Journal of Econometrics 30 (1985) 127–147. North-Holland

TESTS FOR THE CONSISTENCY OF CONSUMER DATA*

W.E. DIEWERT

University of British Columbia, Vancouver, Canada

C. PARKAN

University of Calgary, Calgary, Canada

The paper develops non-parametric tests for the weak separability or the additive separability of a decision-maker's utility function, given a finite body of price and quantity data. The tests generalize some earlier tests proposed by Afriat and Varian. Blackorby, Primont and Russell have pointed out that the econometric literature on testing separability hypotheses is seriously deficient since the econometric tests tend to impose unwanted restrictions on the parametric functional form in addition to the particular separability hypothesis being tested. Hence, the non-parametric tests in the present paper should serve as useful supplements to the usual econometric tests for separability.

1. Introduction

Our basic purpose in writing this paper is twofold: (i) we would like to make Afriat's (1967, 1970, 1972, 1973, 1976) non-parametric method for testing whether a finite body of price and quantity data is consistent with utility maximizing behaviour more accessible, and (ii) we derive some new tests for separability which generalize the earlier work of Afriat (1970) and Varian (1983).

In section 2, we deal with point (i) above. It turns out that the necessity of the Afriat conditions can be established in a very intuitive manner if we make stronger than necessary assumptions on the utility function – namely, concavity and differentiability. We make these restrictive assumptions in section 2 and derive the Afriat conditions.[1] In an appendix, we indicate how the differentiability assumptions can be dropped.[2]

*The research for this paper has been supported by the SSHRC of Canada and by grant SES79-12380 from the National Science Foundation to the University of California at Berkeley. Our thanks to W. Barnett and a referee for helpful comments and to Gerard Debreu for his inspiration and generosity to the first author.

[1] Our exposition is based on Diewert and Parkan (1978). Similar expositions are available in Varian (1982, 1983). However, the basic idea may be traced back to Afriat (1970).

[2] All of our proofs involving the non-differentiable case are relegated to the appendix.

In sections 3 and 4, we utilize the techniques developed in section 2 in order to derive some tests for the utility function to be weakly separable and additively separable respectively. Our tests generalize similar tests developed by Afriat (1970) and Varian (1983) in various directions.

Section 5 concludes.

2. The derivation of the Afriat consistency conditions in the concave case

We wish to determine whether a finite body of price and quantity data is consistent with utility maximizing behaviour.

Suppose that we are given T positive price and quantity vectors, $p^t \gg 0_N$ and $x^t \gg 0_N$ for $t = 1, 2, \ldots, T$.[3] Suppose that a decision-maker has the objective or utility function f and that x^t is a solution to his period t utility maximization problem; i.e., suppose

$$x^t \text{ solves } \max_x \{ f(x) \colon p^t \cdot x \le p^t \cdot x^t,\ x \ge 0_N \} \quad \text{for} \quad t = 1, \ldots, T. \tag{1}$$

Suppose further that the utility function f is: (i) concave,[4] (ii) once differentiable, and (iii) weakly increasing.[5]

Since by assumption, $x^t \gg 0_N$ and so the non-negativity constraints in (1) are not binding, the Kuhn–Tucker necessary conditions for the non-linear programming problems in (1) reduce to:[6] there exist λ^t such that

$$\nabla f(x^t) = \lambda^t p^t, \qquad \lambda^t \ge 0, \qquad t = 1, \ldots, T. \tag{2}$$

The weakly increasing property of f and our assumption that p^t is strictly positive imply that the Lagrange multipliers λ^t appearing in conditions (2) are

[3] Notation: $p \gg 0_N$ means that each component p_n of the N-dimensional vector p is positive; $p \ge 0_N$ means each component of p_n is non-negative; $p > 0_N$ means $p \ge 0_N$ but $p \ne 0_N$; $p \cdot x \equiv \sum_{n=1}^{N} p_n x_n$ denotes the inner product of the vectors p and x and $\nabla_x f(x, y) \equiv [\partial f(x, y)/\partial x_1, \ldots, \partial f(x, y)/\partial x_N]$ denotes the vector of partial derivatives of f with respect to its x components.

[4] f is concave iff $x' \ge 0_N$, $x'' \ge 0_N$, $0 \le \lambda \le 1$ implies $f(\lambda x' + (1 - \lambda) x'') \ge \lambda f(x') + (1 - \lambda) f(x'')$. Throughout this paper we shall assume that concave functions are also continuous over their domain of definition.

[5] In the case of differentiable functions, we say that f is *increasing* if $x > 0_N$ implies $\nabla f(x) \gg 0_N$, *weakly increasing* if $\nabla f(x) > 0_N$, and *non-decreasing* if $\nabla f(x) \ge 0_N$. In the non-differentiable concave case, replace $\nabla f(x)$ in the previous definitions by every $z \in \partial f(x)$ where $\partial f(x)$ denotes the set of supergradients to f at x. z is a supergradient to f at x^0 iff $f(x) \le f(x^0) + z \cdot (x - x^0)$ for every $x \ge 0_N$; see Rockafellar (1970, p. 214).

[6] See Karlin (1959, p. 204) for a convenient exposition of the Kuhn–Tucker Theorem. The positivity of x^t enables us to conclude that the Slater constraint qualification condition [see problem 12 in Karlin (1959, p. 239)] is satisfied for problem t in (1) since $\frac{1}{2} x^t$ satisfies all constraints strictly.

all positive:

$$\lambda^t > 0, \qquad t = 1, \ldots, T. \tag{3}$$

We require one additional fact about concave functions and then we shall have the Afriat conditions at hand. A differentiable concave function has the property [see, for example, Karlin (1959, p. 405)] that any tangent hyperplane to f (i.e., any first-order Taylor series expansion of f) lies above the function. Thus for each t, the following inequality is valid for all $x \ge 0_N$:

$$f(x) \le f(x^t) + \nabla f(x^t) \cdot (x - x^t), \qquad t = 1, \ldots, T. \tag{4}$$

Now define $u^t \equiv f(x^t)$ for $t = 1, \ldots, T$, and replace x in (4) by x^s, $s = 1, \ldots, T$. The inequalities (4) then become the following T^2 inequalities:

$$u^s \le u^t + \lambda^t p^t \cdot (x^s - x^t), \qquad s, t = 1, \ldots, T. \tag{5}$$

Thus we have proven the following theorem, which is a special case of a much stronger theorem due to Afriat (1967, 1973, 1976):

Theorem 1. In order for a body of positive *data,* $\{(p^t, x^t) \colon t = 1, \ldots, T\}$, *to be consistent with utility maximizing behaviour [so that (1) is true] where the utility function f is restricted to be concave, differentiable and weakly increasing, it is* necessary *that there exist* $2T$ *numbers* $u^1, \ldots, u^T$ *(which can be interpreted as utility levels) and* $\lambda^1, \ldots, \lambda^T$ *(which can be interpreted as marginal utilities of income) such that* Afriat revealed preference inequalities *(3) and (5) are satisfied.*

In the appendix, we indicate how the same technique of proof may be modified in order to prove a version of Theorem 1 without assuming that f be differentiable.

