The Le Chatelier Principle in Data Envelopment Analysis

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Summary. The paper gives a brief review of the nonparametric approach to efficiency measurement or Data Envelopment Analysis as it is known in the operations research literature. Inequalities are derived between the efficiency measures when different assumptions are made on the technology sets or on the behavior of managers. Of particular interest is the derivation of a Le Chatelier Principle for measures of allocative inefficiency. Finally, the various inequalities are illustrated using some Canadian data, which is also used to compare DEA methods for measuring the relative efficiency of production units with more traditional index number methods.

Key words: efficiency measurement, data envelopment analysis, Le Chatelier Principle, productivity, nonparametric measurement of technology, index numbers.

Classification code: C14, C43, C61, D61

1 Introduction

Data Envelopment Analysis or (DEA) is the term used by Charnes and Cooper (1985) and their co-workers to denote an area of analysis which is called the nonparameteric approach to production theory[3] or the measurement of the efficiency of production[4] by economists.

In section 2, we will provide an introduction to the theory of benchmarking and the measurement of relative efficiency of production units. Section 3 develops measures of relative efficiency that use only quantity data. These measures are particularly useful in the context of measuring the efficiency of government owned enterprises or units of the general government sector that

[3] See Hanoch and Rothschild (1972), Diewert (1981), Diewert and Parkan (1983) and Varian (1984). It should be noted that in recent times, the term "nonparametric approach to production theory" has sometimes included index number methods for defining the relative efficiency of production units.

[4] See Farrell (1957), Afriat (1972), Färe and Lovell (1978), Färe, Grosskopf and Lovell (1985) and Coelli, Prasada Rao and Battese (1997). The last two books provide a good overview of the subject.

deliver services to the public for free or for prices that do not reflect costs of production. Efficiency measures that use only quantity data (and not price data) are also useful in the regulatory context[5] Section 4 develops measures of relative efficiency for production units in the same industry where reliable price and quantity data are available for the units in the sample. Section 5 notes some relationships between the various efficiency measures developed in the previous two sections. In particular, an efficiency measurement analogue to Samuelson's (1947; 36–39) Le Chatelier Principle is developed in section 5.

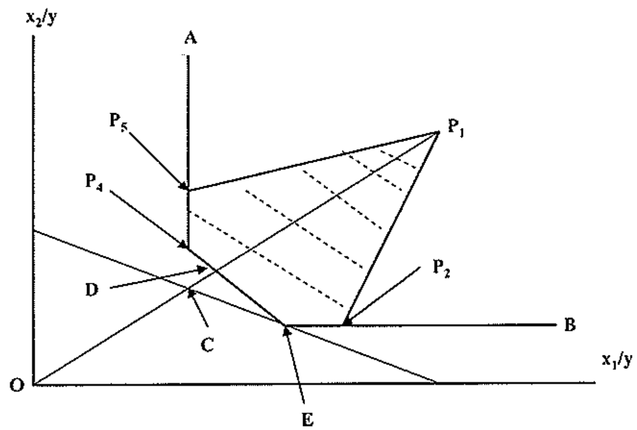
Mendoza (1989) undertook an empirical comparison of 3 different methods for measuring productivity change in the context of time series data for Canada. The 3 different methods of comparison she considered were: (i) a nonparametric or DEA method; (ii) traditional index number methods and (iii) an econometric method based on the estimation of a unit profit function[6] In section 6 we will compare the DEA and index number approaches to efficiency measurement using some more recent aggregate Canadian data and we also illustrate the theoretical Rules developed in previous sections.

Drawing on the empirical and theoretical results reviewed in the previous sections, in section 7 we compare the advantages and disadvantages of DEA methods for measuring the relative efficiency of production units with the more traditional index number and econometric methods.

2 An Introduction to the Nonparametric Measurement of Efficiency

The basic idea in the case of similar firms producing one output and using 2 inputs is due to Farrell (1957; 254). Let there be K firms, denote the output of firm k by $y^k \geq 0$ and denote the amounts of inputs 1 and 2 used by firm k by $x_1^k \geq 0$ and $x_2^k \geq 0$ respectively, for $k = 1, 2, \ldots, K$. Calculate the *input-output coefficients* for each firm defined by x_1^k/y^k and x_2^k/y^k for $k = 1, 2, \ldots$, K. Now plot these pairs of input output coefficients in a two dimensional diagram as in Figure 1 where we have labeled these pairs as the points P^1, $P^2, \ldots, P^5$ (so that $k = 5$).

The convex hull of the 5 data points $P^1, \ldots, P^5$ in Figure 1 is the shaded set: it is the set of all non-negative weighted averages of the 5 points where the weights sum up to 1. The convex free disposal hull of the original 5 points is the shaded set plus all of the points that lie to the north and east of the shaded set.



Farrell took the boundary or frontier of this set as an approximation to the unit output isoquant of the underlying production function[7] In Figure 1, this frontier set is the piecewise linear curve AP^4P^3B. The *Farrell technical efficiency* of the point P^1 was defined to be the ratio of distances OD/OP^1, since this is the fraction (of both inputs) that an efficient firm could use to produce the same output as that produced by Firm 1. A point Pi is regarded as being *technically efficient* if its technical efficiency is unity.

Farrell (1957; 254) noted the formal similarity of his definition of technical efficiency to Debreu's (1951) coefficient of resource utilization.

Farrell (1957; 255) also defined two further efficiency concepts using a diagram similar to Figure 1. Suppose Firm 1 faced the fixed input prices w_1 and w_2 for the two inputs. Then we could form a family of isocost lines with slope w_1/w_2 and find the lowest such isocost line that is just tangent to the free disposal convex hull of the 5 points. In Figure 1, this is the line CE which is tangent to the point P^3. Farrell noted that even if the point P^1 were shrunk in towards the origin to end up at the technically efficient point D, the resulting point would still not be the cost minimizing input combination (which is at P^3). Thus Farrell defined the *price efficiency* of P^1 as the ratio of distances OC/OD. Finally, Farrell (1957; 255) defined the *overall efficiency* of Firm 1 as the ratio of distances OC/OP^1. This measure incorporates both technical and allocative inefficiency. A point Pi is *overall efficient* if its overall efficiency is unity.

There is a problem with Farrell's measure of technical efficiency: Farrell's definition makes the points P^2 and P^5 in Figure 1 technically efficient when it seems clear that they are not: P^2 is dominated by P^3 which uses less of input 1 to produce the same output and P^5 is dominated by P^4 which uses

[5] See Diewert (1981).

[6] For material on variable and unit profit functions, see Diewert (1973) (1974) and Diewert and Wales (1992). Coelli, Prasada Rao and Battese (1997) also compared the three approaches to the measurement of efficiency. Balk (1998; 179–209) also compared the three approaches. Diewert (1980) was perhaps the first to contrast the three approaches and he also included a fourth approach: the Divisia approach. The index number approach was reviewed in detail by Diewert and Nakamura (2003).

