$ as $\mathbf{q}$ changes from $\mathbf{q}_0$ to $\mathbf{q}_1$ the change in $g(\mathbf{q})$ equals the average of the derivatives at the two endpoints times the change in $\mathbf{q}$:

$$g(\mathbf{q}_1) - g(\mathbf{q}_0) = \frac{1}{2}[g_q(\mathbf{q}_0) + g_q(\mathbf{q}_1)]'(\mathbf{q}_1 - \mathbf{q}_0). \tag{3}$$

To rewrite Eq. (3) in terms of observed prices and quantities, note that $f(\mathbf{q})$ is linear homogeneous. Hence, assuming cost minimizing behavior, Wold's theorem implies that:

$$f_q(\mathbf{q}_0)/f(\mathbf{q}_0) = \mathbf{p}_0/\mathbf{p}_0'\mathbf{q}_0 \tag{4}$$

and similarly for $f_q(\mathbf{q}_1)/f(\mathbf{q}_1)$. In addition, since $f(\mathbf{q}_0) = [g(\mathbf{q}_0)]^{1/2}$, $f_q(\mathbf{q}_0) = \frac{1}{2f(\mathbf{q}_0)} g_q(\mathbf{q}_0)$, or

$$\begin{aligned} g_q(\mathbf{q}_0) &= 2f(\mathbf{q}_0)f_q(\mathbf{q}_0) \\ &= 2\mathbf{p}_0[f(\mathbf{q}_0)]^2/\mathbf{p}_0 \cdot \mathbf{q}_0. \end{aligned} \tag{5}$$

Substituting Eq. (5) and the analogous expression for $g_q(\mathbf{q}_1)$ into Eq. (3) then gives:

$$[f(\mathbf{q}_1)]^2 - [f(\mathbf{q}_0)]^2 = [\mathbf{p}_0[f(\mathbf{q}_0)]^2/\mathbf{p}_0'\mathbf{q}_0 + \mathbf{p}_1[f(\mathbf{q}_1)]^2/\mathbf{p}_1'\mathbf{q}_1]'(\mathbf{q}_1 - \mathbf{q}_0). \quad (6)$$

Factoring the left side of Eq. (6), we obtain:

$$f(\mathbf{q}_1) - f(\mathbf{q}_0) = \frac{[\mathbf{p}_0[f(\mathbf{q}_0]^2/\mathbf{p}_0'\mathbf{q}_0 + \mathbf{p}_1[f(\mathbf{q}_1]^2/\mathbf{p}_1'\mathbf{q}_1]'(\mathbf{q}_1 - \mathbf{q}_0)}{f(\mathbf{q}_1 + f(\mathbf{q}_0)} \quad (7)$$

$$\frac{f(\mathbf{q}_1)}{f(\mathbf{q}_0)} - 1 = \frac{\{\mathbf{p}_0 + \mathbf{p}_1[f(\mathbf{q}_1/f(\mathbf{q}_0]^2[\mathbf{p}_0'\mathbf{q}_0/\mathbf{p}_1'\mathbf{q}_1]\}'(\mathbf{q}_1 - \mathbf{q}_0)}{(\mathbf{p}_0'\mathbf{q}_0)[1 + f(\mathbf{q}_1)/f(\mathbf{q}_0)]} \quad (8)$$

The Fisher price index P_F equals $[(\mathbf{p}_1'\mathbf{q}_0/\mathbf{p}_0'\mathbf{q}_0)(\mathbf{p}_1'\mathbf{q}_1/\mathbf{p}_0'\mathbf{q}_1)]^{1/2}$. Because the Fisher index formula satisfies Fisher's "factor reversal" test $P_F Q_F = \mathbf{p}_1'\mathbf{q}_1/\mathbf{p}_0'\mathbf{q}_0$. Substituting Q_F for $f(\mathbf{q}_1)/f(\mathbf{q}_0)$ and simplifying therefore gives:

$$Q_F - 1 = \frac{[\mathbf{p}_0 + \mathbf{p}_1(Q_f/P_F)]'(\mathbf{q}_1 - \mathbf{q}_0)}{(\mathbf{p}_0'\mathbf{q}_0)[1 + Q_F]} \quad (9)$$