Conditions (3) and (5) are also *sufficient* for the given data p^t, x^t to be consistent with utility maximizing behaviour. We have the following result:

Theorem 2 [Afriat (1967)]. Suppose $\{(p^t, x^t) \colon t = 1, \ldots, T\}$ *are given price and quantity vectors with* $p^t \gg 0_N$ *and* $x^t \ge 0_N$,[7] *and there exist* u^t, λ^t *such that conditions (3) and (5) are satisfied. Then there exists a concave, increasing utility function f which rationalizes the data; i.e., conditions (1) are satisfied for this f.*

Proof. Define the increasing linear functions f^t by

$$f^t(x) \equiv u^t + \lambda^t p^t \cdot (x - x^t), \qquad t = 1, \ldots, T. \tag{6}$$

[7] Note that we no longer require that $x^t \gg 0_N$ as in Theorem 1.

Define the increasing, concave function f by

$$f(x) \equiv \min_t \{ f^t(x) : t = 1, \ldots, T \}. \tag{7}$$

Since f defined by (7) is the minimum of the f^s, for each s, we must have $f(x) \leq f^s(x)$ and thus, for $s = 1, \ldots, T$,

$$\begin{aligned} &\max_x \{ f(x) : p^s \cdot x \leq p^s \cdot x^s, x \geq 0_N \} \\ &\leq \max_x \{ f^s(x) : p^s \cdot x \leq p^s \cdot x^s, x \geq 0_N \} \\ &\equiv \max_x \{ u^s + \lambda^s p^s \cdot (x - x^s) : p^s \cdot x \leq p^s \cdot x^s, x \geq 0_N \} \\ &= u^s, \end{aligned} \tag{8}$$

since $\lambda^s > 0$. Also, for $s = 1, \ldots, T$,

$$\begin{aligned} f(x^s) &\equiv \min_t \{ f^t(x^s) : t = 1, \ldots, T \} \\ &\equiv \min_t \{ u^t + \lambda^t p^t \cdot (x^s - x^t) : t = 1, \ldots, T \} \\ &= u^s, \end{aligned} \tag{9}$$

using (5). Thus (8) and (9) show that x^s is a solution to the period s utility maximization problem for $s = 1, \ldots, T$ when f is defined in (7). Q.E.D.

To summarize: the existence of numbers u^t and λ^t such that the inequalities (3) and (5) are true is necessary and sufficient for a given body of data $\{ p^t, x^t \}$ to be consistent with utility maximizing behaviour given our regularity conditions on the underlying utility function f.

The Afriat conditions (3) and (5) may be put into several alternative sets of conditions that are easier to check numerically.

For example, a necessary and sufficient condition for the existence of numbers u^t and λ^t such that (3) and (5) are true for a given body of data $\{(p^t, x^t);\ t = 1, \ldots, T\}$ is that the objective function of the following linear programming problem attain its lower bound of zero:

$$\min \sum_{r=1}^{T} \sum_{t=1}^{T} s^{rt}, \tag{10}$$

subject to

$$s^{rt} = u^t - u^r - \lambda^r p^r \cdot (x^t - x^r) + S^{rt}, \qquad r, t = 1, \ldots, T, \tag{11}$$

$$\lambda^r \geq 1, \qquad r = 1, \ldots, T, \tag{12}$$

$$s^{rt} \geq 0, \quad S^{rt} \geq 0, \qquad r, t = 1, \ldots, T, \tag{13}$$

where the u^r, λ^r, s^{rt} and S^{rt} are variables. The s^{rt} and S^{rt} are non-negative slack variables that are used to convert the inequalities (5) into equalities.[8] If the objective function (10) attains its lower bound of zero, then all of the optimal $s^{rt} = 0$, and it is obvious that (11) and the non-negativity restrictions $S^{rt} \geq 0$ imply (5). Obviously, (12) implies (3). On the other hand, if the inequalities (5) are satisfied with $\lambda^r > 0$ for all r, then all of the numbers u^r and λ^r can be scaled so that the same inequalities (5) are satisfied with $\lambda^r \geq 1$ for $r = 1, \ldots, T$. Hence the linear programming problem (10)–(13) will attain its lower bound of zero in this case.

Another necessary and sufficient condition for the existence of numbers u^t and λ^t such that (3) and (5) are true for a given body of data $\{(p^t, x^t):$ $t = 1, \ldots, T\}$ is that the objective function for the following unconstrained minimization problem attain its lower bound of zero:

$$\min_{u^1, \ldots, u^T, \alpha_1, \ldots, \alpha_T, \sigma_{11}, \ldots, \sigma_{TT}} \left\{ \sum_{r=1}^{T} \sum_{t=1}^{T} [s^{rt}]^2 \right\}, \tag{14}$$

where the T^2 slack variables s^{rt} are defined by

$$s^{rt} \equiv u^t - u^r - (\alpha_r^2 + 1) p^r \cdot (x^t - x^r) + \sigma_{rt}^2, \qquad r, t = 1, \ldots, T. \tag{15}$$

If we define $\lambda^r \equiv \alpha_r^2 + 1 \geq 1$ and $S^{rt} \equiv \sigma_{rt}^2 \geq 0$, it can be seen that eq. (15) corresponds to eq. (11).