[7] Farrell (1957; 254) was assuming constant returns to scale in this part of his paper.

less of input 2 to produce the same output. Charnes, Cooper and Rhodes (1978; 437) and Färe and Lovell (1978; 151) both noticed this problem with Farrell's definition of technical efficiency and suggested remedies. However, in the remainder of this chapter we will stick with Farrell's original definition of technical efficiency, with a few modifications to cover the case of many outputs.

Farrell's basic ideas outlined above for the case of a one output, constant returns to scale technology can be generalized in several ways: (i) we can relax the assumption of constant returns to scale; (ii) we can extend the analysis to the multiple output, multiple input case; (iii) we can generalize the analysis to cover situations where it is reasonable to assume profit maximizing behaviour (or partial profit maximizing behaviour) rather than cost minimizing behaviour and (iv) we can measure inefficiency in different metrics (i.e., instead of measuring technical inefficiency in terms of a proportional shrinkage of the input vector, we could choose to measure the inefficiency in terms of a basket of outputs or a basket of outputs and inputs). Drawing on the work of Mendoza (1989) and others, we shall indicate how the above generalizations (i)–(iii) can be implemented for the case of technologies that produce only 2 outputs and utilize only 2 inputs. The generalization to many outputs and inputs is straightforward. Section 3 below covers approaches that use only quantity data while section 4 describes approaches that utilize both price and quantity data. Section 5 draws on the results of the previous two sections and notes some interesting general relationships between various measures of efficiency loss. Of particular interest is a Le Chatelier Principle for measures of allocative inefficiency.

3 Efficiency Tests Using Only Quantity Data

3.1 The Case of a Convex Technology

Suppose that we have quantity data on k production units that are producing 2 outputs using 2 inputs. Let $y_m^k \geq 0$ denote the amount of output m produced by each production unit (or firm or plant) j for $m = 1, 2$, and let $x_n^k \geq 0$ denote the amount of input n used by firm k for $n = 1, 2$ and $k = 1, 2, \ldots, K$.

We assume that each firm has access to the same basic technology except for efficiency differences. An approximation to the basic technology is defined to be the convex free disposal hull of the observed quantity data $\{(y_1^k, y_2^k, x_1^k, x_2^k) : k = 1, \ldots, K\}$. This technology assumption is consistent with decreasing returns to scale (and constant returns to scale) but it is *not* consistent with increasing returns to scale.

It is necessary to specify a *direction* in which possible inefficiencies are measured; i.e., do we measure the inefficiency of observation i in terms of output m or input n or some combination of outputs and inputs? Mendoza's (1989) methodology allowed for an arbitrary efficiency direction[8] but for simplicity,

[8] See Mendoza (1989; 25–30).

we will restrict ourselves to the Debreu (1951)–Farrell (1957) direction; i.e., we shall measure the inefficiency of observation i by the smallest positive fraction δ_i^* of the ith input vector (x_1^i, x_2^i) which is such that $\{(y_1^k, y_2^k, \delta_i^* x_1^k, \delta_i^* x_2^k)$ is on the efficient frontier spanned by the convex free disposal hull of the k observations. If the ith observation is efficient relative to this frontier, then $\delta_i^* = 1$; the smaller δ_i^* is, then the lower is the efficiency of the ith observation. The number δ_i^* can be determined as the optimal objective function of the following linear programming problem[9]:

$$\delta_i^* = min_{\delta_i \geq 0, \lambda_1 \geq 0, \ldots, \lambda k \geq 0}\{\delta_i : \textstyle\sum_{k=1}^K y_1^k \lambda_k \geq y_1^i; \sum_{k=1}^K y_2^k \lambda k \geq y_2^i; \\ \textstyle\sum_{k=1}^K x_1^k \lambda_k \leq \delta_i x_1^i; \sum_{k=1}^K x_2^k \lambda_k \leq \delta_i x_2^i; {}_{k=1}^K \lambda_k = 1\}. \quad (1)$$

Thus we look for a convex combination of the K data points that can produce at least the observation i combination of outputs (y_1^i, y_2^i) and use only δ_i times the observation i combination of inputs (x_1^i, x_2^i). The smallest such δ_i is δ_i^*.

The linear programming problems (1) are run for each observation i and the resulting $\delta_i^* \geq 0$, serves to measure the relative efficiency of observation i; if $\delta_i^* = 1$, then observation i is efficient. At least one of the J observations will be efficient.

We turn now to the corresponding linear program that tests for efficiency under the maintained hypothesis that the underlying technology is subject to constant returns to scale (in addition to being convex).

3.2 The Case of a Convex, Constant Returns to Scale Technology

In this case, the approximation to the underlying technology set is taken to be the free disposal hull of the convex cone spanned by the K data points. The efficiency of observation i is measured by the positive fraction δ_i^{**} of the ith input vector (x_1^i, x_2^i) which is such that $(y_1^k, y_2^k, \delta_i^{**} x_1^k, \delta_i^{**} x_2^k)$ is on the efficient frontier spanned by the conical convex free disposal hull of the K observations. The efficiency of the ith observation relative to this technology set can be calculated by solving the following linear program:

$$\delta_i^{**} = min_{\delta_i \geq 0, \lambda_1 \geq 0, \ldots, \lambda_K \geq 0}\{\delta_i \text{ subject to: } \textstyle\sum_{k=1}^K y_1^k \lambda_k \geq y_1^i; \sum_{k=1}^K y_2^k \lambda_k \geq y_2^i; \\ \textstyle\sum_{k=1}^K x_1^k \lambda_k \leq \delta_i x_1^k; \sum_{k=1}^K x_2^k \lambda_k \leq \delta_i x_2^k; \}. \quad (2)$$

Note that the LP (2) is the same as (1) except that the constraint $\sum_{k=1}^K \lambda_k = 1$ has been dropped. Thus the optimal solution for (1) is feasible

[9] See Mendoza (1989; 30) for a general version of Test 1. The use of linear programming techniques to calculate nonparametric efficiencies was first suggested by Hoffman (1957; 284) and first used by Farrell and Fieldhouse (1962). Related tests are due to Afriat (1972; 571) and Diewert and Parkan (1983; 141).

for (2) and thus $\delta_i^{**} \le \delta_i^*$; i.e., the constant returns to scale measure of efficiency for observation i will be equal to or less than the convex technology measure of inefficiency for observation i.

We turn now to models that are consistent with increasing returns to scale.

3.3 Quasiconcave Technologies

We first need to define what we mean by a production possibilities set $L(y_1)$ that is conditional on an amount y_1 of output 1. Let S be the set of feasible outputs and inputs. Then $L(y_1)$ is defined to be the set of (y_2, x_1, x_2) such that $(y1, y_2, x_1, x_2)$ belongs to S; i.e., $L(y_1)$ is the set of other outputs y_2 and inputs x_1 and x_2 that are consistent with the production of y_1 units of output 1. We assume that the family of production possibilities sets $L(y_2)$ has the following three properties: (i) for each $y1 \ge 0$, $L(y_1)$ is a closed, convex set[10] (ii) if $y_1' \le y_1''$, then $L(y_1)$ is a subset of $L(y_1)$ and (iii) the sets $L(y_1)$ exhibit free disposal.