Hence, the contribution of the ith item to the change in the Fisher quantity index is:

$$\frac{p_{i0} + [p_{i1}/P_F]Q_F}{\mathbf{p}_0'\mathbf{q}_0 + (\mathbf{p}_0'\mathbf{q}_0)Q_F}(q_{i1} - q_{i0}). \quad (10)$$

An analogous expression exists for the contribution of the ith item to the change in the Fisher price index P_F. Diewert [3] shows that P_F measures the change in the cost function or expenditure function $[\Sigma_i \Sigma_j b_{ij} p_i p_j]^{1/2}$. Since the square of this function is quadratic, its average derivative furnishes the basis for an additive decomposition of the change in P_F in the same way that the average derivative of the squared production or utility function does for the change in Q_F. The contribution of the ith item to the change in the Fisher price index is then:

$$\frac{q_{i0} + q_{i1}(P_F/Q_F)}{\mathbf{p}_0'\mathbf{q}_0 + (\mathbf{p}_0'\mathbf{q}_0)P_F}(p_{i1} - p_{i0}). \quad (11)$$

3. An axiomatic approach to additive decomposition of the fisher index

Christian Ehemann [7] has derived a different solution to the problem of decomposing the change in the Fisher quantity index.[1] The Bureau of Economic Analysis

[1] van IJzeren [11, p. 108–110] [12, p. 5–6] derives the same formula that Ehemann does, but in different ways and in contexts of different problems. Yuri Dikhanov [6] has also suggested this solution.

(BEA) uses this solution to calculate contributions to percent change in real GDP, and contributions to percent change in major components of real GDP such as real personal consumption expenditures [8, p. 16].

The Fisher quantity index is an average of a Laspeyres index, which values the quantity change for the ith item at p_{i0}, and a Paasche index, which values its quantity change at price p_{i1}. An additive decomposition for the change in a Fisher quantity index that uses a weighted average of p_{i0} and p_{i1} to value the ith quantity change can, therefore, be justified on axiomatic grounds. Assuming that all prices are positive, the contributions to the change in Fisher index based on this average price will be bounded by the contributions to the change of the Laspeyres and Paasche indexes given above, just as the Fisher index itself is bounded by the Laspeyres and Paasche indexes.

A weighted average of the Laspeyres and Paasche decompositions that provides additive contributions to the change in a Fisher quantity index exists if there is a λ such that:

$$Q_F - 1 = \frac{\Sigma_i (q_{i1} - q_{i0})(p_{i0} + \lambda p_{i1})}{\Sigma_i q_{i0}(p_{i0} + \lambda p_{i1})}. \tag{12}$$

The solution for λ reveals that the contribution of each quantity change to the change in Q_F equals the amount of the change times an average of the item's period 0 price and its period 1 prices deflated by the overall Fisher price index . To obtain this solution, rewrite (12) as:

$$Q_F[\Sigma_i q_{i0}(p_{i0} + \lambda p_{i1})] = \Sigma_i q_{i1}(p_{i0} + \lambda p_{i1}). \tag{13}$$

Collecting terms that multiply λ gives:

$$Q_F[\Sigma_i p_{i0} q_{i0}] - \Sigma_i p_{i0} q_{i1} = \lambda\{\Sigma_i p_{i1} q_{i1} - Q_F[\Sigma_i p_{i1} q_{i0}]\}. \tag{14}$$

Therefore,

$$\lambda = \frac{Q_F[\Sigma_i p_{i0} q_{i0}] - \Sigma_i p_{i0} q_{i1}}{\Sigma_i p_{i1} q_{i1} - Q_F[\Sigma_i p_{i1} q_{i0}]}. \tag{15}$$