Varian (1982, 1983) has developed extremely efficient algorithms for checking whether a body of data satisfied (3) and (5); however, his procedures do not seem to extend to the more complicated conditions that we are going to develop in sections 3 and 4 below. Until more efficient algorithms are invented, we can only suggest that the reader convert the problem of checking whether a solution exists to a certain system of inequalities into an unconstrained minimization problem in a manner analogous to our conversion of (3) and (5) into (14).

[8] If we are in the context of producer theory rather than consumer theory so that $f(x^t) \equiv u^t$ is output rather than utility, then we may want to add the non-negativity restrictions $u^t \geq 0$, $t = 1, \ldots, T$, to the linear program (10)–(13). Actually, the tests developed in this paper are efficient in the producer theory context only if output is unobserved.

It should be noted that the assumption of concavity is not required in order to derive conditions (3) and (5). Afriat's much stronger (but harder to prove) result is:

Theorem 3 [*Afriat (1967, 1976), Diewert (1973), Varian (1982)*]. *In order for the data* $\{(p^t, x^t)\colon t=1,\ldots,T\}$, *where* $p^t \gg 0_N$ *and* $x^t \geq 0_N$,[9] *to be consistent with utility maximizing behaviour where the utility function* f *is continuous from above*[10] *and is subject to local non-station,*[11] *it is necessary that there exist* u^t *and* λ^t *such that (3) and (5) are satisfied.*

It should be noted that conditions (5) provide a generalization of Samuelson's (1938) weak axiom and Houthakker's (1950) strong axiom of revealed preference theory; see Afriat (1976). Furthermore, conditions (5) are also related to the integrability of demand functions literature; see Mas-Colell (1978).

There is one additional test for consistency that we wish to survey in this section. Frequently, one wishes to know whether a given data set is consistent with utility maximizing behaviour where the utility function f is restricted to be *homothetic*. Shephard (1953) defines a non-negative function $f(x)$ defined over $x \geq 0_N$ to be homothetic if there exists an increasing function of one variable g such that $g[f(x)]$ is positively, linearly homogeneous; i.e., for every $\lambda > 0$, $x \geq 0_N$, $g[f(\lambda x)] = \lambda g[f(x)]$. It can be shown [see Afriat (1972) or Diewert (1973, p. 424)] that a necessary and sufficient condition for the existence of a homothetic, continuous from above and locally non-satiated utility function f which is consistent with the data $\{(p^t, x^t)\colon t=1,\ldots,T\}$ in the sense that x^t is a solution to the period t maximization problem in (1) for $t=1,\ldots,T$ is that there exist numbers u^t and λ^t such that (3) and (5) hold and in addition:

$$u^t = \lambda^t p^t \cdot x^t, \qquad t=1,\ldots,T. \tag{16}$$

The effect of the additional restrictions (16) is to make each of the linear functions f^t defined by (6) pass through the origin. Thus f defined by (7) will be the surface of a convex, polyhedral cone and so will be linearly homogeneous (and thus homothetic).

Thus, to test for the existence of a homothetic utility function f, we need only solve the linear programming problem (10)–(13) with the additional restrictions (16) and see whether the resulting optimal objective function is zero or not.[12]

For a geometrical interpretation of conditions (3), (5) and (16) for the case of two observations ($T=2$) and two goods ($N=2$), see Afriat (1977). For alternative formulations of (3), (5) and (16) in the general case plus an extremely efficient algorithm for checking whether a given data set satisfied (3), (5) and (16), see Varian (1982, 1983).

[9] Note that we do not require that $x^t \gg 0_N$ as in Theorem 1.

[10] This means that, for each u, the set $\{x\colon f(x) \geq u\}$ is closed. This assumption is required to ensure that solutions to the utility maximization problems in (1) exist.

[11] This means that for every $x \geq 0_N$ and $\varepsilon > 0$, there exists $x^\varepsilon \geq 0_N$ such that $(x-x^\varepsilon)\cdot(x-x^\varepsilon) \leq \varepsilon^2$ and $f(x^\varepsilon) > f(x)$.

3. Tests for weak separability

Assume the decision-maker has preferences defined over $N^1 + N^2 + \cdots + N^H + M$ goods. We suppose that these preferences can be represented by a utility function $F(x_1,\ldots,x_H, y)$ where $x_h \geq 0_{N^h}$ is a non-negative vector of dimension N^h and $y \geq 0_M$ is a non-negative vector of dimension M. We wish to know whether the decision-maker's utility function F can be written as $F(x_1,\ldots,x_H,y) = f[g_1(x_1),\ldots,g_H(x_H),y]$, where f is a function of $H+M$ variables and g_h is a function of N^h variables for $h=1,\ldots,H$; i.e., we wish to know if F is *weakly separable* with respect to its $x_1,\ldots,x_H$ arguments.[13] The function f is the macro function, while the functions g_h are called micro functions.[14] Assuming that the micro and macro functions are all concave leads to a simple set of necessary and sufficient conditions for the consistency of a given data set with the hypothesis of utility maximizing behaviour where the utility function is restricted to be weakly separable.

Theorem 4 [*Generalization of Afriat (1970) and Varian (1983)*].[15] *Suppose the macro function f is (i) concave, (ii) once differentiable, and (iii) increasing in its first H arguments and non-decreasing in its last M arguments where* $H \geq 1$ *and* $M \geq 1$. *Suppose the micro functions* g_h, $h=1,\ldots,H$, *are (i) concave, (ii) once differentiable, and (iii) weakly increasing in their arguments. Suppose further that we are given T positive price vectors,* $(p_1^t,\ldots,p_H^t,q^t) \gg (0_{N^1},\ldots,0_{N^H},0_M)$, *and T* positive *quantity vectors,* $(x_1^t,\ldots,x_H^t,y^t) \gg (0_{N^1},\ldots,0_{N^H},0_M)$, *for* $t=1,\ldots,T$. *Finally, we suppose that the given data are consistent with weakly separable utility maximizing behaviour; i.e., for* $t=1,\ldots,T$,

[12] Alternatively, use eq. (16) to eliminate the u^t from (15) and solve the resulting unconstrained minimization problem (14).

[13] For alternative representations of the separability concept and extensive references to the literature, see Blackorby, Primont and Russell (1978).

[14] This terminology follows that of Diewert (1974, pp. 150–153).