For each observation i, define the following set of indexes:

$$I_1^i \equiv \{k : y_1^k \ge y_1^i, k = 1, 2, \ldots, K\}; \tag{3}$$

i.e., I_1^i is the set of observations k such that the amount of output 1 produced by observation k is equal to or greater than the amount of output 1 produced by observation i. Note that observation i must belong to I_1^i.

Given our assumptions on the underlying technology, it can be seen that the free disposal convex hull of the points $(y_2^j, x_1^j, x_2^j), j \in I_1^i$, forms an approximation to the set $L(y_1^i)$. The frontier of this set is taken to be the efficient set. As usual, we measure the efficiency of observation i by the positive fraction δ_i^{**} of the ith input vector (x_1^i, x_2^i) which is such that $\{(y_2^k, \delta_i^{**}x_1^k, \delta_i^{**}x_2^k)$ is on the efficient frontier defined above. The number can be calculated by solving the following linear program[11]:

$$\delta_i^{**} = min_{\delta_i \ge 0, \lambda_1 \ge 0, \ldots, \lambda_K \ge 0}\{\delta_i : \textstyle\sum_{k \in I_1^i} y_2^k \lambda_k \ge y_2^i; \sum_{k \in I_1^i} x_1^k \lambda_k \le \delta_i x_1^i; \\ \textstyle\sum_{k \in I_1^i} x_2^k \lambda_k \le \delta_i x_2^i; \sum_{k \in I_1^i} \lambda_k = 1\} \quad . \tag{4}$$

On the left hand side of each constraint in (4), the indexes k must belong to the index set I_1^i defined by (3) above.

10 If we represent the underlying technology by means of the production function $y_1 = f(y_2, x_1, x_2)$, assumption (i) implies that f is a quasiconcave function.

11 See Mendoza (1989; 54) for a general version of (4) which she called Test 3. The one output quasiconcavity test is due to Hanoch and Rothschild (1972; 259–261). Diewert (1980; 264)(1981) and Diewert and Parkan (1983; 140) developed alternative methods for dealing with a quasiconcave technology but the present method seems preferable.

Denote the optimal k for (4) above by λ_k^{***} for $k \in I_1^i$. By the last constraint in (4), we have

$$\sum_{k \in I_1^i} \lambda_k^{***} = 1; \tag{5}$$

Using definition (3), $\lambda_k^{***} \ge 0$ and (5), it can be seen that

$$\sum_{k \in I_1^i} y_1^k \lambda_k \ge y_1^i. \tag{6}$$

Using (1), (4) and (6), we see that the optimal solution for (4) is feasible for (1) and thus we must have $\delta_i^* \le \delta_i^{***}$. Recall that we showed that $\delta_i^{**} \le \delta_i^*$ and so we have

$$0 \le \delta_i^{**} \le \delta_i^* \le \delta_i^{***}. \tag{7}$$

Thus the efficiency measures generally *increase* (or remain constant) as we make *weaker* assumptions on the underlying technology: the biggest efficiency measure δ_i^{***} corresponds to a quasiconcave (in output 1)[12] technology, the next measure δ_i^* corresponds to a convex technology, and the smallest efficiency measure δ_i^{**} corresponds to a constant returns to scale convex technology.

In definition (3) and in the LP (4), output 1 was singled out to play a special role. Obviously, analogues to (3) and (4) could be constructed where output 2 played the asymmetric role. In this latter case, the underlying technological assumption is that the $y_2 = f(y_1, x_1, x_2)$ production function is quasiconcave. This is a somewhat different technological assumption than our initial one, but both assumptions are consistent with an increasing returns to scale technology[13].

The last paragraph raises two questions:

- What is the motivation for imposing quasiconcavity on all of the inputs and all but one of the outputs?
- How exactly is the researcher to choose which output is to be singled out to play an asymmetric role in the above efficiency measure?

These are difficult to answer questions. In the one output, many input context, we routinely assume quasiconcave technologies, at least in part, because a non quasiconcave technology cannot be identified using observable price and quantity data if producers are competitively minimizing costs. If we carry this line of reasoning over to the case of many outputs, then if the production units in the relevant peer group are competitively minimizing costs and competitively selling all of their outputs except one, then that non competitively supplied

12 Thus δ_i^{***} should be more accurately denoted by δ_{i1}^{***} in order to indicate that we are assuming quasiconcavity with respect to output 1.

13 Mendoza's (1989; 54) Test 3 can also be modified to model quasiconcave technologies of the form $x_1 = g(y_1, y_2, x_2)$, where g is now a factor requirements function.

output should be singled out in the above test to play the asymmetric role. However, strictly speaking, under these hypotheses, we should move on to the tests for efficiency in subsequent sections, where we assume some form of competitive pricing behavior. In general, we cannot offer definitive advice on which output should be singled out to play an asymmetric role in the above efficiency test· the researcher will perhaps have to rely on engineering considerations to single out the output which is most likely to be subject to increasing returns to scale or perhaps just pick the most important output (in terms of market share) as the numeraire output.

This completes our overview of nonparametric efficiency tests that involve the use of quantity data. We now turn to tests that involve both price and quantity data so that overall efficiency measures can be constructed in place of the technical efficiency measures of this section.

4 Efficiency Tests Using Price and Quantity Data

4.1 The Convex Technology Case

We make the same assumptions on the underlying technology as in section 3.1 above. However, we now assume that each producer may be either minimizing cost or maximizing profits[14] We consider each case in turn.

Case (i): Cost Minimization:

We assume that producer k faces the input prices $(w_1k,\ w_2k)$ for the two inputs. To determine whether producer i is minimizing cost subject to our convex technology assumptions, we solve the following linear program[15]:

$$min_{\delta_i\geq 0,\lambda_1\geq 0,\ldots,\lambda_K\geq 0}\{w_1^i(\sum_{k=1}^{K}x_1^k\lambda_k)+w_2^i(\sum_{k=1}^{K}x_2^k\lambda_k):\sum_{k=1}^{K}y_1^k\lambda_k\geq y_1^i;$$

$$\sum_{k=1}^{K}y_2^k\lambda_k\geq y_2^i;\sum_{k=1}^{K}\lambda_k=1\} \tag{8}$$

$$\equiv \epsilon_i^*[w_1^ix_1^i+w_2^ix_2^i]. \tag{9}$$

The meaning of (9) is that we define the overall efficiency measure ϵ_i^* for observation i by equating (9) to the optimized objective function in (8). If we set $\lambda_i = 1$ and the other $\lambda_k = 0$, we have a feasible solution for (8) which yields a value of the objective function equal to $w_1^ix_1^i + w_2^ix_2^i$. Thus $0 < \epsilon_i^* \leq 1$. The number ϵ_i^* can be interpreted as the fraction of (x_1^i, x_2^i) which is such that

[14] In contrast to the technical efficiency measures defined in section 2 where at least one observation had to be efficient (with an efficiency score of 1), in this section, it can be the case that no observation is efficient.