Recall that $P_F Q_F = \mathbf{p}_1' \mathbf{q}_1 / \mathbf{p}_0' \mathbf{q}_0$. Hence dividing the numerator and denominator of Eq. (15) by $Q_F[\Sigma_i p_{i0} q_{i0}]$ yields:

$$\lambda = \frac{1}{P_F} \frac{1 - Q_L/Q_F}{1 - P_L/P_F} \tag{16}$$

Any value of λ will solve Eq. (12) if the Fisher and Laspeyres indexes are identical. Otherwise, the numerator and denominator of the second fraction on the right side of Eq. (16) cancel out because

$$[Q_L/Q_F]^2 = [\mathbf{p}_0' \mathbf{q}_1 / \mathbf{p}_0' \mathbf{q}_0][\mathbf{p}_1' \mathbf{q}_0 / \mathbf{p}_1' \mathbf{q}_1]$$

and

$$[P_L/P_F]^2 = [\mathbf{p}_1'\mathbf{q}_0/\mathbf{p}_0'\mathbf{q}_0][\mathbf{p}_0'\mathbf{q}_1/\mathbf{p}_1'\mathbf{q}_1].$$

Equation (16) therefore simplifies to $\lambda = 1/P_F$. Substituting $1/P_F$ for λ in Eq. (12), the contribution of the ith item to the change in the Fisher quantity index is:

$$\frac{p_{i0} + p_{i1}/P_F}{\mathbf{p}_0'\mathbf{q}_0 + (\mathbf{p}_1'\mathbf{q}_0)/P_F}(q_{i1} - q_{i0}). \tag{17}$$

4. Similarity of the axiomatic and economic decompositions of the Fisher index

The appendix shows that the axiomatic decomposition in Eq. (17) and the economic decomposition in Eq. (10) approximate each other to the second order. This approximation property is not surprising in light of the similarity of the two formulas. Equation (10) differs from Eq. (17) mainly by placing more weight on the prices from the period with the higher level of consumption as measured by the quantity index Q_F. This difference in weighting has a negligible effect unless quantities and relative prices undergo large changes.

As a numerical check of the agreement between the economic and the axiomatic decompositions, we used both formulas to calculate contributions to percent change in real personal consumption expenditures in every year from 1987 to 1998. (Table 8.3 in the Survey of Current Business, August 2001, page 109, displays such contributions for 1997 to 2000.) The values of the two formulas never differed by as much as 0.001 percentage points. Both the analytical results in the appendix and the numerical tests therefore indicate that the axiomatic decomposition of the change in the Fisher index is also a satisfactory measure of the contributions of individual quantity changes to the change in the production or utility function.

5. Additive decompositions for geometric mean and Törnqvist indexes

The Törnqvist index is a kind of geometric mean (or "log change") index. A geometric mean price index P_G has the form:

$$P_G \equiv \prod_i (p_{i1}/p_{i0})^{\sigma_i}, \tag{18}$$

where the σ_i are positive and sum to 1. In the case of the Törnqvist price index P_T, σ_i equals the average of the time 0 and time 1 expenditure shares for item i, or $0.5(p_{i0}q_{i0}/\mathbf{p}_0'\mathbf{q}_0 + p_{i1}q_{i1}/\mathbf{p}_1'\mathbf{q}_1)$.[2]

[2] The exposition can be done either in terms of a price index or in terms of a quantity index. In this section we use a price index because the Bureau of Labor Statistics plans to use a Törnqvist formula to construct an alternative consumer price index. In the previous section we focused on the quantity index because the Bureau of Economic Analysis publishes contributions to change for the Fisher quantity index.

Diewert [3] shows that the Törnqvist price index measures the change in a cost or expenditure function that has the translog functional form, and that the Törnqvist quantity index measures the change in a translog utility or production function. The translog function cannot be transformed into a function that is quadratic in its original price or quantity arguments. Hence, the translog production (or cost) function lacks an additive decomposition that would correspond to the decomposition of the function underlying the Fisher index.[3]

Seeking an additive decomposition that lies between the Laspeyres and Paasche decompositions is also not a useful approach in the case of the Törnqvist index. The Törnqvist index cannot have a decomposition with this property because it is not bounded by the Laspeyres and Paasche indexes.

