EXACT AND SUPERLATIVE WELFARE CHANGE INDICATORS

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An exact welfare change indicator is defined to be a known function of the price and quantity data for two periods (for a utility maximizing consumer) which is exactly equal to the Hicksian equivalent variation. A welfare change indicator is termed superlative if it is exactly equal to the equivalent variation for an expenditure function which has a second-order approximation property.

The paper exhibits a number of superlative welfare change indicators, and also reviews the earlier attempts of Hicks and Weitzman to obtain equivalent variation measures that had a second-order approximation property.

I. INTRODUCTION

For a large number of policy purposes, it is useful to have an ex post measure of welfare change for a single consumer (or for a group of homogeneous consumers) that depends only on the observed price and quantity data for the two time periods under consideration. More specifically, suppose the consumer faces the price vector $p^t \equiv (p_1^t,\ldots,p_N^t)$ in period t and consumes the quantity vector $x^t = (x_1^t,\ldots,x_N^t)$ in period t for $t = 0,1$. We would like to have an accurate measure of welfare change, $W(p^0,p^1,x^0,x^1)$, that depends only on the observed price and quantity vectors for the two periods.

The above problem has been addressed by many economists over the years. Perhaps the first contributors to the subject were Dupuit [1844] and Marshall [1890] who developed the familiar consumer surplus (or area under the demand curve) measure of welfare change for a single market. This single-market measure of welfare change was generalized to the many market case by Hotelling [1938, 253–4], Hicks [1941–42, 134; 1945–46, 73] and Harberger [1971, 788] who obtained the following measure of welfare change:[1]

$$(1)\qquad W_H(p^0,p^1,x^0,x^1) \equiv p^0 \cdot (x^1 - x^0) + (1/2)(p^1 - p^0) \cdot (x^1 - x^0) = (1/2)(p^0 + p^1) \cdot (x^1 - x^0)$$

where

$$p \cdot x = \sum_{n=1}^{N} p_n x_n$$

denotes the inner product between the vectors p and x. All three of the above authors justified their welfare change measures as being some sort of quadratic approximation to the true welfare change.

Weitzman [1988] recently re-examined the exact nature of the approximation

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1. Hotelling did not directly obtain the welfare change measure (1), but it can be deduced from his work with some additional assumptions. As shall be seen in section II, Hicks [1941-42] derived the right-hand side of (1) as an approximation to an average of his equivalent and compensating variations under the additional hypothesis that $p^0 \cdot x^0 = p^1 \cdot x^1$.

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properties of W_H defined by (1). He noted that there is a problem with the consumer surplus measure defined by (1): the period 1 price vector p^1 needs to be *normalized* in some fashion (to eliminate the effects of inflation for example) in order to make it comparable with the period 0 price vector p^0. Weitzman then went on to show how an appropriately modified version of formula (1), with p^1 replaced by $\theta^{-1}p^1$ where θ is an appropriate scalar deflator, could provide a second-order approximation to a cardinal measure of welfare change, namely equivalent variation.[2]

In section II, I formally define the Hicksian equivalent and compensating variations and provide alternative derivations for the first- and second-order approximations first obtained by Hicks [1941–42].

In section III, I discuss Weitzman's approximation results and provide my own approximation results using his methodological approach.

I shall argue that the second-order approximation results of Hicks and Weitzman are flawed. Hence, in the remainder of the paper, I develop an alternative approach to the derivation of second-order approximations.

The equivalent and compensating variations can be defined in terms of the consumer's expenditure function. The alternative approach to welfare change measurement developed in sections IV and V is analogous to the theory of exact and superlative index numbers.[3] I look for functional forms for the consumer's expenditure function which have the property that the equivalent variation is *exactly* equal to a known function of the observed data p^0,p^1,x^0,x^1 for the two periods under consideration. The known function of the data is called a superlative welfare change indicator if the functional form for the expenditure function is *flexible*.[4]

In section IV, I find a superlative welfare change indicator that is closely related to Irving Fisher's [1922] ideal quantity index. In section V, an entire family of superlative welfare change indicators is found. Each member of this family is exact for a certain normalized quadratic expenditure function.[5]

Section VI shows that the superlative welfare change indicators defined in this paper numerically approximate each other to the second order and hence, for empirical applications, it generally will not matter which indicator is used.[6]

Section VII concludes. Proofs of the theorems and corollaries are given in the appendix.

II. HICKSIAN APPROACHES TO THE MEASUREMENT OF EX POST WELFARE CHANGE

Consider a consumer whose preferences over N goods can be represented by a utility function f. Thus the consumer's level of welfare is $u = f(x)$ where $x \equiv (x_1, \ldots, x_N)$ is the vector of quantities consumed during the period.

The expenditure function $e(u,p)$ dual to f is defined as

2. The normalization issue has not gone unnoticed. Harberger [1971, 793] suggested that θ could be chosen so that a single numeraire price would be constant in the two periods but his preferred normalization seems to have been $\theta = p^1 \cdot x^1 / p^0 \cdot x^0$. Diewert [1976b, 145] noted that the normalization issue was critical: if the observed data (p^0,p^1,x^0,x^1) are such that we are in Samuelson's [1947, 148-152] region of indeterminancy where neither point is revealed preferred to the other, then W_H can be made positive or negative simply by scaling prices in either period.

3. See Diewert [1976a].

4. More precisely, we ask that the expenditure function be flexible in the class of expenditure functions that exhibit money metric scaling. On the concept of flexibility, see Diewert [1974, 115] and Barnett [1983]. The term money metric scaling is due to Samuelson [1974].

5. See Diewert and Wales [1988a; 1988b] on the properties of the normalized quadratic expenditure function.

6. This property is analogous to a corresponding property possessed by a superlative index number formula; see Diewert [1978].

(2) $e(u,p) = \min_x(p \cdot x : f(x) \geq u).$

Under normal conditions, the expenditure function e can characterize preferences just as well as the utility function f.[7]

Suppose that we can observe the consumption vectors x^0 and x^1 during periods 0 and 1 when the consumer faces prices p^0 and p^1 respectively. Suppose that the consumer engages in expenditure minimizing behaviour during the two periods and u_0 and u_1 denote the welfare levels attained during periods 0 and 1 respectively. Then we have:

(3) $$e(u_t,p^t) = p^t \cdot x^t, \qquad t = 0,1.$$

In addition, if e is differentiable with respect to commodity prices, we have[8]

(4) $$x^0 = \nabla_p e(u_0,p^0) \quad \text{and} \quad x^1 = \nabla_p e(u_1,p^1)$$

where

$$\nabla_p e(u,p) \equiv [\delta e(u,p)/\delta p_1,\ldots,\delta e(u,p)/\delta p_N]$$

denotes the vector of derivatives of $e(u,p)$ with respect to the components of p.

Two measures of the ex post welfare gain going from the period 0 situation to the period 1 situation are the *equivalent variation* defined as

(5) $$V(p^0,x^0,x^1) \equiv e[f(x^1),p^0] - e[f(x^0),p^0] = e(u_1,p^0) - e(u_0,p^0).$$

and the *compensating variation* defined as

(6) $$V(p^1,x^0,x^1) \equiv e[f(x^1),p^1] - e[f(x^0),p^1] = e(u_1,p^1) - e(u_0,p^1).$$

The above two theoretical measures of welfare change use cardinal measures of utility that are proportional to the size of certain budget sets using a fixed set of reference prices. The equivalent variation uses the base period reference prices p^0, and the cardinal measure of utility that corresponds to the consumption vector x is $e[f(x),p^0]$ which, in turn, is the "income" required to just attain the indifference surface through the point x. On the other hand, the compensating variation uses the current period reference prices p^1, and the cardinal measure of utility that corresponds to the consumption vector x is $e[f(x),p^1]$.[9]

The welfare change measures (5) and (6) were first introduced in the $N = 2$ case by Henderson [1940-41, 120] and in the general case by Hicks [1941–42]. The terms compensating and equivalent variation are due to Hicks [1940–41, 110; 1941–42, 128].[10]

The main task of this paper is to obtain an approximation to the welfare change measures (5) and (6) that utilizes only the observed data for the two periods, p^0, p^1, x^0 and x^1.