[15] See Mendoza (1989; 67) for a general version of (8) which she called Test 4.

$\epsilon_i^*(x_1^i, x_2^i)$ on the minimum cost isocost line for observation i; i.e., ϵ_i^* is an analogue to the overall efficiency measure OC/OP[1] which occurred in Figure 1. Comparing (1) and (8), it can be seen that the optimal λ_k^* solution for (1) is a feasible solution for (8) and thus:

$$0 < \epsilon_i^* \leq \delta_i^*. \tag{10}$$

The second inequality in (10) simply reflects the fact that overall efficiency ϵ_i^* is equal to or less than technical efficiency δ_i^* (recall Figure 1 again).

Case (ii): Profit Maximization:

We now assume that firm i also faces the positive output prices (p_1^i, p_2^i) for the two outputs. To determine whether producer i is maximizing profits subject to our convex technology assumptions; we solve the following linear program[16]:

$$max_{\lambda_1\geq 0,\ldots,\lambda_K\geq 0}\{\sum_{m=1}^{2}p_m^i(\sum_{k=1}^{K}y_m^k\lambda_k)-(\sum_{k=1}^{K}x_n^k\lambda_k):\sum_{k=1}^{K}\lambda_k=1\} \tag{11}$$

$$\equiv p_1^iy_1^i+p_2^iy_2^i-\alpha_i^*[w_1^ix_1^i+w_2^ix_2^i]. \tag{12}$$

Equating (11) to (12) defines the efficiency measure α_i^* for observation i. If we set $\lambda_i = 1$ in (11) and the other $\lambda_k = 0$, we obtain a feasible value for the objective function equal to $p_1^iy_1^i + p_2^iy_2^i - [w_1^ix_1^i + w_2^ix_2^i]$. Thus $\alpha_i^* = 1$. If $\alpha_i^* = 1$, then observation i is efficient relative to our assumptions on the technology and relative to the hypothesis of complete profit maximization. The interpretation of α_i^* is similar to that of ϵ_i^* defined above by (9).

It can be seen that the optimal $\lambda_k^* = 0$ solution to (8) is feasible for (11). Using this fact and the inequalities in (8), we have[17]

$$\alpha_i^* \leq \delta_i^*. \tag{13}$$

Thus when we assume that the underlying technology set is convex and calculate the efficiency of observation i, ϵ_i^*, under the assumption of cost minimizing behavior and compare this efficiency level to the relative efficiency of observation i, α_i^*, calculated under the assumption of profit maximizing behavior, we find that the relative efficiency level under the profit maximizing assumption will be equal to or less than the relative efficiency level under the cost minimizing assumption.

We now turn to the corresponding linear programs that test for the efficiency of observation i under the maintained hypothesis that the underlying technology is subject to constant returns to scale.

[16] This is Mendoza's (1989; 88) Test 7. It is also a special case of her Test 4. Since there is only one constraint in the problem, the solution to (11) is $max_k\sum_{m=1}^{2}p_m^iy_m^i - \sum_{n=1}^{2}w_n^ix_n^i; k = 1, 2, \ldots, K$. For related tests, see Afriat (1972; 594) for the single output case and Hanoch and Rothschild (1972; 268–270) and Diewert and Parkan (1983; 151) for the multiple output case.

[17] Mendoza (1989; 76–77) showed this.

4.2 The Convex Conical Technology Case

Case (i):Cost Minimization:

Guided by the results of section 3.2, it can be seen that all we have to do is to drop the constraint $\sum_{k=1}^{K}\lambda_k = 1$ from (8). The resulting optimized objective function is set equal to $\epsilon_i^{**}[w_1^i x_1^i + w_2^i x_2^i]$. Since the new LP has one less constraint than (8), it will generally attain a smaller optimized objective function and so ϵ_i^{**} will generally be smaller than ϵ_i^*; i.e.,

$$\epsilon_i^{**} \leq \epsilon_i^*. \tag{14}$$

By comparing the new LP to (2), we can also show

$$\delta_i^{**} \geq \epsilon_i^{**}. \tag{15}$$

The inequality (14) shows that making *stronger* assumptions on the underlying technology tends to *decrease* the efficiency measure; i.e., the constant returns to scale measure of the efficiency of observation i, ϵ_i^{**}, will be equal to or less that the convex technology measure of the efficiency of observation i,ϵ_i^*. The inequality (15) shows that assuming cost minimizing behaviour tends to decrease the efficiency of observation i, ϵ_i^{**}, compared to the measure of technical efficiency that we obtained earlier for observation i, δ_i^{**}.[18]

Case (ii):Profit Maximization:

As in section 2.2, we could approximate the underlying technology set by the free disposal hull of the convex cone spanned by the K data points. To determine whether observation i is on the frontier of this set, we could attempt to solve the LP problem (11) after dropping the constraint $\sum_{k=1}^{K}\lambda_k = 1$. Unfortunately, the resulting optimal objective function is either 0 or plus infinity. Hence a different approach is required.

In order to obtain an operational approach, we consider a *conditional profit maximization problem* in place of the full profit maximization problem that appears in the objective function of (11); i.e., we allow firm i to maximize profits but we assume that the level of one input is *fixed* in the short run. Thus if the fixed input is input 2, to determine whether producer i is maximizing (variable) profits subject to our convex, conical technology assumptions, we solve the following linear programming problem[19]:

$$max_{\lambda_1\geq 0\lambda_2\geq 0,\ldots,\lambda_k\geq 0}\{\sum_{m=1}^{2} p_m^i(\sum_{k=1}^{K} y_m^k\lambda_k) - \sum_{n=1}^{2} w_n^i(\sum_{k=1}^{K} x_1^k\lambda_k) : \sum_{k=1}^{K} x_2^k\lambda_k \leq x_2^i\} \tag{16}$$

[18] These results and the appropriate general test may be found in Mendoza (1989; 78), which she called Test 5.