The Törnqvist index is, however, approximately bounded by the Laspeyres and Paasche indexes. Furthermore, it has an additive decomposition that is approximately bounded by the Laspeyres and Paasche decompositions. This decomposition also has the property that when the additive change in the Törnqvist index equals its logarithmic change – which occurs when price decreases for some items exactly offset the effect of price increases for others so that $P_T - 1 = \log P_T = 0$ – the additive decomposition of $P_T - 1$ is the same as the additive decomposition of log P_T. That is, when $P_T = 1$, the contribution of item i to the change in the Törnqvist index equals its weight in the Törnqvist formula σ_i times its logarithmic price change $\log(p_{i1}/p_{i0})$.

The decomposition with these properties weights the change in price of the ith item in a Törnqvist price index by the item's average expenditure share divided by the logarithmic mean of p_{i0} and p_{i1}/P_T. The logarithmic mean function $m(a, b)$ is defined for positive a and b as $(a - b)/(\log a - \log b)$, or as a if $a = b$. Properties of this function are discussed in Reinsdorf [9, p. 104] and Törnqvist et al. [10].

Although its property of being approximately bounded by the Laspeyres and Paasche decompositions depends on whether the Törnqvist formula is used to calculate the σ_i, our decomposition does not depend on the method of calculation of the σ_i. Hence, it applies to geometric mean indexes in general and not just to P_T.

Let $q_i^* \equiv \sigma_i/m(p_{i1}/P_G, p_{i0})$ or $\sigma_i/m(p_{i1}, P_G p_{i0})$ and let $\mathbf{q}^*$ be the vector of the q_i^*. Then:

$$P_G - 1 = \frac{\Sigma_i (p_{i1} - p_{i0}) q_i^*}{\mathbf{p}_0' \mathbf{q}^*}. \tag{19}$$

Equation (19) is equivalent to:

$$P_G = \frac{\Sigma_i p_{i1} \sigma_i / m(p_{i1}, P_G p_{i0})}{\Sigma_i p_{i0} \sigma_i / m(p_{i1}, P_G p_{i0})}. \tag{20}$$

[3] Diewert [5, p. 17] shows, however, that the Törnqvist index does have a multiplicative decomposition that reflects contributions to change in its underlying cost or production function.

To demonstrate that the right hand side of Eq. (20) can be simplified to P_G, let $r_i \equiv p_{i1}/p_{i0}$. If $r_i = P_G$ for all i, the denominator of Eq. (20) simplifies to $1/P_G$ and the numerator of Eq. (20) simplifies to 1. Otherwise, the denominator of Eq. (20) can be written as:

$$\Sigma_i p_{i0}\sigma_i/m(p_{i1}, P_G p_{i0}) = \Sigma_i \sigma_i \frac{\log r_i - \log P_G}{r_i - P_G}. \tag{21}$$

Similarly, the numerator of Eq. (20) can be written as:

$$\Sigma_i p_{i1}\sigma_i/m(p_{i1}, P_G p_{i0}) = \Sigma_i \sigma_i \frac{r_i(\log r_i - \log P_G)}{r_i - P_G}. \tag{22}$$

To factor P_G out of the right side of Eq. (22), add and also subtract $P_G \log r_i/(r_i - P_G)$ and then substitute log P_G for $\Sigma_i \sigma_i \log r_i$:

$$\begin{aligned} &\Sigma_i \sigma_i \frac{r_i(\log r_i - \log P_G)}{r_i - P_G} \\ &= \Sigma_i \sigma_i \frac{P_G \log r_i + (r_i - P_G)\log r_i}{r_i - P_G} - \log P_G \Sigma_i \sigma_i \frac{r_i}{r_i - P_G} \\ &= \Sigma_i \sigma_i \frac{P_G \log r_i}{r_i - P_G} + \log P_G - \log P_G \Sigma_i \sigma_i \frac{r_i}{r_i - P_G}. \end{aligned} \tag{23}$$