[15] Afriat and Varian consider the case where $H=1$. Afriat also considers the case where $H=2$ and $M=0$.

$(x_1^t,\dots,x_H^t,y^t)$ solves the following utility maximization problem:

$$\max_{x_1,\dots,x_H,y}\left\{f[g_1(x_1),\dots,g_H(x_H),y]:\right.$$

$$\left.\sum_{h=1}^{H} p_h^t\cdot x_h+q^t\cdot y\le Y^t,\ x_1\ge 0_{N^1},\dots,x_H\ge 0_{N^H},\ y\ge 0_M\right\}, \tag{17}$$

where period t 'income' Y^t is defined by

$$Y^t\equiv\sum_{h=1}^{H} p_h^t\cdot x_h^t+q^t\cdot y^t,\qquad t=1,\dots,T. \tag{18}$$

Then there exist 2HT scalars X_h^t,β_h^t, $h=1,\dots,H$, $t=1,\dots,T$, such that the following inequalities hold:

$$X_h^s\le X_h^t+\beta_h^t p_h^t\cdot(x_h^s-x_h^t), \tag{19}$$

$$\beta_h^t>0,\qquad s,t=1,\dots,T,\quad h=1,\dots,H.$$

There also exist 2T scalars u^t,λ^t, $t=1,\dots,T$, such that

$$u^s\le u^t+\lambda^t\left[\sum_{h=1}^{H}(\beta_h^t)^{-1}(X_h^s-X_h^t)\right]+\lambda^t q^t\cdot(y^s-y^t), \tag{20}$$

$$\lambda^t>0,\qquad s,t=1,\dots,T.$$

Proof. Since f is increasing in its first H arguments, by (17) x_h^t must maximize $g_h(x_h)$ subject to a period t expenditure constraint on the x_h goods; i.e., we must have

$$x_h^t\gg 0_{N^h}\quad\text{solves}\quad\max_{x_h}\{g_h(x_h):\ p_h^t\cdot x_h\le p_h^t\cdot x_h^t,\ x_h\ge 0_{N^h}\}, \tag{21}$$

$$h=1,\dots,H,\quad t=1,\dots,T.$$

Now repeat the argument used in Theorem 1 applied to (21), and we obtain the existence of the X_h^t and β_h^t that satisfy (19), where

$$X_h^t\equiv g_h(x_h^t),\qquad h=1,\dots,H,\quad t=1,\dots,T. \tag{22}$$

The counterparts to the Kuhn–Tucker conditions (2) and (3) now are

$$\nabla_x g_h(x_h^t)=\beta_h^t p_h^t,\qquad \beta_h^t>0,\qquad h=1,\dots,H,\quad t=1,\dots,T. \tag{23}$$

Define the level and first-order partial derivatives of f by

$$u^t\equiv f[X_1^t,\dots,X_H^t,y^t],\qquad t=1,\dots,T, \tag{24}$$

$$\alpha_h^t\equiv\partial f[X_1^t,\dots,X_H^t,y^t]/\partial X_h>0,\quad h=1,\dots,H,\quad t=1,\dots,T, \tag{25}$$

$$\gamma^t\equiv\nabla_y f[X_1^t,\dots,X_H^t,y^t]\ge 0_M,\qquad t=1,\dots,T. \tag{26}$$

The inequalities in (25) and (26) follow from our monotonicity assumptions on f.

The concavity of f implies, using (4), (24), (25) and (26), that for each X_h in the range of g_h and $y\ge 0_M$,

$$f[X_1,\dots,X_H,y]\le u^t+\sum_{h=1}^{H}\alpha_h^t(X_h-X_h^t)+\gamma^t\cdot(y-y^t), \tag{27}$$

$$t=1,\dots,T.$$

Substitute $(X_1,\dots,X_H,y)=(X_1^s,\dots,X_H^s,y^s)$ for $s=1,\dots,T$ into (27), and we obtain (using (24))

$$u^s\le u^t+\sum_{h=1}^{H}\alpha_h^t(X_h^s-X_h^t)+\gamma^t\cdot(y^s-y^t),\qquad s,t=1,\dots,T. \tag{28}$$

The Kuhn–Tucker conditions for the tth utility maximization problem in (21) and definitions (25) and (26) yield the existence of λ^t such that

$$\alpha_h^t\nabla_x g_h(x_h^t)=\lambda^t p_h^t,\qquad \lambda^t>0,\qquad t=1,\dots,T, \tag{29}$$

$$\gamma^t=\lambda^t q^t,\qquad t=1,\dots,T. \tag{30}$$

Eqs. (23), (25) and (29) show that $\alpha_h^t=\lambda^t(\beta_h^t)^{-1}$ for $t=1,\dots,T$, $h=1,\dots,H$. Substitution of these equalities and the equalities in (30) into (28) yields (20). Q.E.D.

Somewhat surprisingly, we are unable to modify the above proof to cover the non-differentiable case (as was done in the case of Theorem 1).[16] However,

[16] We are unable to establish (29) in the non-differentiable case. In general, it is difficult to characterize the set of supergradients of a composite function like $f[g_1(x_1),\dots,g_H(x_H),y]$ in terms of the supergradients of $f,g_1,\dots,g_H$; see Rockafellar (1978) for a survey of the literature.

in Theorem 6 below, we indicate how a non-differentiable version of Theorem 4 can be obtained if we make use of the duality between utility and cost functions. Before proceeding to the non-differentiable case, we first show that conditions (19) and (20) are *sufficient* for the given data set to be consistent with weakly separable utility maximizing behaviour.