[19] The constraint in (16) will hold as an equality in the optimal solution. Hence the nonnegative λ_k^* which solve (16) serve to define a weighted combination of the K data points which uses the observation i amount of input 2, x_2^i, and maximizes profits at the prices of observation i.

$$= max_k\{[\sum_{m=1}^{2} p_m^i y_m^k - (\sum_{n=1}^{2} w_n^i x_1^k)][x_2^i/x_2^k] : k = 1,2,,K\}^{20} \tag{17}$$

$$\equiv p_1^i y_1^i + p_2^i y_2^i - \alpha_i^{**}[w_1^i x_1^i + w_2^i x_2^i] \tag{18}$$

where (18) serves to define the observation i efficiency measure α_i^{**}. Note that $\lambda_i^{**} = 1$ and the other $\lambda_k = 0$ is a feasible solution for (16) and this implies that $\alpha_i^{**} \leq 1$.[21]

The simple maximization problem defined by (17) can be written in the following instructive way:

$$\begin{aligned} &max_k\{[\textstyle\sum_{m=1}^{2} p_m^i y_m^k - (\sum_{n=1}^{2} w_n^i x_1^k][x_2^i/x_2^k] : k = 1,2,\ldots,K\} \\ &= x_2^i max_k\{\textstyle\sum_{m=1}^{2} p_m^i[y_m^k/x_2^k] - (\sum_{n=1}^{2} w_n^i[x_n^k/x_2^k]) : k = 1,2,\ldots,K\}. \end{aligned} \tag{19}$$

Note that the points $[y_1^k/x_2^k, y_2^k/x_2^k, x_1^k/x_2^k, x_2^k/x_2^k] = [y_1^k/x_2^k, y_2^k/x_2^k, x_1^k/x_2^k, 1]$ are feasible output and input vectors under our constant returns to scale assumption but where the amount of input 2 is fixed at the level 1. Thus the maximization problem in (19) scales each observed output-input vector k so that the resulting scaled last input level is equal to 1 and then we take the output and input prices faced by production unit i, $[p_1^i, p_2^i, w_1^i, w_2^i]$, evaluate unit profits at these prices for each scaled output-input vector k, $p_1^i[y_1^k/x_2^k] + p_2^i[y_2^k/x_2^k] - w_1^i[x_1^k/x_2^k] - w_2^i[x_1^k/x_2^k]$, take the maximum over k of these hypothetical profits and then scale the resulting hypothetical profits by the observation i level of the "fixed" input, which is equal to x_2^i.

Comparison of (2) and (16) shows that the optimal solution to (2) generates a feasible solution for (16) and thus

$$\delta_i^{**} \geq \alpha_i^{**}; \tag{20}$$

i.e., the observation i technical efficiency measure δ_i^{**} is always equal to or greater than the overall observation i (conditional on input 2) profit maximization efficiency measure α_i^{**}.

Since the LP problem (16) does not simply drop the constraint $\sum_{k=1}^{K}\lambda_k = 1$, the single constraint in the convex technology LP problem (11), we cannot develop an inequality between the solution to (16) and the solution to (11). However, since both problems use *all* of the price and quantity data pertaining to the K observations, typically the solutions to (11) and (16) will be similar in that the efficiencies generated by these models will tend to be much lower than the technical efficiencies generated by the models presented in section 3.

[20] We require $x_2^k > 0$ for $k = 1,2,\ldots,K$ in order to derive (17) from (16).

[21] A sufficient condition to ensure that the solution to (16) is finite is $x_2^k > 0$ for $k = 1,\ldots,K$.

4.3 The Quasiconcave Technology Case

We consider only the cost minimization case[22]

We make the same technology assumptions as were made in section 3.3. Recall the index set I_1^i defined by (3). To determine whether producer i is minimizing cost subject to our quasiconcave technology in output 1 assumption, we solve the following linear program:

$$min_{\lambda_1\geq 0 \lambda_2 \geq 0,\ldots,\lambda_k\geq 0}\{w_1^i(\textstyle\sum_{k\in I_1^i} x_1^k\lambda_k) + w_2^i(\sum_{k\in I_1^i} x_2^k\lambda_k) : \sum_{k\in I_1^i} y_2^k\lambda_k \geq y_2^i; \sum_{k\in I_1^i}\lambda_k = 1\} \tag{21}$$

$$\equiv \epsilon_i^{***}[w_1^i x_1^i + w_2^i x_2^i]. \tag{22}$$

As usual, ϵ_i^{***} is our measure of overall efficiency for observation i under our present assumptions on the technology and on the producer's behaviour. Since the index i belongs to the index set I_1^i (recall (3)), it can be seen that $\lambda_i = 1$ and the other $\lambda_k = 0$ is feasible for the LP(21) and gives rise to the feasible value for the objective function equal to $w_1^i x_1^i + w_2^i x_2^i$. Thus $\epsilon_i^{***} \leq 1$. It is also possible to see that the optimal δ_i^{***}, λ_i^{***} solution to (4) is a feasible ϵ_i,λ_k solution for (21). Thus

$$0 \leq \epsilon_i^{***} \leq \delta_i^{***}; \tag{23}$$

i.e., the (quasiconcave in output 1) cost minimizing overall efficiency for observation i, ϵ_i^{***}, will be equal to or less than the corresponding (quasiconcave in output 1) technical efficiency loss for observation i, δ_i^{***}.

Comparing (21) with (8) and using the definition of the index set I_1^i (recall (3)), it can be seen that the optimal λ_i^{***}, ϵ_i^{***} solution for (21) is a feasible solution for (8[23] Thus

$$\epsilon_i^{***} \geq \epsilon_i^{*}; \tag{24}$$

i.e., the observation i efficiency measure assuming a quasiconcave technology and cost minimizing behaviour ϵ_i^{***} will be equal to or greater than the observation i efficiency measure assuming a convex technology and cost minimizing behaviour ϵ_i^{*}.

5 Relationships between the Efficiency Measures

The inequalities derived in the previous two sections can be summarized by two rules. Note that all efficiency measures are measured in the same metric.

Rule 1: The nonparametric efficiency measures tend to *fall* as we make more restrictive technological assumptions; i.e., the quasiconcave technology efficiency measure will be equal to or greater than the corresponding convex technology efficiency measure which in turn will be equal to or greater than the corresponding convex conical technology loss measure.

Rule 2: The nonparametric efficiency measures tend to *fall* as we assume optimizing behaviour over a larger number of goods; i.e., the technical efficiency measure will be equal to or greater than the corresponding cost minimizing efficiency measure which will be equal to or greater than the corresponding profit maximizing efficiency measure. This is Mendoza's (1989; 76–77) Le Chatelier Principle for measures of allocative efficiency.

We illustrate some of the above points using some Canadian data in the following section.

6 An Empirical Comparison of Alternative Efficiency Measures for Canada

We use National Accounts and OECD data for Canada for the years 1980-2004 in order to illustrate the above programs[24] Producer data on three (net) outputs and two primary inputs are used. The three net outputs are: domestic output, y_1 (C + G + I); exports, y_2; and minus imports, y_3. The two primary inputs are: labour, x_1 and reproducible capital, x_2. These data are listed in Table 1. The corresponding producer prices, p_1, p_2, p_3 for net outputs and w_1 and w_2 for primary inputs are listed in Table 2[25].

The tests for technical efficiency of each observation, (1) and (2) in sections 3.1 and 3.2, were run using the quantity data listed in Table 1 above[26] The relative technical efficiencies of the year i observation assuming a convex technology set, δ_i^*, and assuming a convex, constant returns to scale technology set, δ_i^{**}, are listed in Table 3 below. The cost minimization relative efficiencies ϵ_i^* defined by (8) and (9) in section 4.1 for the case of a convex technology and ϵ_i^{**} defined in section 4.2 for the case of a convex, constant returns to scale technology are also listed in Table 3 below. The profit maximization relative efficiencies α_i^* defined by (11) and (12) in section 4.1 for the case of a convex technology and α_i^{**} defined by (16) and (18) in section 4.2 for the case of a convex, constant returns to scale technology (with capital fixed) are also listed in Table 3 below.