Under the assumption of optimizing behaviour on the part of the consumer, we have from (3) that $e(u_0,p^0) = p^0 \cdot x^0$ and $e(u_1,p^1) = p^1 \cdot x^1$. First-order approximations to the unobservable terms, $e(u_1,p^0)$ and $e(u_0,p^1)$, can be obtained using Taylor's Theorem. Thus we have the following *linear approximations* to $V(p^0,x^0,x^1)$ and $V(p^1,x^0,x^1)$:

7. See Diewert [1974] for an appropriate set of regularity conditions.

8. See Hicks [1946, 331].

9. Samuelson [1974] refers to such cardinal measures of utility as money metric measures of utility.

10. From Hicks' [1941–42, 128] diagram and definitions, it is clear that Hicks defined the equivalent and compensating variations as (5) and (6). However, later Hicks [1946, 331–32] appears to have defined the compensating variation as $e(u_0,p^1) - e(u_0,p^0)$ and the equivalent variation as $e(u_1,p^1) - e(u_1,p^0)$. Needless to say, these alternative definitions have caused a considerable amount of confusion in the literature over the past forty-five years. I will use the original definition in this paper.

$$V(p^0,x^0,x^1) \cong [e(u_1,p^1) + \nabla_p e(u_1,p^1) \cdot (p^0 - p^1)] - e(u_0,p^0)$$

$$= [p^1 \cdot x^1 + x^1 \cdot (p^0 - p^1)] - p^0 \cdot x^0$$

using (3) and (4)

$$(7) \qquad = p^0 \cdot (x^1 - x^0);$$

$$V(p^1,x^0,x^1) \cong e(u_1,p^1) - [e(u_0,p^0) + \nabla_p e(u_0,p^0) \cdot (p^1 - p^0)]$$

$$= p^1 \cdot x^1 - [p^0 \cdot x^0 + x^0 \cdot (p^1 - p^0)]$$

using (3) and (4)

$$(8) \qquad = p^1 \cdot (x^1 - x^0).$$

Thus the equivalent variation is approximately equal to the sum of the quantity changes weighted by the base period prices p^0, while the compensating variation is approximately equal to the sum of the quantity changes weighted by the current period prices p^1. These linear approximations are essentially due to Hicks [1941–42, 134].[11]

Instead of approximating $e(u_1,p^0)$ and $e(u_0,p^1)$ to the first order, we may use Taylor's Theorem to obtain second-order approximations. This results in the following valid second-order approximations to the equivalent and compensating variations:

$$V(p^0,x^0,x^1) \cong [e(u_1,p^1) + \nabla_p e(u_1,p^1) \cdot (p^0 - p^1) + (1/2)(p^0 - p^1)^T S^1 (p^0 - p^1)] - e(u_0, p^0)$$

$$(9) \qquad = p^0 \cdot (x^1 - x^0) + (1/2)(p^1 - p^0)^T S^1 (p^1 - p^0);$$

$$V(p^1,x^0,x^1) \cong e(u_1,p^1) - [e(u_0,p^0) + \nabla_p e(u_0,p^0) \cdot (p^1 - p^0) + (1/2)(p^1 - p^0)^T S^0 (p^1 - p^0)]$$

$$(10) \qquad = p^1 \cdot (x^1 - x^0) + (1/2)(p^1 - p^0)^T S^0 (p^1 - p^0)$$

where

$$S^t \equiv \nabla^2_{pp} e(u_t,p_t), \qquad t = 0,1,$$

is the matrix of second-order partial derivatives of the expenditure function with respect to prices and $(p^1 - p^0)^T$ denotes the transpose of the column vector $(p^1 - p^0)$. Since expenditure functions are concave in prices, S^t is a negative semidefinite symmetric matrix for $t = 0,1$. In economic terms, S^t is the Slutsky substitution matrix for period t. The quadratic approximations are again essentially due to Hicks although he used somewhat different techniques.[12]

11. Hicks [1941–42, 132] defined the Laspeyres variation algebraically as $LV \equiv -x^0 \cdot (p^1-p^0)$ and the Paasche variation as $PV \equiv -x^1 \cdot (p^1-p^0)$ (although the minus sign is missing on page 132 due to a typographical error). These definitions do not appear to agree with the geometric definitions of LV and PV made by Hicks [1941–42, 127–128] which reduce to $LV \equiv p^1 \cdot (x^1-x^0)$ and $PV \equiv p^0 \cdot (x^1-x^0)$. However, Hicks assumed that expenditure in the two periods was identical so that $p^0 \cdot x^0 = p^1 \cdot x^1$. Using this last equality, it can be verified that the algebraic and geometric definitions of LV and PV coincide. Finally Hicks [1941–42, 134] obtained the following second-order approximations: $V(p^1,x^0,x^1) \approx LV + (1/2)S$ and $V(p^0,x^0,x^1) \approx PV-(1/2)S$ where S is a nonnegative second-order "generalized substitution effect." If we omit the second-order terms, the Hicksian approximations become $CV \equiv V(p^1,x^0,x^1) \approx LV$ which is (8) and $EV \equiv V(p^0,x^0,x^1) \approx PV$ which is (7).

12. See Hicks [1941–42, 133–134] and the previous footnote, but note that Hicks did not distinguish between S^0 and S^1; i.e., his generalized substitution effect was defined as $S \equiv (p^1-p^0)^T \nabla^2_{pp} e(u,p)\,(p^1-p^0)$. Also, in his early work on consumer surplus, Hicks did not have duality theory to draw on to simplify the computations. However, in his later work, Hicks [1946, 331] clearly makes use of the following valid second-order approximation which uses only the expenditure function:

$$e(u_0,p^1) - e(u_0,p^0) \approx \nabla_p e(u_0,p^0) \cdot (p^1-p^0) + (1/2)(p^1-p^0)^T \nabla^2_{pp} e(u_0,p^0)(p^1 - p^0).$$

The problem with the second-order approximations (9) and (10) is that they contain the unobserved substitution matrices S^1 and S^0. However, if we could assume that $S^0 = S^1$, then taking an arithmetic average of the two approximations (9) and (10) would yield the following second-order approximations to the average of the equivalent and compensating variations:

$$(11)\quad (1/2)V(p^0,x^0,x^1) + (1/2)V(p^1,x^0,x^1) \cong (1/2)(p^0 + p^1) \cdot (x^1 - x^0).$$

Note that the right-hand side of (11) is equal to the right-hand side of (1). The approximation (11) was obtained by Hicks [1941–42, 134; 1945–46, 73], and he identified the right-hand side of (11) as a generalization to many markets of the Marshallian change in consumer surplus measure.[13]

Although (11) is a valid first-order approximation, it is not a valid second-order approximation since it rests on the assumption that the substitution matrices S^0 and S^1 be at least approximately equal. Unfortunately, the matrices $S^t = \nabla^2_{pp}e(u_t,p^t)$ are not invariant to changes in the degree of inflation; i.e., since $e(u,p)$ is homogeneous of degree one in prices, the substitution matrices will be homogeneous of degree minus one in prices so that $\nabla^2_{pp}e(u_t,\lambda p^t) = \lambda^{-1}\nabla^2_{pp}e(u_t,p^t)$ for all scalars $\lambda > 0$. Hence, for large differences in the scale of prices in the two periods, it is unlikely that S^0 will equal S^1, even approximately. Moreover, if u_0 and u_1 are far apart, then it is likely that S^0 and S^1 will differ substantially even if prices are approximately constant.[14]

For many purposes in applied welfare economics we do not want an average of the compensating and equivalent variations: we want one or the other (or both).[15] Thus in the remainder of this paper, I will concentrate on the problems involved in obtaining observable approximations to the equivalent variation (5). The problems involved in obtaining approximations for the compensating variation (6) are similar and moreover, as Hicks [1941–42, 128] observed, the compensating variation using the original data is equal to the negative of the equivalent variation using time period permuted data.