Next, collect terms on $\log P_G$:

$$\begin{aligned} &\Sigma_i \sigma_i \frac{P_G \log r_i}{r_i - P_G} + \log P_G - \log P_G \Sigma_i \sigma_i \frac{r_i}{r_i - P_G} \\ &= \Sigma_i \sigma_i \frac{P_G \log r_i}{r_i - P_G} - \log P_G \Sigma_i \sigma_i \left[\frac{r_i}{r_i - P_G} - \frac{r_i - P_G}{r_i - P_G}\right] \\ &= \Sigma_i \sigma_i \frac{P_G \log r_i}{r_i - P_G} - \log P_G \Sigma_i \sigma_i \frac{P_G}{r_i - P_G} \\ &= P_G \Sigma_i \sigma_i \frac{\log r_i - \log P_G}{r_i - P_G}. \end{aligned} \tag{24}$$

Finally, divide Eq. (24) by the right side of Eq. (21) and cancel like terms to obtain:

$$\frac{\Sigma_i p_{i1}\sigma_i/m(p_{i1}, P_G p_{i0})}{\Sigma_i p_{i0}\sigma_i/m(p_{i1}, P_G p_{i0})} = P_G. \tag{25}$$

6. Decomposition of the geometric mean index version of the Fisher index

Equation (20) implies that Laspeyres and Paasche indexes can be written in the form of geometric mean indexes. Let P_L denote the Laspeyres price index. Substitute

P_L for P_G in Eq. (20) and solve $\sigma_i/m(p_{i1}, P_G p_{i0}) = \lambda q_{i0}$ where λ is a constant of proportionality such that the σ_i sum to 1:

$$\sigma_i = \lambda q_{i0} m(p_{i1}, P_L p_{i0}) = \frac{s_{i0} m(r_i, P_L)}{\Sigma_j s_{j0} m(r_j, P_L)} \tag{26}$$

where $s_{i0} = p_{i0} q_{i0}/\mathbf{p}_0' \mathbf{q}_0$, the budget share of item i in period 0. The Laspeyres price index therefore weights the ith log price change in proportion to the logarithmic mean of its own value and the value of P_L times s_{i0}:

$$\log P_L = \Sigma_i \frac{s_{i0} m(r_i, P_L)}{\Sigma_j s_{j0} m(r_j, P_L)} \log r_i \tag{27}$$

A similar substitution of q_{i1} into Eq. (20) shows that the Paasche price index $P_P = [\Sigma_i s_{i1}(1/r_i)]^{-1}$ has a logarithm of:

$$\log P_P = \Sigma_i \frac{s_{i1} m(1/r_i, 1/P_P)}{\Sigma_j s_{j1} m(1/r_j, 1/P_P)} \log r_i. \tag{28}$$

Averaging the weights on the $\log r_i$ in Eqs (27) and (28) gives the σ_i that allow the Fisher index to be expressed in the form of a geometric mean index. These σ_i can, in turn, be transformed into a third possible additive decomposition of the Fisher index using Eq. (20). Experimentation shows that this decomposition generally matches the decomposition implied by Eq. (17), which averages q_{i0} and q_{i1}/Q_F, to five or more decimal places. The two decompositions seem, therefore, to be numerically equivalent to each other. Note however, that whereas negative quantities pose no problem for Eq. (17), the geometric decomposition of a Fisher quantity index exists only when all values are positive.

7. Conclusion

Since the change in a Fisher index is not a simple combination of changes in its constituent items, how to calculate additive contributions of individual changes to the overall change in the index is not obvious. This paper has provided three formulas for additive decompositions for the change in the Fisher index. Fortunately, they all give very similar answers. Hence, the method that BEA uses for calculating contribution to change in real GDP can be justified on both axiomatic and economic grounds.