Theorem 5 [*Generalization of Afriat (1970) and Varian (1983)*]. *Suppose that we are given T positive price vectors,* $(p_1^t,\ldots,p_H^t,q^t) \gg (0_{N^1},\ldots,0_{N^H},0_M)$, *and t non-negative quantity vectors,* $(x_1^t,\ldots,x_H^t,y^t) \geq (0_{N^1},\ldots,0_{N^H},0_M)$, *for* $t=1,\ldots,T$. *Suppose that there exist* $2T(H+1)$ *numbers* $X_h^t, \beta_h^t, u^t, \lambda^t$, $h=1,\ldots,H$, $t=1,\ldots,T$, *such that these numbers and the given data satisfy the inequalities (19) and (20). Then there exists a concave, increasing macro function f of H + M variables and there exist H concave, increasing micro functions* g_h, $h=1,\ldots,H$, *such that (17) and (21) holds for these functions.*

The proof is analogous to the proof of Theorem 2. The g_h are defined by

$$g_h(x_h) \equiv \min_t\{g_h^t(x_h):\ t=1,\ldots,T\}, \qquad h=1,\ldots,H,$$

where the TH linear functions g_h^t are defined by

$$g_h^t(x_h) \equiv X_h^t + \beta_h^t p_h^t \cdot (x_h - x_h^t).$$

The functions f and f^t are defined by

$$f^t(X_1,\ldots,X_H,y) \equiv u^t + \lambda^t \sum_{h=1}^{H} (\beta_h^t)^{-1}(X_h - X_h^t) + \lambda^t q^t \cdot (y - y^t),$$

$$f(X_1,\ldots,X_H,y) \equiv \min_t\{f^t(X_1,\ldots,X_H,y):\ t=1,\ldots,T\}.$$

The inequalities (19) and (20) play the same role in this section that the inequalities (3) and (5) played in the previous section. Unfortunately, the inequalities (19) and (20) are not linear in the 'unknowns', X_h^t, β_h^t, u^t and λ^t, so that linear programming techniques cannot be used to test whether a given body of price and quantity data is consistent with weakly separable utility maximizing behaviour. However, the unconstrained minimization technique associated with (14) in the previous section can be adapted to determine whether the inequalities (19) and (20) have a solution. The details are left to the reader.

Comparing Theorem 3 with Theorem 1, it seems evident that we should be able to weaken our regularity conditions on the macro and micro functions and establish a stronger version of Theorem 4. The following theorem is a step in this direction: the concavity assumption on the macro function f is replaced by a quasi-concavity[17] assumption.

The proof of Theorem 6 requires some knowledge about the duality between cost and utility functions. Define the cost function C_h that is dual to the micro function g_h for $h=1,2,\ldots,H$ by

$$C_h(X_h,p_h) \equiv \min_{x_h}\{p_h \cdot x_h:\ g_h(x_h) \geq X_h,\ x \geq 0_{N^h}\}, \tag{31}$$

where $p_h \gg 0_{N^h}$ and X_h belongs to the range of g_h. If g_h is continuous, quasi-concave, and weakly increasing, then C_h will satisfy various regularity conditions, including continuity and increasingness in its X_h argument.[18] If in addition, g_h is concave, then C_h will be convex in its X_h argument.[19] In the following theorem, in order to simplify the proof, we shall assume that each $C_h(X_h,p_h)$ is differentiable with respect to its X_h argument.[20] The non-differentiable case is proven in the appendix.

Theorem 6 (*Generalization of Theorem 4*). *Suppose the macro function f is (i) continuous from above, (ii) subject to local non-satiation, and (iii) quasi-concave. Suppose the micro functions* g_h, $h=1,\ldots,H$, *are (i) concave, (ii) weakly increasing, and (iii) have dual cost functions* $C_h(X_h,p_h)$ *that are differentiable with respect to their* X_h *arguments. Suppose further that we are given T positive price vectors,* $(p_1^t,\ldots,p_H^t,q^t) \gg (0_{N^1},\ldots,0_{N^H},0_M)$; *T non-negative quantity vectors,* $y^t \geq 0_M$; *HT non-negative but non-zero quantity vectors,* $x_h^t > 0_{N^h}$, *for* $h=1,\ldots,H$, $t=1,\ldots,T$. *Finally, suppose that the data are consistent with (weakly) separable utility maximizing behaviour; i.e., (17) holds. Then there exist numbers* X_h^t, β_h^t, u^t *and* λ^t *such that (19) and (20) hold.*

Proof. Define

$$X_h^t \equiv g_h(x^t), \qquad h=1,\ldots,H, \quad t=1,\ldots,T. \tag{32}$$

[17] A function f is quasi-concave iff the upper level sets $L(u) \equiv \{x:\ f(x) \geq u\}$ are convex for each level u in the range of f. A set S is convex iff $x' \in S$, $x'' \in S$, $0<\lambda<1$ implies $\lambda x' + (1-\lambda)x'' \in S$.

[18] See Diewert (1982).

[19] See Diewert (1978, p. 33). A function f is convex iff $-f$ is concave.

[20] Blackorby and Diewert (1979) provide sufficient conditions on g_h which guarantee the differentiability of $C_h(X_h,p)$ with respect to X_h.

Using (31) and (32), the relations (17) may be rewritten as

$X_1^t, \ldots, X_H^t, y^t$ solves

$$\max_{X_1,\ldots,X_H,y} \left\{ f(X_1,\ldots,X_H,y): \sum_{h=1}^{H} C_h(X_h, p_h^t) + q^t \cdot y \le Y^t, \right.$$
$$\left. y \ge 0_M,\ X_h \in \text{range } g_h,\ h=1,\ldots,H \right\}. \tag{33}$$

For $t=1,\ldots,T$, define the sets R^t and S^t by

$$R^t \equiv \left\{ (X_1,\ldots,X_H,y): Y^t - \sum_{h=1}^{H} C_h(X_h, p_h^t) - q^t \cdot y \ge 0, \right.$$
$$\left. X_h \in \text{range } g_h,\ h=1,\ldots,H \right\}, \tag{34}$$

$$S^t \equiv \{ (X_1,\ldots,X_H,y): f(X_1,\ldots,X_H,y) \ge f(X_1^t,\ldots,X_H^t,y^t),$$
$$y \ge 0_M,\ X_h \in \text{range } g_h,\ h=1,\ldots,H \}. \tag{35}$$

S^t is a convex set since f is quasi-concave. R^t is also a convex set, since the functions $-C_h(X_h, p_h^t)$ are concave in X_h.[21] The relations (33) show that S^t and R^t have no interior points in common, and hence by the Separating Hyperplane Theorem [Karlin (1959, p. 398)], there exists a hyperplane that separates the two sets; i.e., for $t=1,\ldots,T$, there exist vectors $A^t \equiv (A_1^t,\ldots,A_H^t)$ and $a^t \equiv (a_1^t,\ldots,a_M^t)$ (at least one of these two vectors is non-zero) and a scalar α^t such that

$$(X,y) \in R^t \quad \text{implies} \quad A^t \cdot X + a^t \cdot y \le \alpha^t, \tag{36}$$

$$(X,y) \in S^t \quad \text{implies} \quad A^t \cdot X + a^t \cdot y \ge \alpha^t. \tag{37}$$

Since (X^t, y^t) belongs to both R^t and S^t, we have

$$\alpha^t = A^t \cdot X^t + a^t \cdot y^t, \qquad t=1,\ldots,T. \tag{38}$$

[21] Since the g_h are concave, the C_h are convex in their X_h arguments and hence the functions $-C_h$ are concave.