Having concluded that the Hicksian approach to obtaining operational second-order approximations to the measurement of welfare change failed, we turn now to the approach pioneered by Weitzman.

III. WEITZMAN'S APPROACH TO THE MEASUREMENT OF WELFARE CHANGE

As was noted in the previous section, the problem with the welfare change measure (1) is that it is not invariant to scale changes in period 1 prices. Weitzman [1988] explicitly addressed this scaling problem in a recent paper. In this section, I shall follow the spirit of his approach and derive a welfare change indicator

13. In the two papers cited, Hicks assumed that $p^0 \cdot x^0 = p^1 \cdot x^1$ and this assumption is needed to derive (11) from his work.

14. Indeed, if preferences are homothetic so that $e(u,p) = uc(1,p)$ and $p^0 = p^1$, then $S^1 = u_1\nabla^2_{pp}e(1,p^1) = (u_1/u_0)S^0$ where $S^0 = u_0\nabla^2_{pp}e(1,p^0)$. Thus if u_1 is very different from u_0, S^0 will generally be very different from S^1.

15. This is the problem with a result due to Diewert [1987] who showed that if the consumer's expenditure function had a certain normalized quadratic functional form and normalized prices $v^i = p^i/p^i \cdot \alpha$ were used in place of prices p^i then the following analogue to (11) held exactly: $(1/2)V(v^0,x^0,x^1)+(1/2)V(v^1,x^0,x^1) = (1/2)(v^0+v^1) \cdot (x^1-x^0)$. The problem is that the welfare change indicator $(1/2)(v^0+v^1) \cdot (x^1-x^0)$ is not even approximately invariant to the investigator's choice of the parameter vector α, which can be arbitrary.

using only the properties of the expenditure function $e(u,p)$.[16]

Recall the quadratic expansion (9) where $e(u_1,p^0)$ was approximated around the point (u_1,p^0). In this section, my strategy will be to approximate $e(u_1,p^0)$ around the point $(u_1,\theta^{-1}p^1)$ where θ is a scalar deflator to be determined later. Since $e(u,p)$ is linearly homogeneous in p, Euler's Theorem on homogeneous functions yields the following equations:

$$(12) \qquad e(u_1,\theta^{-1}p^1) = \theta^{-1}p^1 \cdot x^1;$$

$$(13) \qquad \nabla_p e(u_1,\theta^{-1}p^1) = \nabla_p e(u_1,p^1) = x^1;$$

$$(14) \qquad \nabla^2_{pp} e(u_1,\theta^{-1}p^1) = \theta\nabla^2_{pp} e(u_1,p^1);$$

$$(15) \qquad \nabla^2_{pu} e(u_1,\theta^{-1}p^1) = \nabla^2_{pu} e(u_1,p^1);$$

where we have used (3) in (12), (4) in (13) and

$$\nabla^2_{pu} e(u_1,p^1) \equiv [\partial^2 e(u_1,p^1)/\partial p_1 \partial u, \ldots, \partial^2 e(u_1,p^1)/\partial p_N \partial u].$$

As a preliminary result, we approximate the vector of price derivatives $\nabla_p e(u_0,p^0)$ to the first order around the point $(u_1,\theta^{-1}p^1)$:

$$(16) \qquad \nabla_p e(u_0,p^0) \cong \nabla_p e(u_1,\theta^{-1}p^1) + \nabla^2_{pp} e(u_1,\theta^{-1}p^1)(p^0 - \theta^{-1}p^1) + \nabla^2_{pu} e(u_1,\theta^{-1}p^1)(u_0 - u_1).$$

Premultiply both sides of (16) by the row vector $(p^0 - \theta^{-1}p^1)^T$. Using (4) and (13), the resulting equation yields the following first-order approximation:

$$(17) \qquad (p^0 - \theta^{-1}p^1)^T \nabla^2_{pp} e(u_1,\theta^{-1}p^1) \cdot (p^0 - \theta^{-1}p^1) \cong (p^0 - \theta^{-1}p^1) \cdot (x^0 - x^1) - (p^0 - \theta^{-1}p^1) \cdot \nabla^2_{pu} e(u_1,\theta^{-1}p^1)(u_0 - u_1).$$

Recall (9), our earlier second-order approximation to the Hicksian equivalent variation $V(p^0,x^0,x^1)$. Instead of approximating $e(u_1,p^0)$ to the second order around the point (u_1,p^1), we shall now approximate it around the point $(u_1,\theta^{-1}p^1)$ and obtain the following second-order approximation:

$$(18) \qquad V(p^0,x^0,x^1) \cong [e(u_1,\theta^{-1}p^1) + \nabla_p e(u_1,\theta^{-1}p^1)(p^0 - \theta^{-1}p^1) + (1/2)(p^0 - \theta^{-1}p^1)^T \nabla^2_{pp} e(u_1,\theta^{-1}p^1) \cdot (p^0 - \theta^{-1}p^1)] - e(u_0,p^0)$$
$$= p^0 \cdot (x^1 - x^0) + (1/2)(p^0 - \theta^{-1}p^1)^T \nabla^2_{pp} e(u_1,\theta^{-1}p^1)(p^0 - \theta^{-1}p^1)$$

using (3), (12) and (13)

$$(19) \qquad \cong p^0 \cdot (x^1 - x^0) + (1/2)(p^0 - \theta^{-1}p^1) \cdot (x^0 - x^1) - (1/2)(p^0 - \theta^{-1}p^1) \cdot \nabla^2_{pu} e(u_1,\theta^{-1}p^1)(u_0 - u_1)$$

where the first-order approximation (17) was substituted into (18) to obtain (19).

Following Weitzman's [1988, 550] strategy, I shall attempt to choose a deflator θ which will make the last term on the

16. Weitzman [1988, 549] used a mixture of primal and dual techniques; i.e., he defined the indirect compensation function as $w(x) \equiv e[f(x),p^0]$ and made use of the following quadratic expansion using quantities as the independent variables:

$$W(x^1) \cong W(x^0) + \nabla_x W(x^0) \cdot (x^1 - x^0) + (1/2)(x^1 - x^0)^T \nabla^2_{xx} W(x^0)(x^1 - x^0).$$

right-hand side of (19) equal to zero. Using (15) and assuming that $u_0 \neq u_1$, this zero condition leads to the following equation:

$$(20) \qquad (p^0 - \theta^{-1}p^1) \cdot \nabla^2_{pu}e(u_1,p^1) = 0.$$

Unfortunately, in general, we do not know what $\nabla^2_{pu}e(u_1,p^1) = \partial x(u_1,p^1)/\partial u$, the vector of real income derivatives of demand evaluated at (u_1,p^1), is. However, in the *homothetic case* when the following condition is satisfied,

$$(21) \qquad e(u,p) = ue(1,p)$$

for all positive u and p, it can be seen that $\nabla^2_{pu}e(u_1,p^1) = u_1^{-1}x^1$. Thus in the homothetic case, solving (20) leads to the following θ solution:

$$(22) \qquad \theta = p^1 \cdot x^1/p^0 \cdot x^1.$$

Note that θ is the Paasche [1874] price index.