This paper also provides a method for calculating contributions of individual changes in prices (or quantities) to the change in a Törnqvist or geometric mean price (or quantity) index. We hope that economic researchers and statistical agencies will make use of our results.

Appendix

In this Appendix, we show that the economic and axiomatic percentage change decompositions for the Fisher ideal quantity index approximate each other to the second order around any point where the period 0 and 1 price vectors are equal (so that $\mathbf{p}_0 = \mathbf{p}_1 = \mathbf{p} \equiv (p_1, p_2, \ldots, p_n)'$ say) and where the period 0 and 1 quantity vectors are equal (so that $\mathbf{q}_0 = \mathbf{q}_1 = \mathbf{q} \equiv (q_1, q_2, \ldots, q_n)'$ say). Thus for "normal" time series data, where the change in prices and quantities are small going from one period to the next, we would expect the economic and axiomatic decompositions to be quite close to each other.

Define the ith term in the economic decomposition of the Fisher quantity index to be

$$E_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1) \equiv e_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)(q_{i1} - q_{i0)}; \quad i = 1, 2, \ldots, n \tag{A1}$$

where the function e_i is defined as follows:

$$e_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1) \equiv \{p_{i0} + p_{i1} Q_F [P_F]^{-1}\}/\mathbf{p}_0'\mathbf{q}_0 + \mathbf{p}_0'\mathbf{q}_0 Q_F\}; \quad i = 1, 2, \ldots, n. \tag{A2}$$

Recall that $P_F = P_F(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)$ and $Q_F = Q_F(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)$ are the Fisher price and quantity indexes respectively. Define the ith term in the axiomatic decomposition of the Fisher quantity index to be

$$A_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1) \equiv a_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)(q_{i1} - q_{i0}); \quad i = 1, 2, \ldots, n \tag{A3}$$

where the function ai is defined as follows:

$$a_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1) \equiv \{p_{i0} + p_{i1}[P_F]^{-1}\}/\{\mathbf{p}_0'\mathbf{q}_0 + \mathbf{p}_1'\mathbf{q}_0[P_F]^{-1}\}; \quad i = 1, 2, \ldots, n. \tag{A4}$$

Applying Lemma 2 in Diewert [4, p. 887], it can be seen that $A_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)$ will approximate $E_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)$ to the second order around a point where the two price vectors are equal and the two quantity vectors are equal provided that $a_i(\mathbf{p}, \mathbf{p}, \mathbf{q}, \mathbf{q}) = e_i(\mathbf{p}, \mathbf{p}, \mathbf{q}, \mathbf{q})$ for each i and all first order partial derivatives of $a_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)$ equal the corresponding derivatives of $e_i(\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}_0, \mathbf{q}_1)$ when both functions are evaluated at $\mathbf{p}_0 = \mathbf{p}_1 = \mathbf{p}$ and $\mathbf{q}_0 = \mathbf{q}_1 = \mathbf{q}$. Routine calculations show that:

$$a_i(\mathbf{p}, \mathbf{p}, \mathbf{q}, \mathbf{q}) = e_i(\mathbf{p}, \mathbf{p}, \mathbf{q}, \mathbf{q}) = p_i/\mathbf{p}'\mathbf{q} \tag{A5}$$

$$\begin{aligned} \partial a_i(\mathbf{p}, \mathbf{p}, \mathbf{q}, \mathbf{q})/\partial p_{i0} &= \partial e_i(\mathbf{p}, \mathbf{p}, \mathbf{q}, \mathbf{q})/\partial p_{i0} \\ &= \{1/2\mathbf{p}'\mathbf{q}\} - \{p_i q_i/2(\mathbf{p}'\mathbf{q})^2\}; \end{aligned} \tag{A6}$$

$$\partial a_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{i1} = \partial e_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{i1} \\ = \{1/2\mathbf{p}'\mathbf{q}\} - \{p_i q_i/2(\mathbf{p}'\mathbf{q})^2\}; \tag{A7}$$