Finally, we use the data in Tables 1 and 2 to construct:

- a chained Fisher (1922) ideal index of net outputs, Yt for year t;
- a chained Fisher ideal index of primary inputs Xt for year t and
- a measure of *index number productivity* in year t equal to t Yt/Xt.

[22] Mendoza (1989; 83) considered more general cases in her Test 6.

[23] Using definition (3), $\lambda_i^{***} \geq 0$ and (5), it can be seen that $\sum_{k\in I_1^i} y_1^k \lambda_k^{***} \geq y_1^i$.

[24] We did not compute the quasiconcavity efficiencies since these tend to be close to 1 and are not very informative.

[25] All prices were normalized to equal 1 in the year 1960.

[26] We have three (net) outputs instead of two outputs but the reader need only modify the tests in the obvious ways.

Table 1. Quantity Data on Net Outputs and Primary Inputs for Canada, 1980–2004

Year	y_1	y_2	y_3	x_1	x_2
1980	88.22	23.23	25.38	42.36	36.83
1981	91.73	23.62	26.02	44.11	38.24
1982	85.45	23.20	21.80	42.68	40.07
1983	89.07	24.63	24.01	42.90	40.60
1984	93.38	29.21	28.12	43.97	41.52
1985	98.49	30.63	30.48	45.27	42.82
1986	101.71	31.97	32.68	46.76	44.38
1987	106.53	32.95	34.43	47.92	46.02
1988	112.55	35.95	39.10	49.44	48.04
1989	116.91	36.24	41.39	50.53	50.46
1990	116.22	37.97	42.23	50.88	53.07
1991	114.12	38.66	43.28	49.91	54.92
1992	114.44	41.45	45.31	49.47	56.14
1993	115.95	45.97	48.66	49.68	57.03
1994	119.46	51.83	52.58	50.64	57.94
1995	121.39	56.22	55.60	51.60	59.29
1996	122.85	59.40	58.42	51.87	60.72
1997	130.60	64.35	66.78	52.95	62.06
1998	133.68	70.18	70.19	54.25	64.51
1999	139.19	77.75	75.66	55.87	66.80
2000	145.42	84.61	81.75	57.50	69.42
2001	147.83	81.96	77.62	58.36	72.42
2002	155.53	82.19	78.29	59.70	74.95
2003	162.32	81.51	82.07	60.93	77.73
2004	168.06	85.49	88.78	61.75	80.95

In order to make the resulting index number estimates of Canada's productivity for the years 1980–2004, we normalize the productivities by dividing by Prod^{2002}. This makes the resulting normalized index number estimates of productivity, γ^i, comparable to the profit maximizing estimates of relative efficiency listed in Table 3, since we had $\alpha^*_{2002} = \alpha^{**}_{2002} = 1$ and the year 2002 was the only efficient observation for both α^*_i and α^{**}_i. The normalized index number estimates of productivity are listed in the last column of Table 3.

Looking at Table 3, it can be seen that the various efficiency measures satisfy the following inequalities, which we showed in sections 3 and 4 must be satisfied:

$$\delta^{**}_i \le \delta^*_i; \tag{25}$$

$$\epsilon^{**}_i \le \epsilon^*_i; \tag{26}$$

$$\alpha^*_i \le \epsilon^*_i \le \delta^*_i; \tag{27}$$

$$\epsilon^{**}_i \le \delta^{**}_i; \tag{28}$$

$$\alpha^{**}_i \le \delta^{**}_i. \tag{29}$$

Table 2. Price Data on Net Outputs and Primary Inputs for Canada, 1980–2004

Year	p_1	p_2	p_3	w_1	w_2
1980	3.0783	3.7382	3.3640	4.3250	2.8210
1981	3.4053	4.0361	3.7466	4.6735	3.1366
1982	3.7361	4.1491	3.9089	5.1695	3.2346
1983	3.9537	4.1960	3.9273	5.4053	3.3299
1984	4.1081	4.3480	4.1334	5.6786	3.4856
1985	4.2730	4.4370	4.2510	5.9370	3.5477
1986	4.4630	4.4283	4.3272	6.1151	3.6578
1987	4.6241	4.5167	4.2734	6.5117	3.8049
1988	4.8124	4.5288	4.1715	6.9206	3.9791
1989	5.0277	4.6281	4.1734	7.2986	4.1243
1990	5.2515	4.5938	4.2160	7.6279	4.1099
1991	5.4192	4.4235	4.1456	8.0047	4.0010
1992	5.5112	4.5500	4.3145	8.2749	4.0053
1993	5.6198	4.7522	4.5751	8.4190	4.1214
1994	5.7082	5.0337	4.8661	8.4753	4.3004
1995	5.7797	5.3564	5.0152	8.5948	4.3944
1996	5.8433	5.3863	4.9523	8.7775	4.4662
1997	5.9309	5.3945	4.9883	9.1036	4.6262
1998	5.9969	5.3772	5.1623	9.3415	4.6238
1999	6.0794	5.4361	5.1471	9.5678	4.6415
2000	6.2151	5.7743	5.2620	10.0450	4.7601
2001	6.3336	5.8617	5.4230	10.3032	4.7135
2002	6.4492	5.7705	5.4544	10.4646	4.7970
2003	6.5617	5.6630	5.0758	10.6265	4.8548
2004	6.6784	5.7843	4.9621	10.8718	5.0059

For the Canadian data set, we also find *empirically* that

$$\alpha^{**}_i \le \epsilon^{**}_i. \tag{30}$$

However, we cannot establish the inequality (30) as a *theoretical* certainty. Looking at α^*_i versus α^{**}_i, for the Canadian data, it can be seen that for the most part, $\alpha^*_i \le \alpha^{**}_i$ and sometimes α^*_i is substantially below α^{**}_i; i.e., the relative efficiency of an observation when we assume profit maximizing behavior and a convex technology, α^*_i, is generally less than the corresponding relative efficiency of an observation when we assume profit maximizing behavior subject to a fixed capital constraint and a convex, constant returns to scale technology, α^{**}_i. However, for the years 2003 and 2004, this relationship does not hold.