Substituting (22) and (20) back into (19) yields the following approximate measure for the equivalent variation:

$$(23) \qquad V(p^0,x^0,x^1) \equiv p^0 \cdot (x^1 - x^0)$$
$$+ (1/2)[p^0 - (p^0 \cdot x^1/p^1 \cdot x^1)p^1] \cdot [x^0 - x^1]$$
$$= p^0 \cdot x^1 - (1/2)p^0 \cdot x^0$$
$$- (1/2)p^0 \cdot x^1 p^1 \cdot x^0/p^1 \cdot x^1$$
$$= W_P(p^0,p^1,x^0,x^1)$$

where the right-hand side of (23) serves to define the *Weitzman-Paasche welfare change indicator* W_P.

The arguments that Weitzman [1988, 551] used to obtain a welfare change indicator were somewhat different, but he ended up using the right-hand side of (19) after dropping the last term. In the case of homothetic preferences, he recommended the use of the Laspeyres [1871] price index for θ:

$$(24) \qquad \theta \equiv p^1 \cdot x^0/p^0 \cdot x^0.$$

If we substitute (24) into (19) after dropping the last term, we obtain the *Weitzman-Laspeyres welfare change indicator* W_L:

$$(25) \qquad W_L(p^0,p^1,x^0,x^1) \equiv p^0 \cdot (x^1-x^0)$$
$$+ (1/2)[p^0 - (p^0 \cdot x^0/p^1 \cdot x^0)p^1] \cdot [x^0-x^1]$$
$$= [(1/2)(p^0 \cdot x^1/p^0 \cdot x^0)$$
$$+ (1/2)(p^1 \cdot x^1/p^1 \cdot x^0) - 1]p^0 \cdot x^0.$$

Note that $p^0 \cdot x^1/p^0 \cdot x^0$ is the Laspeyres quantity index, $p^1 \cdot x^1/p^1 \cdot x^0$ is the Paasche quantity index and $p^0 \cdot x^0$ is base period expenditure. Hence, if we use the indicator W_L to measure welfare change, the welfare of the consumer increases going from period 0 to period 1 if and only if the arithmetic average of the Paasche and Laspeyres quantity indexes is greater than one.

There are at least two difficulties with using the Weitzman strategy to obtain the concrete welfare change measures W_P and W_L: (i) it is necessary to assume homothetic preferences to obtain a concrete result and (ii) W_P and W_L do not closely approximate each other which casts some doubt on the accuracy of the approximation techniques being used.[17] The problem is that the first-order approximation (16) was used in order to derive an equation (17) which was later used to eliminate a term in the second-order approximation (18). The procedure does not seem to lead to an overall second-order approximation. Similarly, Weitzman [1988, 550] approxi-

17. W_L and W_P do not approximate each other to the second order in a certain sense to be made precise in section VI below.

mates a key term in his second-order approximation by means of a first-order approximation.

In the remainder of the paper, I shall give up on trying to obtain direct second-order approximations to the equivalent variation. Instead, I shall pursue an indirect second-order approximation route. I shall look for functional forms for $e(u,p)$ that can approximate arbitrary preferences to the second order and which are sufficiently simple so that there is an exact formula for the equivalent variation, $e(u_1,p^0) - e(u_0,p^0)$, that can be computed using just the observed price and quantity data, p^0, p^1, x^0 and x^1, for the two periods. I call the resulting objective welfare change measure $W(p^0,p^1,x^0,x^1)$ a superlative welfare change indicator in analogy with the index number literature.[18]

IV. TRANSFORMED QUADRATIC PREFERENCES AND EXACT WELFARE CHANGE MEASURES

Before I present the main result on the measurement of welfare change, it is first necessary to review some material on second-order approximations and flexible functional forms.

If the expenditure function $e(u,p)$ can approximate an arbitrary twice continuously differentiable expenditure function $e^*(u,p)$ to the second order at a point (u_0,p^0), then e must satisfy the following equations:

$$(26)\quad e(u_0,p^0) = e^*(u_0,p^0);$$

1 equation

$$(27)\quad \nabla_u e(u_0,p^0) = \nabla_u e^*(u_0,p^0)$$

1 equation

$$(28)\quad \nabla_p e(u_0,p^0) = \nabla_p e^*(u_0,p^0)$$

N equations

$$(29)\quad \nabla^2_{uu} e(u_0,p^0) = \nabla^2_{uu} e^*(u_0,p^0)$$

1 equation

$$(30)\quad \nabla^2_{pp} e(u_0,p^0) = \nabla^2_{uu} e^*(u_0,p^0)$$

$N\ (N+1)/2$ independent equations

$$(31)\quad \nabla^2_{pu} e(u_0,p^0) = \nabla^2_{pu} e^*(u_0,p^0)$$

N equations.

Since expenditure functions must be homogeneous of degree one in prices, not all of the equations (26)–(31) are independent. If e and e^* are homogeneous of degree one in prices, then the derivatives of these functions will satisfy the following equations:

$$(32)\quad e^*(u_0,p^0) = p^0 \cdot \nabla_p e^*(u_0,p^0);$$

$$e(u_0,p^0) = p^0 \cdot \nabla_p e(u_0,p^0);$$

1 equation

$$(33)\quad \nabla^2_{pp} e^*(u_0,p^0)p^0 = 0_N;$$

$$\nabla^2_{pp} e(u_0,p^0)p^0 = 0_N;$$

N equations

$$(34)\quad \nabla_u e^*(u_0,p^0) = p^0 \cdot \nabla^2_{pu} e^*(u_0,p^0);$$

$$\nabla_u e(u_0,p^0) = p^0 \cdot \nabla^2_{pu} e(u_0,p^0);$$

1 equation.

where 0_N is a zero vector of dimension N.

In section II, we noted that the equivalent variation used the cardinal measure of utility, $e[f(x),p^0]$, which is the income which will allow the consumer to just attain the utility level indexed by the consumption vector x if he or she faces the base period prices $p \equiv (p^0_1,\ldots,p^0_N)$. Since I am going to be using this money metric utility scaling, there is no harm in trans-

18. See Diewert [1976a].

forming the utility scale so that the expenditure function satisfies the following equation:[19]

(35) $e(u,p^0) = u$ for all $u > 0$.

If both e and e^* satisfy the money metric scaling assumption (35), then these functions and their derivatives will satisfy the following restrictions:

(36) $e^*(u_0,p^0) = u_0;$

$e(u_0,p^0) = u_0;$

1 equation

(37) $\nabla_u e^*(u_0,p^0) = 1;$

$\nabla_u e(u_0,p^0) = 1;$

1 equation

(38) $\nabla^2_{uu} e^*(u_0,p^0) = 0;$

$\nabla^2_{uu} e(u_0,p^0) = 0$

1 equation.

Putting all of the above material together, it can be seen that if $e(u,p)$ is twice continuously differentiable at (u_0,p^0), linearly homogeneous in prices and possesses the money metric utility scaling property (35), then in order for e to approximate a differentiable expenditure function e^* (which also satisfies the money metric utility scaling property) to the second order, we need only satisfy N-1 of the equations (28), N-1 of the equations (31) and the $N(N$-1$)/2$ equations in the upper triangle of the N^2 equations (30); the restrictions (32)–(34) and (36)–(38) will imply that the remaining equations in (26)–(31) will be satisfied.[20] If e satisfies the N-1 + N-1 + $N(N$-1$)/2$ equations mentioned above, then we say that the expenditure function e is *flexible*.