$$\partial a_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial q_{i0} = \partial e_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial q_{i0} = -\{p_{i2}/2(\mathbf{p}'\mathbf{q})^2\}; \tag{A8}$$

$$\partial a_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial q_{i1} = \partial e_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q}/\partial q_{i1} = 0 \tag{A9}$$

$$\partial a_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{j0} = \partial e_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{j0} = -\{p_i q_j/2(\mathbf{p}'\mathbf{q})^2\}; \\ i \neq j; \tag{A10}$$

$$\partial a_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{j1} = \partial e_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{j1} = -\{p_i q_j/2(\mathbf{p}'\mathbf{q})^2\}; \\ i \neq j; \tag{A11}$$

$$\partial a_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{j0} = \partial e_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial p_{j0} = -\{p_i q_j/(\mathbf{p}'\mathbf{q})^2\}; \\ i \neq j; \tag{A12}$$

$$\partial a_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q})/\partial q_{i1} = \partial e_i(\mathbf{p},\mathbf{p},\mathbf{q},\mathbf{q}/\partial q_{i1} = 0; \quad i \neq j. \tag{A13}$$

Thus Lemma 2 in Diewert [4] and Eqs (A5)–(A13) show that the axiomatic and economic decompositions of the Fisher quantity index approximate each other to the second order around an equal price and quantity point. An analogous result holds for the economic and axiomatic decompositions for the Fisher price index due to the symmetry of the Fisher price and quantity indexes; the levels and derivatives of the price decomposition terms at an equal price and quantity point are the same as Eqs (A5)–(A13) except that the role of prices and quantities is permuted.

References

[1] J. Dalén, Computing Elementary Aggregates in the Swedish Consumer Price Index, *Journal of Official Statistics* **8** (1992), 129–147.

[2] J. Dalén, Reply, *Journal of Official Statistics* **10** (1994), 107–108.

[3] W.E. Diewert, Exact and Superlative Index Numbers, *Journal of Econometrics* **4** (1976), 114–145. Reprinted in *Essays in Index Number Theory,* (Vol. 1), W.E. Diewert and A.O. Nakamura, eds, Elsevier Science Publishers, Amsterdam, 1993.

[4] W.E. Diewert, Superlative Index Numbers and Consistency in Aggregation, *Econometrica* **46** (1978), 883–900. Reprinted in *Essays in Index Number Theory,* (Vol. 1), W.E. Diewert and A.O. Nakamura, eds, Elsevier Science Publishers, Amsterdam, 1993.

[5] W.E. Diewert, The Quadratic Approximation Lemma and Decompositions of Superlative Indexes, Discussion Paper 00-15, Department of Economics, University of British Columbia, Vancouver, Canada, V6T 1Z1, 2000.

[6] Y. Dikhanov, The Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures, mimeo, International Economics Department, The World Bank, Washington, DC, January 1997.

[7] C. Ehemann, Analyzing the Chained-Dollar Measure of Output: Contributions of Components to Level and Change. Mimeo, Bureau of Economic Analysis, Washington, DC, 1997.

[8] B.R. Moulton and E.P. Seskin, A Preview of the 1999 Comprehensive Revision of the National Income and Product Accounts Statistical Changes, *Survey of Current Business* **79**(10) (October 1999), 6–17.

[9] M.B. Reinsdorf, A New Functional Form for Price Indexes' Elementary Aggregates, *Journal of Official Statistics* **10** (1994), 103–108.

[10] L. Törnqvist, P. Vartia and Y.O. Vartia, How Should Relative Changes Be Measured? *The American Statistician* **39** (1985), 43–46.

[11] J. van IJzeren, Over de Plausibiliteit van Fisher's Ideale Indices, (On the Plausibility of Fisher's Ideal Indices), Statistische en Econometrische Onderzoekingen (Centraal Bureau voor de Statistiek), *Nieuwe Reeks* **7**(2) (1952), 104–115.

[12] J. van IJzeren, *Bias in International Index Numbers: A Mathematical Elucidation,* Eindhoven, the Netherlands, 1987.