Hence $(X_1^t,\ldots,X_H^t,y^t)$ solves the concave programming problem (39) and the quasi-concave programming problem (40) below:

$$\max_{X_1,\ldots,X_H,y} \left\{ \sum_{h=1}^{H} A_h^t X_h + a^t \cdot y: Y^t - \sum_{h=1}^{H} C_h(X_h, p_h^t) + q^t \cdot y \ge 0, \right.$$
$$\left. X_h \in \text{range } g_h,\ h=1,\ldots,H \right\}, \tag{39}$$

$$\min_{X_1,\ldots,X_H,y} \left\{ \sum_{h=1}^{H} A_h^t X_h + a^t \cdot y: f(X_1,\ldots,X_H,y) \ge f(X_1^t,\ldots,X_H^t,y^t), \right.$$
$$\left. y \ge 0_M,\ X_h \in \text{range } g_h,\ h=1,\ldots,H \right\}. \tag{40}$$

Since $X_1^t,\ldots,X_H^t,y^t$ solves (39) and since the constraints $X_h \in$ range g_h are not binding at this solution,[22] the Kuhn–Tucker conditions for (39) yield the existence of a Lagrange multiplier $\xi^t \ge 0$ such that the following first-order conditions hold:

$$A_h^t = \xi^t \partial C_h(X_h^t, p_h^t)/\partial X_h \equiv \xi^t P_h^t, \tag{41}$$
$$h=1,\ldots,H, \quad t=1,\ldots,T,$$

$$a^t = \xi^t q^t, \qquad t=1,\ldots,T. \tag{42}$$

Since $(A^t, a^t) \ne 0_{H+M}$, we must have $\xi^t \ne 0$ and hence

$$\xi^t > 0, \qquad t=1,\ldots,T. \tag{43}$$

The substitution of (41) and (42) into (40) and the use of (43) means that, for $t=1,\ldots,T$,

$X_t^1,\ldots,X_H^t,y^t$ solves

$$\min_{X_1,\ldots,X_H,y} \left\{ \sum_{h=1}^{H} P_h^t X_h + q^t \cdot y: f(X_1,\ldots,X_H,y) \ge f(X_1^t,\ldots,X_H^t,y^t), \right.$$
$$\left. y \ge 0_M,\ X_h \in \text{range } g_h,\ h=1,\ldots,H \right\}. \tag{44}$$

[22] We need the weakly increasing property of the g_h and $x_h^t > 0_{N^h}$ to establish this. We can also establish that the Slater constraint qualification condition holds.

Since the problem of minimizing the cost of achieving a given utility level is dual to the problem of maximizing utility subject to an expenditure constraint (when the utility function is subject to local non-satiation), (44) implies that, for $t = 1,\ldots,T$,

$X_1^t,\ldots,X_H^t, y^t$ solves

$$\max_{X_1,\ldots,X_H,y}\left\{f(X_1,\ldots,X_H,y):\ \sum_{h=1}^{H} P_h^t X_h + q^t\cdot y \le \sum_{h=1}^{H} P_h^t X_h^t + q^t\cdot y^t,\right.$$
$$\left. y \ge 0_M,\ X_h \in \text{range } g_h,\ h=1,\ldots,H\right\}. \tag{45}$$

The maximization problems that occur in (45) are exactly the kind of utility maximization problems that occurred in Theorem 3. Hence, applying Theorem 3, there exists λ^t, u^t such that

$$u^s \le u^t + \lambda^t \sum_{h=1}^{H} P_h^t (X_h^s - X_h^t) + \lambda^t q^t\cdot(y^s - y^t),$$
$$\lambda^t > 0, \qquad s,t = 1,\ldots,T, \tag{46}$$

which is (20) if we define β_h^t by

$$(\beta_h^t)^{-1} \equiv P_h^t \equiv \partial C_h(X_h^t, p_h^t)/\partial X_h > 0,$$
$$h=1,\ldots,H, \quad t=1,\ldots,T. \tag{47}$$

We have yet to establish (19). Since the g_h are concave, the C_h are convex in their quantity arguments. Hence applying the negative of the inequality (4), we have, for $t=1,\ldots,T$ and $h=1,\ldots,H$,

$$C_h[g_h(x_h), p_h^t] - C_h[g_h(x_h^t), p_h^t]$$
$$\ge [\partial C_h[g_h(x_h^t, p_h^t]/\partial X_h][g_h(x_h) - g_h(x_h^t)]. \tag{48}$$

Let $x_h = x_h^s$ for $s=1,\ldots,T$ in (48) and use (32) and (47) to obtain

$$C_h[X_h^s, p_h^t] - C_h[X_h^t, p_h^t] \ge P_h^t(X_h^s - X_h^t),$$
$$h=1,\ldots,H, \quad s,t=1,\ldots,T. \tag{49}$$

We also have

$$C_h(X_h^t, p_h^t) = p_h^t\cdot x_h^t, \qquad h=1,\ldots,H, \quad t=1,\ldots,T, \tag{50}$$

$$C_h(X_h^s, p_h^t) \equiv \min_x\{p_h^t:\ g_h(x) \ge g_h(x_h^s) \equiv X_h^s,\ x \ge 0_{N^h}\}$$
$$\le p_h^t\cdot x_h^s, \qquad h=1,\ldots,H, \quad s,t=1,\ldots,T, \tag{51}$$

where the last inequality follows from the feasibility of x_h^s for the cost minimization problem. Eqs. (49) to (51) yield

$$p_h^t\cdot(x_h^s - x_h^t) \ge P_h^t(X_h^s - X_h^t),$$
$$h=1,\ldots,H, \quad s,t=1,\ldots,T, \tag{52}$$

which is (19) using (47). Q.E.D.