Consider the following functional form for an expenditure function e:

(39) $$e(u,p) \equiv [p^T Apu^2 + 2b \cdot pc \cdot p(u-\beta)u-\gamma)]^{1/2}$$

where the matrix of parameters A $[a_{ij}]$ is symmetric, b and c are N dimensional parameter vectors and β and γ are scalar parameters. Note that the functional form defined by (39) is essentially the square root of a quadratic form in prices. Hence I call the resulting expenditure function the transformed quadratic expenditure function. Note that it is homogeneous of degree one in p.

Theorem 1: Let $p^0 >> 0_N$,$u_0 > 0$ and set $\beta = u_0$. Then for every vector c and scalar γ such that

(40) $$c \cdot p^0 \neq 0, \gamma \neq u_0,$$

there exists a symmetric matrix A and a vector b such that

(41) $$b \cdot p^0 = 0$$

and

(42) $$p^{0T} Ap^0 = 1$$

and the resulting transformed quadratic expenditure function e defined by (39) is flexible at (u_0,p^0).

Thus under the very mild restrictions (40), the transformed quadratic expendi-

19. Note the difference between (21) (which characterizes homotheticity) and (35) (which characterizes money metric utility scaling using base period prices p^0 as reference prices): (21) must hold for all prices whereas (35) holds only for $p = p^0$.

20. If $\partial^2 e^*(u_0,p^0)/\partial p_i \partial p_j = \partial^2 e(u_0,p^0)/\partial p_i \partial p_j$ for $1 \leq i < j \leq N$, then by Young's theorem, the same equalities will be satisfied for $1 \leq j < i \leq N$.

ture function e defined by (39), (41) and (42) is flexible at (u_0,p^0) for all c and γ, if β is set equal to u_0.

It turns out that we will be able to relate the transformed quadratic expenditure function defined by (39) to the Fisher [1922] Ideal Quantity Index Q_F which is defined as follows:

$$(43) \qquad Q_F(p^0,p^1,x^0,x^1) \equiv [p^1 \cdot x^1 p^0 \cdot x^1 / p^1 \cdot x^0 p^0 \cdot x^0]^{1/2}.$$

Note that Q_F is the geometric mean of the product of the Paasche quantity index, $Q_P \equiv p^1 \cdot x^1 / p^1 \cdot x^0$, and the Laspeyres quantity index, $Q_L \equiv p^0 \cdot x^1 / p^0 \cdot x^0$.

The Fisher Ideal Quantity Index Q_F defined by (43) and the corresponding Fisher Ideal Price Index defined as $P_F(p^0,p^1,x^0,x^1) \equiv p^1 \cdot x^1 / p^0 \cdot x^0 \; Q_F(p^0,p^1,x^0,x^1)$ have a long and distinguished history in the annals of index number theory. Irving Fisher [1922] attempted to characterize P_F as being the best price index number formula from the viewpoint of the test or axiomatic approach to index number theory, but Eichhorn [1976] and others showed that Fisher's analysis was flawed. However, satisfactory axiomatic characterizations of P_F (or Q_F were finally obtained by Funke and Voeller [1979] and Diewert [1992]. Diewert also showed that the Fisher Ideal Indexes P_F and Q_F seemed best from the viewpoint of the axiomatic approach to index number theory since these indexes satisfy more commonly used tests than any of their leading competitors. A totally different approach to justifying the use of P_F and Q_F based on economic optimizing behavior was provided by Konüs and Byushgens [1926]. Diewert [1976a, 133–37] generalized their approach and showed that the use of P_F and Q_F seemed preferred from the viewpoint of the economic approach to index number theory. One subsidiary purpose of the present paper is to bring to the attention of applied welfare economists this literature on index number theory. The following theorem provides yet another justification for the use of the Fisher quantity index in applied welfare economics.

Theorem 2: Suppose that a consumer faces prices p^t in period t for $t = 0,1$ and engages in expenditure minimizing behaviour during the two periods (so that (3) and (4) hold). Let the (observed) consumption vector in period t be x^t and let the (unobserved) utility level be u_t for $t = 0,1$. Finally, suppose that the consumer's preferences can be represented by a transformed quadratic expenditure function defined by (39) with $\beta = u_0$ and $\gamma = u_1$. Then the ratio of the utility levels is equal to the Fisher quantity index;[21] i.e.,

$$(44) \qquad u^1 / u^0 = Q_F(p^0,p^1,x^0,x^1).$$

Corollary: Suppose that in addition to the hypothesis of the theorem, the vector b satisfies the normalization (41) and the matrix A satisfies the normalization (42). Then the equivalent variation $V(p^0,x^0,x^1)$ defined by (5) is equal to the following expression that depends only on observable price and quantity data:

$$(45) \; V(p^0,x^0,x^1) = [Q_F(p^0,p^1,x^0,x^1)-1]p^0 \cdot x^0 \equiv W_F(p^0,p^1,x^0,x^1).$$

In view of the above results, I shall call the observable indicator of welfare change on the right-hand side of (45) the Fisher Welfare Change Indicator, $W_F(p^0,p^1,x^0,x^1)$. If preferences can be represented by the transformed quadratic expenditure func-

21. If $b = 0_N$ or $c = O_N$, then this result reduces to a theorem established by Konüs and Byushgens [1926] many years ago.

tion e defined by (39), where b and A satisfy (41) and (42) and β and γ satisfy

$$(46) \qquad \beta = u_0, \quad \gamma = u_1,$$

then (45) tells us that the Fisher welfare change indicator is *exactly* equal to the equivalent variation so that we have an *exact* measure of welfare change.

We shall define an exact measure of welfare change to be *superlative* if the functional form for the underlying expenditure function is flexible at the base point (u_0,p^0).[22] Using theorem 1, it can be seen that the expenditure function defined by (39), (41) and (46) is flexible at (u_0,p^0) provided that we choose c to be such that $c \cdot p^0 \neq 0$ (this is no problem) and provided that

$$(47) \qquad u_0 \neq u_1.$$

Unfortunately, (47) need not be satisfied in general. However, if (47) is satisfied so that $u_0 = u_1$, then there is no need for the expenditure function $e(u,p)$ to be completely flexible: all we require is that $e(u,p)$ be able to approximate $e^*(u,p)$ to the second order around the point (u_0,p^0) with respect to the price arguments only. It can be verified that e defined by (39), (41), (42) and (46) does have this restricted flexibility property in the case where $u_0 = u_1$. Thus I shall call the Fisher measure of welfare change W_F defined by the right-hand side of (45) a superlative welfare change indicator, since it is exactly equal to the equivalent variation for an expenditure function which has an appropriate second-order approximation property.

We turn now to another class of preferences which lead to superlative measures of welfare change.

22. This terminology is analogous to that used in index number theory; see Diewert [1976a, 117] and Fisher [1922, 247].

V. NORMALIZED QUADRATIC PREFERENCES AND EXACT WELFARE CHANGE MEASURES

Consider the following *normalized quadratic* functional form for an *expenditure function*:

$$(48) \quad e(u,p) = [b \cdot p + (1/2)(\alpha \cdot p)^{-1}p^TAp]u + c \cdot p(u - \beta)(u - \gamma)$$

where α, b and c are N dimensional parameter vectors, A is a symmetric parameter matrix and β and γ are scalar parameters. If $c = 0_N$, then the expenditure function defined by (48) collapses to the (homothetic) normalized quadratic expenditure function introduced by Diewert and Wales [1988a; 1988b].

The following theorem is a counterpart to theorem 1 above and shows that the normalized quadratic expenditure function has the same approximation properties as the transformed quadratic expenditure function considered in the previous section.