Perhaps the most interesting thing to note about the results listed in Table 3 is that with the exception of the first two years, the index number

Table 3. Relative Efficiencies for Canada, 1980–2004

Year i	δ_i^*	δ_i^{**}	ϵ_i^*	ϵ_i^{**}	α_i^*	α_i^{**}	γ_i
1980	1.0000	1.0000	1.0000	0.9977	0.8308	0.8847	0.8629
1981	1.0000	1.0000	1.0000	1.0000	0.8480	0.8922	0.8604
1982	1.0000	1.0000	1.0000	1.0000	0.7574	0.8438	0.8422
1983	1.0000	1.0000	1.0000	1.0000	0.7659	0.8659	0.8630
1984	1.0000	1.0000	1.0000	1.0000	0.8163	0.8982	0.8894
1985	1.0000	1.0000	1.0000	1.0000	0.8345	0.9121	0.9015
1986	0.9912	0.9909	0.9893	0.9880	0.8343	0.9072	0.8929
1987	1.0000	1.0000	1.0000	1.0000	0.8465	0.9114	0.9026
1988	1.0000	1.0000	1.0000	1.0000	0.8600	0.9156	0.9095
1989	1.0000	1.0000	1.0000	1.0000	0.8528	0.9042	0.9021
1990	0.9844	0.9810	0.9728	0.9706	0.8345	0.8830	0.8833
1991	0.9824	0.9666	0.9596	0.9437	0.8170	0.8619	0.8655
1992	0.9874	0.9635	0.9601	0.9432	0.8273	0.8665	0.8717
1993	0.9890	0.9525	0.9632	0.9406	0.8457	0.8805	0.8844
1994	0.9924	0.9502	0.9732	0.9497	0.8767	0.9075	0.9088
1995	0.9882	0.9479	0.9704	0.9435	0.8804	0.9113	0.9147
1996	0.9922	0.9449	0.9701	0.9372	0.8857	0.9132	0.9179
1997	1.0000	0.9807	0.9955	0.9526	0.9147	0.9337	0.9355
1998	0.9978	0.9752	0.9892	0.9534	0.9322	0.9436	0.9457
1999	0.9992	0.9982	0.9945	0.9791	0.9580	0.9671	0.9675
2000	1.0000	1.0000	1.0000	1.0000	0.9795	0.9854	0.9838
2001	1.0000	1.0000	1.0000	1.0000	0.9780	0.9806	0.9812
2002	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2003	1.0000	1.0000	1.0000	1.0000	0.9951	0.9918	0.9926
2004	1.0000	1.0000	1.0000	1.0000	0.9985	0.9910	0.9928

estimates of efficiency, γ_i, are reasonably close to the efficiency estimates, α_i^{**} , which are based on a (variable) profit maximizing model where we assume capital is fixed and assume that there is a convex, constant returns to scale technology. These results are similar to the results obtained by Mendoza (1989; 111), who obtained nonparametric productivity indexes that were quite similar to the corresponding index number measures of productivity[27].

[27] Mendoza (1989; 129–134) also obtained econometric estimates of sectoral technical change for Canada and she compared these estimates with her nonparametric estimates of sectoral technical change. Her results showed that the econometric estimates of efficiency change are simply a highly smoothed version of the corresponding nonparametric estimates. Diewert and Wales (1992; 718) and Fox (1996) showed that econometric estimates of efficiency change were approximately equal to smoothed versions of index number estimates of productivity growth.

7 A Comparison of the Alternative Methods for Measuring Productive Efficiency

We summarize our comparison of alternative methods for measuring the relative efficiency of a number of production units in the same industry in point form.

- Nonparametric or DEA techniques have an overwhelming advantage over index number and econometric methods when *only* quantity data are available. Index number methods cannot be implemented without a complete set of price and quantity data. Econometric methods (i.e., production function methods) are not likely to be successful if only quantity data are available due to limited degrees of freedom[28].
- The relative efficiency of any single observation will tend to decrease as the sample size increases. All three methods have this problem.
- Nonparametric and econometric efficiency scores will tend to increase as we make less restrictive assumptions on the underlying technology; i.e., a quasiconcave technology set is less restrictive than a convex technology set which in turn is less restrictive than a constant returns to scale convex technology set. Index number estimates of efficiency remain unchanged as we change our assumptions on the technology.
- Nonparametric and economic efficiency scores will tend to decrease as we make stronger assumptions about the optimizing behaviour of producers; recall Rule 2 in section 5. It is not clear what will happen to econometric based efficiency scores under the same conditions. Since index number methods are based on the assumption of complete optimizing behaviour we cannot vary our assumptions on optimizing behaviour when using index number methods.
- If we hold the number of observations in our sample constant but disaggregate the data so that the number of inputs or outputs is increased, then nonparametric efficiency scores will tend to increase.[29] However, index number efficiency scores will generally remain unaffected by increasing disaggregation[30] It is not clear what will happen using econometric methods.

[28] Diewert (1992) discusses this point at some length.

[29] As we disaggregate, the objective functions of the various linear programming problems will remain unchanged but the feasible regions for the problems become more constrained or smaller and hence the objective function minimums for the linear programming problems will become larger. Hence, the loss measures will decrease or remain constant and thus efficiency will tend to increase as we disaggregate. This point was first made by Nunamaker (1985). The profit maximization problems (11) and (16) are not affected by disaggregation.

[30] This follows from the approximate consistency in aggregation property of superlative index number formulae like the Fisher and Trnqvist formulae; see Diewert (1978; 889, 895).

- The cost of computing index number estimates of relative efficiency is extremely low; the cost of the nonparametric estimates is low and the cost of computing econometric estimates can be very high if the number of goods exceeds 20 and flexible functional form techniques are used[31].
- When complete price and quantity data are available, the nonparametric estimates based on a constant returns to scale technology and profit maximizing behaviour (subject to one input being fixed) are approximately equal to the corresponding index number estimates. Econometric estimates based on the same assumptions will tend to be similar to the first two sets of estimates (but much smoother in the time series context).
- Nonparametric techniques can be adapted to deal with situations where input prices are available but not output prices. Econometric techniques can also deal with this situation but index number methods cannot be used in this situation[32].
- Nonparametric methods may be severely biased due to measurement errors; i.e., the best or most efficient observation in a DEA study may be best simply because some output was greatly overstated or some important input was greatly understated. Index number methods are also subject to measurement errors but econometric methods may be adapted to deal with gross outliers.

Our overall conclusion is that DEA methods for measuring relative efficiency can be used profitably in a wide variety of situations when other methods are not practical or are impossible to use.

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[31] The cost of estimating a fully flexible or semiflexible functional form can be high in terms of the analyst's time in doing the econometric estimation. When curvature conditions are imposed using the normalized quadratic functional form and the number of commodities are large, then in order to ensure convergence of the nonlinear regression using Shazam, it is necessary to gradually increase the rank of the substitution matrix by adding an additional rank one matrix to the already estimated substitution matrix and then rerun the model using the finishing parameter values of the previous model as starting values for the new model and so on. The procedure terminates after an iteration where the log likelihood of the model does not increase significantly.

[32] An exception occurs if there is only one output.

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Finding Common Ground: Efficiency Indices

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1 Introduction

The last two decades have witnessed a revival in interest in the measurement of productive efficiency pioneered by Farrell (1957) and Debreu (1951). 1978 was a watershed year in this revival with the christening of DEA by Charnes, Cooper and Rhodes (1978) and the critique of Farrell technical efficiency in terms of axiomatic production and index number theory in Färe and Lovell (1978). These papers have inspired many others to apply these methods and to add to the debate on how best to define technical efficiency.