Comparing the hypotheses of Theorem 6 with those of Theorem 4, we see that the hypotheses of the latter theorem are considerably weaker: (i) the concavity and monotonicity assumptions on the macro function f have been weakened to quasi-concavity and local non-satiation, (ii) the differentiability hypotheses on both the macro and micro functions have been dropped, and (iii) the positivity restrictions on the quantity data have been considerably weakened.

The hypotheses of Theorem 6 can be weakened and the result will still be true. Let $\tilde{g}_h$ denote the free disposal, quasi-concave hull of the micro function g_h for $h=1,\ldots,H$.[23] Then all we require is that $\tilde{g}_h$ be concave and weakly increasing for $h=1,\ldots,H$.[24] These hypotheses can be weakened even further: all we require is that the $\tilde{g}_h$ be transformable into concave functions.[25]

Note that even under our weakened hypotheses, we still seem to require some sort of curvature restriction on both macro and micro functions. However, the original Afriat result, Theorem 3, required no curvature restrictions at

[23] $\tilde{g}_h$ can be constructed from g_h as follows. Define the cost function C_h dual to g_h by (38). Define $\tilde{g}_h$ in terms of C_h for $x_h \ge 0_{N^h}$ by $\tilde{g}_h(x_h) \equiv \max_{X_h,p_h}\{X_h:\ C_h(X_h,p_h) \le p_h\cdot x_h$ for every $p_h \gg 0_{N^h}\}$.

[24] If x_h^* solves the competitive cost minimization problem (31), then x_h^* will also be a solution to $\min_{x_h}\{p_h\cdot x_h:\ \tilde{g}_h(x_h) \ge X_h,\ x \ge 0_{N^h}\}$; see Diewert (1982, p. 554) for more details.

[25] Necessary and sufficient conditions for the existence of such concavifying transformations are surveyed and developed in Kannai (1981).

all.[26] This suggests to us that the hypotheses of Theorem 6 could be further weakened,[27] but we leave this task to other (more competent) researchers.

We turn now to tests for additive separability.

4. Tests for additive separability

Assume the same framework as in section 3, except that now we wish to derive necessary and sufficient conditions for a data set to be consistent with utility maximizing behaviour where the utility function F has the following additively separable form: $F(x_1,\dots,x_H,y)=f[\sum_{h=1}^H g_h(x_h),y]$ where the macro function f is a function of $1+M$ variables and the micro function g_h is a function of N^h variables for $h=1,\dots,H$.

Theorem 7 [*Generalization of Afriat (1970) and Varian (1983)*].[28] *Suppose the macro function f is (i) concave, (ii) once differentiable, and (iii) increasing in its first argument and non-decreasing in its last M arguments. Suppose $H\ge 2$ and the micro functions g_h, $h=1,\dots,H$, are (i) concave, (ii) once differentiable, and (iii) weakly increasing in their arguments. Suppose further that we are given T positive price vectors, $(p_1^t,\dots,p_H^t,q^t)$, and T positive quantity vectors, $(x_1^t,\dots,x_H^t,y^t)$, for $t=1,\dots,T$. Suppose the data are consistent with additively separable utility maximizing behaviour; i.e., for $t=1,\dots,T$,*

$$(x_1^t,\dots,x_H^t,y^t) \text{ solves}$$

$$\max_{x_1,\dots,x_H,y}\left\{f\left[\sum_{h=1}^H g_h(x_h),y\right]:\ \sum_{h=1}^H p_h^t\cdot x_h+q^t\cdot y\le Y^t,\right.$$

$$\left. x_1\ge 0_{N^1},\dots,x_H\ge 0_{N^H},\ y\ge 0_M\right\}, \tag{53}$$

where Y^t is defined by (18). Then there exist HT scalars, X_h^t, $h=1,\dots,H$, $t=1,\dots,T$, and 3T scalars, β^t,λ^t,u^t, $t=1,\dots,T$, such that the following inequalities hold:

$$X_h^s\le X_h^t+\beta^t p_h^t\cdot(x_h^s-x_h^t), \tag{54}$$

$$\beta^t>0,\qquad h=1,\dots,H,\quad s,t=1,\dots,T,$$

$$u^s\le u^t+\lambda^t(\beta^t)^{-1}\sum_{h=1}^H(X_h^s-X_h^t)+\lambda^t q^t\cdot(y^s-y^t), \tag{55}$$

$$\lambda^t>0,\qquad s,t=1,\dots,T.$$

Proof. Repeat the proof of Theorem 4, except that (i) $f[\sum_{h=1}^H X_h^t,y^t]$ replaces $f[X_1^t,\dots,X_H^t,y^t]$, (ii) α^t replaces α_h^t, and (iii) we deduce that $\beta_h^t=\lambda^t/\alpha^t\equiv\beta^t$ for $h=1,\dots,H$. Hence conditions (19) and (20) become (54) and (55). Q.E.D.

Corollary 7.1. Suppose $f[\sum_{h=1}^H g_h(x_h),y]$ is replaced by the following completely additive structure: $\sum_{h=1}^H g_h(x_h)$. Then there exist HT scalars, X_h^t, $h=1,\dots,H$, $t=1,\dots,T$, and T scalars, λ^t, $t=1,\dots,T$, such that the following inequalities hold:

$$X_h^s\le X_h^t+\lambda^t p_h^t\cdot(x_h^s-x_h^t), \tag{56}$$

$$\lambda^t>0,\qquad h=1,\dots,H,\quad s,t=1,\dots,T.$$

Proof. Under the present hypotheses, we find that $\alpha^t=\alpha_h^t=1$ for $h=1,\dots,H$, $t=1,\dots,T$, and hence $\beta^t=\lambda^t/\alpha^t=\lambda^t$ for $t=1,\dots,T$. Suppressing the y variables and substituting $\beta^t=\lambda^t$ into (54) yields (56).[29] Q.E.D.

The interesting point to notice about the inequalities (56) is that they are linear in the 'unknowns' X_h^t and λ^t, and hence linear programming techniques may be used in order to check whether a given data set has a solution satisfying (56).

Theorem 8 below shows that (54) and (55) are sufficient for the given data set to be consistent with additively separable utility maximizing behaviour.