Theorem 3: Let $p^0 \gg 0_N$, $u_0 > 0$ and set $\beta = u_0$. Then for every vector α and scalar γ such that

$$(49) \qquad \alpha \cdot p^0 \neq 0, \gamma \neq u_0,$$

a symmetric matrix A and vectors b and c exist which satisfy the following restrictions:

$$(50) \qquad Ap^0 = 0_n;$$

$$(51) \qquad b \cdot p^0 = 1;$$

$$(52) \qquad c \cdot p^0 = 0$$

and the resulting normalized quadratic expenditure function e defined by (48) is flexible at (u_0,p^0) for an arbitrary twice continuously differentiable expenditure function e^* which has money metric utility scaling at the prices p^0.

Thus under the mild restrictions (49), the normalized quadratic expenditure function e defined by (48) and (50)–(51) is flexible at (u_0,p^0) for all α and γ, if β is set equal to u_0.

The following theorem is a counterpart to theorem 2.

Theorem 4: Let p^0 and p^1 be the positive price vectors faced by an expenditure minimizing consumer in periods 0 and 1 and let x^0 and x^1 be the nonnegative nonzero corresponding (observed) consumption vectors. Let the (unobserved) utility levels in the two periods be $u_0 > 0$ and u_1. Suppose that the consumer's preferences can be represented by a normalized quadratic expenditure function defined by (48) with $\beta = u_0$ and $\gamma = u_1$. Finally, suppose that the parameter vector α satisfies the following restrictions:

$$(53) \qquad \alpha \cdot p^0 \neq 0,\ \alpha \cdot p^1 \neq 0.$$

Then the ratio of the utility levels in the two periods, u_1/u_0, is given by the following observable expression:

$$(54)\ u_1/u_0 = (p^0 \cdot \alpha p^1 \cdot x^1 + p^1 \cdot \alpha p^0 \cdot x^1) / (p^0 \cdot \alpha p^1 \cdot x^0 + p^1 \cdot \alpha p^0 \cdot x^0) \equiv Q_\alpha(p^0,p^1,x^0,x^1)$$

where the normalized quadratic quantity index Q_α is defined by the right-hand side of (54).

Corollary: Suppose that in addition to the hypotheses of the theorem, A, b and c satisfy (50)–(52) so that the expenditure function e satisfies money metric utility scaling at the prices p^0. Then the equivalent variation (5) is equal to the following expression which depends only on the parameter vector α and on the price and quantity data for the two periods:

$$(55)\ V(p^0,x^0,x^1) = [Q_\alpha(p^0,p^1,x^0,x^1)-1]p^0 \cdot x^0 \equiv W_\alpha(p^0,p^1,x^0,x^1)$$

where the quantity index Q_α is defined by the right-hand side of (54).

Thus the normalized quadratic measure of welfare change W_α defined by the right-hand side of (55) is an exact measure of welfare change for each parameter vector which satisfies (53). Is W_α also a superlative welfare change indicator?

Theorem 3 tells us that the answer to the above question is yes provided that $u_0 \neq u_1$ since, in this case, the underlying expenditure function e is flexible. However, in the case where $u_0 = u_1$, I argue, as in the previous section, that there is no need for the expenditure function $e(u,p)$ to be completely flexible: all that is required is that $e(u,p)$ defined by (48), (50)-(51) and $\beta = \gamma \equiv u_0$ be able to approximate an arbitrary expenditure function $e^*(u,p)$ to the second order around the point (u_0p^0) with respect to the price arguments only, and e does have this approximation property. Hence for each nonnegative, nonzero parameter vector α, $W_\alpha(p^0,p^1,x^0,x^1)$ is a superlative welfare change indicator.[23] Thus, we have an entire family of superlative indicators. In the following section, I shall show that the members of this family numerically approximate each other fairly closely.

In order to obtain an explicit measure of welfare change, it will be necessary to make an explicit choice for the parameter

23. These restrictions will ensure that the normalized quadratic expenditure function $e(u,p)$ is well defined for all strictly positive price vectors p. To ensure that $e(u,p)$ is globally concave in p, we also require that the matrix A be negative semidefinite; see Diewert and Wales [1987, 53].

vector α. Two natural choices for α are $\alpha \equiv x^0$ and $\alpha \equiv x^1$. I evaluate W_α under these alternative choices (letting $W_{x0} \equiv W_0$ and $W_{x1} \equiv W_1$):

$$(56) \quad W_0(p^0,p^1,x^0,x^1)$$
$$\equiv [(p^0 \cdot x^0 p^1 \cdot x^1 + p^1 \cdot x^0 p^0 \cdot x^1)$$
$$/(p^0 \cdot x^0 p^1 \cdot x^0 + p^1 \cdot x^0 p^0 \cdot x^0) - 1]p^0 \cdot x^0$$
$$= [(1/2)(p^1 \cdot x^1/p^1 \cdot x^0)$$
$$+ (1/2)(p^0 \cdot x^1/p^0 \cdot x^0) - 1]p^0 \cdot x^0$$
$$= [(1/2)Q_P + (1/2)Q_L - 1]p^0 \cdot x^0;$$
$$W_1(p^0,p^1,x^0,x^1)$$
$$\equiv [(p^0 \cdot x^1 p^1 \cdot x^1 + p^1 \cdot x^1 p^0 \cdot x^1)$$
$$/(p^0 \cdot x^1 p^1 \cdot x^0 + p^1 \cdot x^1 p^0 \cdot x^0) - 1]p^0 \cdot x^0$$
$$(57) \quad = [((1/2)Q^{-1}_P + (1/2)Q^{-1}_L)^{-1} - 1]\, p^0 \cdot x^0$$

where Q_P and Q_L are the Paasche and Laspeyres quantity indexes. Thus the u_1/u_0 term in W_0 is equal to the arithmetic mean of the Paasche and Laspeyres quantity indexes,[24] while the u_1/u_0 term in W_1 is equal to the harmonic mean of the Paasche and Laspeyres quantity indexes.

Recall the Fisher welfare change indicator W_F defined in (45). The u_1/u_0 term in W_F was equal to the Fisher quantity index Q_F which in turn is equal to the geometric mean of the Paasche and Laspeyres quantity indexes. Thus using the properties of means of order r, we have the following numerical inequalities between our three superlative welfare change indicators:[25]

$$(58) \quad W_1(p^0,p^1,x^0,x^1) \leq W_F(p^0,p^1,x^0,x^1)$$
$$\leq W_0(p^0,p^1,x^0,x^1).$$

The results of the following section indicate that the three welfare change indicators listed above in (58) will tend to be numerically very close to each other in empirical applications.

VI. THE NUMERICAL APPROXIMATION PROPERTIES OF SUPERLATIVE INDICATORS

If we examine the Fisher welfare change indicator W_F defined in (45) and the two numerical quadratic welfare change indicators W_0 and W_1 defined by (56) and (57), it can be seen that these three functions have exactly the same form except for the quantity index part which represents u_1/u_0. These quantity indexes are the geometric, arithmetic and harmonic means of the Paasche and Laspeyres quantity indexes, and hence we would expect that if these indicators are evaluated at an equal price and equal quantity point (i.e., if $p^0 = p^1$ and $x^0 = x^1$), then the level, all $4N$ first derivatives and all $(4N)^2$ second-order partial derivatives of W_F, W_0 and W_1 will coincide.[26] The following theorem extends this result by showing that W_F approximates W_α to the second order for all α satisfying (53); not only for $\alpha = x^0$ (recall that this choice of α led to the welfare change indicator W_0) or for $\alpha \equiv x^1$ (which led to W_1).