In this paper we try to pull together some of the variants that have arisen over these decades and show when they are equivalent. The specific cases we take up include: 1) the original Debreu-Farrell measure versus the Russell measure—the latter introduced by Färe and Lovell, and 2) the directional distance function and the additive measure. The former was introduced by Luenberger (1992) and the latter by Charnes, Cooper, Golany and Seiford (1985). We also provide a discussion of the associated cost interpretations. The findings are that the common ground is "small" in the sense of the function satisfying it.

2 Basic Production Theory Details

In this section we introduce the basic production theory that we employ in this paper. We will be focusing on the input based efficiency measures here, but the analysis could readily be extended to the output oriented case as well.

To begin, technology may be represented by its input requirement sets

$$L(y) = \{ x : x \text{ can produce } y \} , \ y \in \Re^M_+ , \tag{1}$$

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AGGREGATION, EFFICIENCY, AND MEASUREMENT

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Preface

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In addition to Daniel Primont's tribute and overview of the contributions of R. Robert Russell to the study of aggregation, efficiency and measurement, this volume consists of nine original papers, which cannot readily be organized into disjoint groups. As Russell pointed out to us when asked how to organize the contributions, defining groups of papers would 'entail serious aggregation error.' We thus chose to include the contribtutions in alphabetical order by first author's surname.

Most of the papers included here were presented and discussed at a symposium held at University of California at Riverside whose title—Conference on Aggregation, Efficiency and Measurement: In Honor of Professor R. Robert Russell—is the basis of the title of this volume. This conference was the brainchild of Taradas Bandyopadhyay, who is a colleague of Russell. Each of the papers in this volume was reviewed by two anonymous referees, to whom we are very grateful. We would also like to thank Lisa Duke and Xiuying Jin for their help in typesetting the manuscript.

What follows is a brief overview of each paper.

Blackorby and Brett study Pareto optima in an overlapping-generations model. In this setting they find that the standard results that obtain in a static general equilibrium model are overturned. In the usual static model Pareto optimality requires the equality of producer and consumer prices; commodity taxes that create a divergence of producer and consumer prices lead to a suboptimal outcome. However, Blackorby and Brett show that in the OLG model in which the government can levy commodity taxes and make generation-specific transfers almost all Pareto optima will involve commodity taxation, subsidies, and taxes on either savings or on capital inputs. Thus, in the OLG setting the government has an important role to play that it does not have in the one-period static case.

Hudgins and Primont show how the usual comparative static results that arise in a model of competitive profit maximization can be derived when using the directional technology distance function as the representation of the firm's technology. They also provide a summary of the derivative restrictions that should be satisfied by a directional technology distance function. Many of these restrictions are the standard ones implied by the assumptions of monotonicity and curvature. However, the restrictions implied by the translation property are unique to directional distance functions and they are the focus of a more detailed scrutiny in the final section of their contribution. These derivative restrictions are useful in formulating econometric models of directional distance functions.

Campbell and Marino distinguish three motivations for mergers, namely 1) market power, 2) technological or efficiency gains from shared fixed inputs and 3) managers' own utility. The first two are included in what the authors call synergistic merger. In contrast to synergy, the authors note that managers' utility can be linked to the 'observability problem' which arises from the principal-agent model. The principal may lose observability of the agents as a consequence of merger. They provide testable predictions of when such mergers are profitable despite the observability problem.

Diewert and Mendoza present a sequence of Data Envelopment Analysis (DEA) models that are used to compute various measures of input efficiency that are in the family of Debreu-Farrell measures that were advocated by R. Robert Russell. They theoretically demonstrate two types of Le-Chatelier results for these measures. In particular, they show that if stronger technological assumptions are imposed on the DEA model then measures of technical input efficiency will decrease and they show that if stronger behavioral assumptions are imposed then overall measures of input efficiency will decrease.

Recent Canadian time-series data are used to illustrate these Le-Chatelier effects. Inspired by Mendoza (1989), their analysis is extended to a comparison of three methods for computing annual rates of productivity change and measures of efficiency loss for each year in the 1961–1980 data set using 1) DEA techniques 2) superlative index numbers and 3) statistical estimation of unit profit functions. The strengths and weaknesses of each approach are highlighted. They conclude that the DEA method can be fruitful, particularly when the other two methods are not practical (or possible.)

In their contribution, Färe, Grosskopf and Zelenyuk try to relate four of the many versions of technical efficiency that have been introduced over the years. They discuss the conditions under which the Farrell measure of technical efficiency and the so-called Russell measure yield the same result. They also study the relationship between the directional distance function and what is referred to as the additive measure in the operations research literature, both of which have an additive structure.

In results that would not surprise Russell, for the two 'multiplicative' measures—Farrell and Russell—to yield the same score the technology must be input homothetic with the input component consistent with equal-weighted Cobb-Douglas form. For the directional distance function to yield the same score as the additive measure, technology must be translation input homothetic with the input aggregator specified as an arithmetic mean.

Grosskopf, Hayes and Taylor provide an empirical interlude by applying the decomposition of labor productivity growth (introduced by Kumar and Russell) to U.S. state manufacturing in the 1990s. They find that the three components: technical change (innovation), efficiency change (diffusion) and capital deepening all played a role, with innovation the primary determinant of manufacturing productivity growth in all states. Capital deepening contributed to labor productivity growth in all but three states, and explains at least half of the labor productivity growth in a dozen states.

In a second stage, these components were related to various policy variables; a growing technology sector is a strong contributor to labor productivity growth, while a growing public sector is largely a drag. Improvements in labor force quality appear to have had little impact on the pace of technical change or the diffusion of technology, but capital deepening was significantly greater in states with a more highly educated population.

Daniel Henderson focuses on technical efficiency and measurement, by providing nonparametric techniques to estimate or measure higher-order moments of technical efficiency. The nonparametric approach allows estimation of these moments without restrictive assumptions on the distribution of inefficiency, which plagued earlier efforts in the stochastic frontier literature. He also provides an empirical example; the estimators are applied to a panel of 17 railway companies over a 14 year time period.

In his contribution, Bill Schworm studies intellectual property rights, efficiency and productivity in a model with endogenous innovation. He uses a stylized version of the Rivera-Batiz and Romer model, which allows him to study equilibria under alternative regimes using standard measures of technical and allocative efficiency. This allows him to compare the efficiency of economies with and without patent rights.

Thijs ten Raa continues the technical efficiency theme, starting with a discussion of the difference between the Farrell (1957) and Debreu (1951) efficiency measures. He chooses the Debreu approach and shows how Debreu's efficiency measure for an economy may be disaggregated into production unit inefficiencies. This contribution gracefully touches on all three issues which unify this volume: aggregation, efficiency and measurement.