Theorem 5: Let the Fisher welfare change indicator, $W_F(p^0,p^1,x^0,x^1)$, be defined by the

24. This quantity index corresponds to the price index number formula 8054 listed in Irving Fisher [1922, 487].

25. See Schlömilch's inequality in Hardy, Littlewood and Polya [1934, 26].

26. See Diewert [1978, 897] and note that the geometric, arithmetic and harmonic means of Q_P and Q_L are all symmetric means.

right-hand side of (45) and let the normalized quadratic welfare change indicator conditional on the parameter vector α, $W_\alpha(p^0,p^1,x^0,x^1)$, be defined by the right-hand side of (56).[27] Suppose that the period 1 prices are a multiple of the period 0 prices; i.e., $p^1 = \theta p^0$ where $\theta > 0$. Suppose also that the period 1 quantities are a multiple of the period 0 quantities; i.e., $x^1 = \lambda x^0$ where $\lambda > 0$ is a positive scalar. Then W_F and W_α approximate each other to the second order around such a price quantity point; i.e., we have for every α

$$(59) \quad W_F(p^0,\theta p^0,x^0,\lambda x^0) = W_\alpha(p^0,\theta p^0,x^0,\lambda x^0);$$

$$(60) \quad \nabla W_F(p^0,p^1,x^0,x^1) = \nabla W_\alpha(p^0,p^1,x^0,x^1);$$

$$(61) \quad \nabla^2 W_F(p^0,p^1,x^0,x^1) = \nabla^2 W_\alpha(p^0,p^1,x^0,x^1)$$

where ∇W denotes the $4N$ dimensional vector of first-order derivatives of W with respect to its arguments and $\nabla^2 W$ denotes the $4N$ by $4N$ matrix of second-order partial derivatives of the function W.

The above theorem indicates that under "normal" conditions the welfare change indicators W_F, W_0 and W_1 will all closely approximate each other.[28]

Recall (23) and (25) which defined the Weitzman-Paasche and Weitzman-Laspeyres welfare change indicators, W_P and W_L, respectively. Since W_L coincides with the normalized quadratic indicator W_0, it can be seen that W_L is superlative. However, straightforward computations show that although W_P approximates W_F to the first order around a point where $p^1 = \theta p^0$ and $x^1 = \lambda x^0$, it does not approximate it to the second order; i.e., we do not get the equality (61) when W_α is replaced by W_P. Hence it seems unlikely that W_P could be a superlative welfare change indicator.

VII. CONCLUSION

The goal of this paper has been to obtain a known function of the price and quantity data of an optimizing consumer for two periods that will indicate whether the consumer's welfare has improved going from period 0 to period 1. I called such a known function a welfare change indicator and termed it a superlative indicator if it exactly equalled the Hicksian equivalent variation for an expenditure function that had an appropriate second-order approximation property.

In section IV, we found one such superlative indicator, W_F, the Fisher welfare change indicator. In section V, we found an entire family of superlative indicators, W_α, one for each nonnegative, nonzero vector α. We found it convenient to set α equal to either x^0 (the base period quantity vector) or x^1 (the current period quantity vector) and this led to the welfare change indicators W_0 and W_1; see definitions (56) and (57).

In section VI, we found that all of the superlative welfare change indicators, W_F, W_α, W_0 and W_1, will numerically approximate each other rather closely and so the choice of a particular superlative indicator probably will not matter in empirical applications.[29] If forced to make a choice

27. I assume that α satisfies the restrictions (53). I also require that $p^i \cdot x^j > 0$ for $i,j = 0,1$ so that the Fisher quantity index is well defined.

28. Diewert [1978, 894] found that using Canadian consumer data, superlative price and quantity indexes (using the chain principle) approximated each other to within .1 percent approximately. I would expect a similar close degree of approximation with respect to these superlative welfare change indicators using data pertaining to consecutive periods.

29. This parallels the situation in index number theory where it can be shown that all known superlative index number formulae approximate each other to the second order around an equal price, equal quantity point; see Diewert [1978, 888]. Incidentally, theorems 3, 4 and 5 can be utilized to show that the normalized quadratic quantity index Q_α, defined by the right-hand side of (54), is a superlative quantity index for each vector $\alpha > 0_N$.

among the various superlative indicators,[30] I would tend to choose W_F since W_F relies on the use of the Fisher quantity index Q_F and, from the viewpoint of the axiomatic or test approach to index number theory, the use of Q_F can be given a strong justification.[31]

This paper demonstrates that there is a close connection between the measurement of welfare change, using only the price and quantity data pertaining to two periods of time, and index number theory. However, this insight is not new: Hicks [1941–42] noted this close correspondence almost fifty years ago.

30. It should be noted that W_F, W_0 and W_1 are all consistent with Samuelson's [1947, 146–163] revealed preference theory: (i) if $Q_L < 1$ and $Q_P \leq 1$, then x^0 is preferred to x^1; (ii) if $Q_L = 1$ and $Q_P = 1$, then x^0 is revealed to be indifferent to x^1; (iii) if $Q_L \geq 1$ and $Q_P > 1$, then x^1 is preferred to x^0. Under the conditions of cases (i)–(iii), the reader can verify that W_F, W_0 and W_1 all indicate the correct direction of welfare change.

31. See Diewert [1992, section 2].

APPENDIX

Proof of Theorem 1

Note that in view of (42), there are $[N(N+1)/2]-1$ independent parameters in the symmetric matrix A and in view of (41), there are N-1 independent parameters in the vector b. Thus, there are $2N-2 + N(N-1)/2$ independent parameters in all (given that c and γ are predetermined), which is just the minimal required number. Note also that if the restrictions (41) and (42) are satisfied, then e will satisfy the money metric scaling property (35).

Define the vector x^0 in terms of the derivatives of e^* as

$$x^0 \equiv \nabla_p e^*(u_0, p^0). \tag{A-1}$$

Using (32) and (35), we have

$$u_0 = p^0 \cdot x^0. \tag{A-2}$$

Define the matrix A in terms of the derivatives of e^* as follows:

$$A \equiv \nabla^2_{pp} e^*(u_0, p^0)(p^0 \cdot x^0)^{-1} + x^0 x^{0T}(p^0 \cdot x^0)^{-2} \tag{A-3}$$

where x^{0T} denotes the transpose of the column vector x^0. Premultiply both sides of (A-3) by p^{0T} and postmultiply by p^0. Making use of (33) yields the following equation:

$$p^{0T} A p^0 = 0 + p^0 \cdot x^0 x^0 \cdot p^0 (p^0 \cdot x^0)^{-2} = 1$$

and thus A satisfies the restriction (42).

Define the vector of parameters b in terms of the derivatives of e^* (and c and γ which are predetermined) as follows:

$$b \equiv (p^0 \cdot c)^{-1}(u^0 - \gamma)^{-1} \cdot [\nabla^2_{pu} e^*(u_0, p^0) u_0 - x^0]. \tag{A-4}$$

The restrictions (40) are required to ensure that b is well defined. Premultiply both sides of (47) by p^{0T}. Using (34), (37), (A-1) and (A-2), we find that

$$p^0 \cdot b = (p^0 \cdot c)^{-1}(u_0 - \gamma)^{-1} \cdot [p^0 \cdot \nabla^2_{pu} e^*(u_0, p^0) u_0 - p^0 \cdot x^0]$$

$$= (p^0 \cdot c)^{-1}(u_0 - \gamma)^{-1}(u_0 - u_0)$$

$$= 0$$

and thus b satisfies the restriction (41). To complete the proof, we need only show that e satisfies (28), (30), (31). Using the restrictions (41) and (42), we find that

$$\text{(A-5)} \qquad \nabla_p e(u_0,p^0) = Ap^0 u_0$$

$$= x^0(p^0 \cdot x^0)^{-1} u_0 \qquad \text{using (A-3)}$$

$$= x^0 \qquad \text{using (A-2)}$$

$$= \nabla_p e^*(u_0,p^0) \qquad \text{using (A-1)}$$

and thus equations (28) are satisfied.

Differentiating $e(u_0,p^0)$ and using the restrictions (41) and (42), we obtain the following equations:

$$\text{(A-6)} \qquad \nabla^2_{pu} e(u_0,p^0)$$

$$= Ap^0 + bp^0 \cdot c(u_0 - \gamma)u_0^{-1};$$

$$\text{(A-7)} \qquad \nabla^2_{pp} e(u_0,p^0) = Au^0 - Ap^0(Ap^0)^T u_0.$$

Now set the right-hand side of (A-6) equal to $\nabla^2_{pu} e^*(u_0,p^0)$. Making use of $Ap^0 u_0 = x^0$ (which follows from (A-5)) and (A-2), we find that the resulting equation is equivalent to equation (A-4) which was used to define b. Hence equations (31) are satisfied. Finally, set the right-hand side of (A-7) equal to $\nabla^2_{pp} e^*(u_0,p^0)$. Making use of (A-2) and (A-5), we see that the resulting system of equations is equivalent to (A-3) which was used to define A. Thus equations (30) are also satisfied.

Proof of Theorem 2

Differentiating (39) with $p = u_0$ and $\gamma = u_1$ and making use of (3), we find that equations (4) reduce to

$$\text{(A-8)} \qquad x^0 = Ap^0 u_0^2 / p^0 \cdot x^0;$$

$$x^1 = Ap^1 u_1^2 / p^1 \cdot x^1.$$

Square both sides of (43) and we obtain

$$[Q_F(p^0,p^1,x^0,x^1)]^2$$

$$= (p^0 \cdot x^1 / p^0 \cdot x^0)/(p^1 \cdot x^0 / p^1 \cdot x^1)$$

$$= (p^{0T} Ap^1 u_1^2 / p^1 \cdot x^1 p^0 \cdot x^0)$$

$$/(p^{1T} Ap^0 u_0^2 / p^0 \cdot x^0 p^1 \cdot x^1) \qquad \text{using (A-8)}$$

$$\text{(A-9)} \qquad = u_1^2 / u_0^2$$

where (A-9) follows using the symmetry of the matrix A. Taking positive square roots of both sides of (A-9) yields (44).

Proof of Corollary to Theorem 2

The normalizations (41) and (42) impose money metric scaling on the expenditure function using the base prices p^0 as the fixed reference prices. Hence we have

$$V(p^0,x^0,x^1) = e(u_1,p^0) - e(u_0,p^0)$$

$$= u_1 - u_0$$

$$= [(u_1 / u_0) - 1]u_0$$

$$= [Q_F(p^0,p^1,x^0,x^1) - 1]u_0 \qquad \text{using (44)}$$

$$= [Q_F(p^0,p^1,x^0,x^1) - 1]p^0 \cdot x^0$$

since $u_0 = p^0 \cdot x^0 = e(u_0,p^0)$ by the money metric scaling property of e.

Proof of Theorem 3

I first define A, b and c in terms of the derivatives of e^* and show that the restrictions (50)–(52) are satisfied.

Define the vector x^0 as follows:

$$\text{(A-10)} \qquad x^0 \equiv \nabla_p e^*(u_0,p^0).$$

By the money metric utility scaling property of e^*, we have

$$\text{(A-11)} \qquad p^0 \cdot x^0 = e^*(u_0,p^0) = u_0.$$

Define the vector b as follows:

$$\text{(A-12)} \qquad b \equiv u_0^{-1} x^0.$$

Premultiply both sides of (A-12) by p^{0T} and use (A-11) to obtain

$$p^0 \cdot b = u_0^{-1} p^0 \cdot x^0 = 1$$

and thus b satisfies (51).

Define the vector c in terms of the derivatives of e^* as follows:

$$c \equiv (u_0 - \gamma)^{-1} \cdot [\nabla^2_{pu} e^*(u_0, p^0) - u_0^{-1} x^0]. \quad \text{(A-13)}$$

Using (49), we see that c is well defined. Premultiplying both sides of (A-13) by p^{0T} yields

$$\begin{aligned} p^0 \cdot c &= (u_0 - \gamma)^{-1} \cdot [p^0 \cdot \nabla^2_{pu} e^*(u_0, p^0) - u_0^{-1} p^0 \cdot x^0] \\ &= (u_0 - \gamma)^{-1} [\nabla_u e^*(u_0, p^0) - 1] \quad \text{using (34) and (A-11)} \\ &= (u_0 - \gamma)^{-1} [1 - 1] \quad \text{using (37)} \\ &= 0 \end{aligned}$$

and thus c satisfies (52).

Define A in terms of the derivatives of e^* as follows:

$$A \equiv u_0^{-1} \alpha \cdot p^0 \nabla^2_{pp} e^*(u_0, p^0). \quad \text{(A-14)}$$

Postmultiplying (A-14) by p^0 yields $Ap^0 = 0_N$ using (33), and thus A satisfies (50).

As in the proof of theorem 1, we need only show that equations (28), (30) and (31) are satisfied once we define A, b and c as above. Equating $\nabla_p e(u_0, p^0)$ to $x^0 = \nabla_p e^*(u_0, p^0)$ yields an equation which is equivalent to (A-12), and thus (28) is satisfied. Equating $\nabla^2_{pu} e(u_0, p^0)$ to $\nabla^2_{pu} e^*(u_0, p^0)$ yields an equation which is equivalent to (A-13) if we make use of the restrictions (50)-(52), and thus (31) is satisfied. Finally, equating $\nabla^2_{pp} e(u_0, p^0)$ to $\nabla^2_{pp} e^*(u_0, p^0)$ yields an equation which is equivalent to (A-14) if $\alpha \cdot p^0 \neq 0$ and thus (30) is satisfied.

Proof of Theorem 4

Differentiating (48) with respect to prices with $\beta = u_0$ and $\gamma = u_1$ and making use of (4) yields the following equations:

$$x^0 = [b + (\alpha \cdot p^0)^{-1} A p^0 - (1/2)(\alpha \cdot p^0)^{-2} p^{0T} A p^0 \alpha] u_0 \quad \text{(A-15)}$$

and

$$x^1 = [b + (\alpha \cdot p^1)^{-1} A p^1 - (1/2)(\alpha \cdot p^1)^{-2} p^{1T} A p^1 \alpha] u_1. \quad \text{(A-16)}$$

Substitute (A-15) and (A-16) into the right-hand side of (54), make use of the symmetry of the A matrix, and the result follows.

Proof of Corollary to Theorem 4

The expenditure function e is defined by (48) with $\beta = u_0$ and $\gamma = u_1$. If, in addition, the parameters of e satisfy the restrictions (50)-(52), then e exhibits money metric scaling at the prices p^0 and hence, using (3) and (36), gives us $u_0 = p^0 \cdot x^0 = e(u_0, p^0)$ and $u_1 = e(u_1, p^0)$. Substituting these equalities into definition (5) yields

$$\begin{aligned} V(p^0, x^0, x^1) &= u_1 - u_0 \\ &= [(u_1/u_0) - 1] u_0 \\ &= [(u_1/u_0) - 1] p^0 \cdot x^0 \\ &= [Q_\alpha(p^0, p^1, x^0, x^1) - 1] p^0 \cdot x^0 \quad \text{using (54)} \end{aligned}$$

Proof of Theorem 5

A series of straightforward but lengthy computations.

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