[26] However, we do require linear budget constraints in Theorem 3 and this is what allows us to dispense with curvature restrictions.

[27] One easy way of dispensing with the quasi-concavity assumption on f is to assume (45), instead of deducing it. However, we do not have a simple set of sufficient conditions that will imply this assumption.

[28] Afriat and Varian both consider the case where $H=2$, $M=0$ and $f(X)\equiv X$.

[29] We may also substitute $\beta^t=\lambda^t$ into (55) and deduce the inequalities $u^s\le u^t+\sum_{h=1}^H(X_h^s-X_h^t)$ for $s,t=1,\dots,T$. However, if we define $u^t\equiv\sum_{h=1}^H X_h^t$ for $t=1,\dots,T$, we see that the above inequalities hold identically as equalities, and hence are not of any interest.

Theorem 8 [*Generalization of Afriat (1970) and Varian (1983)*]. *Suppose that we are given T positive price vectors,* $(p_1^t,\dots,p_H^t,q^t)$, *and T non-negative quantity vectors,* $(x_1^t,\dots,x_H^t,y^t)$, $t=1,\dots,T$. *Suppose that there exist* $(H+3)T$ *numbers,* $X_h^t,\beta^t,\lambda^t,u^t$, $h=1,\dots,H$, $t=1,\dots,T$, *such that these numbers and the given data satisfy the inequalities (54) and (55). Then there exists a concave, increasing macro function f of* $1+M$ *variables and there exist H concave, increasing micro functions,* g_h, $h=1,\dots,H$, *such that (53) holds for these functions.*

The proof is analogous to the proof of Theorem 2.

Corollary 8.1. Suppose that there exist $(H+1)T$ *numbers,* X_h^t,λ^t, $h=1,\dots,H$, $t=1,\dots,T$, *such that these numbers and the given data satisfy the inequalities (56). Then there exist H concave, increasing functions* g_h, $h=1,\dots,H$, *such that (53) holds for these functions, except that* $f[\sum_{h=1}^H g_h(x_h),y]$ *in (53) is replaced by* $\sum_{h=1}^H g_h(x_h)$.

Recall how Theorem 6 generalized Theorem 4. We can similarly generalize Theorem 7 and its corollary: the differentiability assumptions on f and the g_h may be dropped, the concavity assumption on f may be replaced by the assumption of quasi-concavity, and the positivity restrictions on the quantity data may be relaxed. The proof of this generalization of Theorem 7 follows along the lines of Theorem 6, except that $f(X_1,\dots,X_H,y)$ is replaced by $f(\sum_{h=1}^H X_h,y)$. The key to the proof is to look at the modified version of (40) and conclude that $X_1^t,\dots,X_H^t$ solves

$$\min_{X_1,\dots,X_H}\left\{\sum_{h=1}^H A_h^t X_h:\ \sum_{h=1}^H X_h\ge\sum_{h=1}^H X_h^t\right\},\tag{57}$$

and hence $A_h^t=A^t$ for $h=1,\dots,H$ and $t=1,\dots,T$. These equalities imply that $P_h^t=P^t\equiv(\beta^t)^{-1}$ for all h and t and the generalization of Theorem 7 follows.

5. Conclusion

We have derived generalizations of the Afriat–Varian necessary and sufficient conditions for a given body of data to be consistent with utility maximizing behaviour under the hypotheses that the utility function is either weakly separable or additively separable. Determining whether the resulting conditions are satisfied by the given data leads to *non-parametric tests for separability*. The tests are non-parametric because they do not depend on the underlying utility function having a particular parametric functional form, as is the case in the econometric literature on testing for separability. Blackorby, Primont and Russell (1978, ch. 8) have pointed out that this econometric literature is seriously deficient from a theoretical point of view, since the econometric tests tend to impose unwanted restrictions on the parametric functional form in addition to the particular separability hypothesis being tested. Thus the non-parametric tests developed here should serve as useful supplements to the usual econometric tests for separability.

Even if the tests developed in this paper fail, the tests may be used to construct utility functions (which satisfy the desired hypotheses) that are approximately consistent with the data. For example, even if the objective function (14) does not attain its lower bound of zero, the u^t and $\lambda^t\equiv(\alpha_t^2+1)$ which solve (14) may be used in definitions (6) and (7) in order to construct a utility function f that is approximately consistent with the data.[30]

The necessity of conditions (3) and (5), or (19) and (20), or (54) and (55), were very easy to derive, provided that we assumed positive data and that the relevant utility functions were both concave and differentiable. The resulting conditions turned out to be sufficient as well. Somewhat surprisingly, it proved to be difficult to relax the positivity and differentiability assumptions: we were forced to resort to rather complex arguments involving cost functions.

We have not derived all possible combinations of tests in this paper. For example, if we wish to impose linear homogeneity on the micro function g_h, then add the following restrictions to (19) and (54):

$$X_h^t=\beta_h^t p^t\cdot x^t,\qquad t=1,\dots,T.\tag{58}$$

If we wish to impose linear homogeneity on the macro function f, then add the following restrictions:

$$u^t=\lambda^t\sum_{h=1}^H\left(\beta_h^t\right)^{-1}X_h^t+\lambda^t q^t\cdot y^t,\qquad t=1,\dots,T,\tag{59}$$

to (20) or (55) [except that $\beta_h^t=\beta^t$ in (55)]. The restrictions (58) and (59) have the effect of forcing the linear functions g_h^t and f^t defined in Theorem 5 through the origin, and hence the g_h and f defined in Theorem 5 will be linearly homogeneous functions.

Finally, we note that Theorems 1–2, 4–5 and 7–8 may be adapted to yield theorems in the context of producer theory if we interpret utility as output. If output is *unobservable*, then the above theorems are applicable, except that we substitute the hypothesis of cost minimizing behaviour rather than utility maximizing behaviour. If in addition, output is *observable*, then the 'unknowns' u^t in (5), (20) and (54) become known output levels, and the above

[30]See Afriat (1973) for a further development of this approximate consistency idea.

theorems must be modified accordingly. Other tests for a separable production structure could be developed, but these developments are outside the scope of this consumer-oriented paper.

Appendix: Proofs of theorems in the non-differentiable case

Proof of Theorem 1. Apply the Uzawa (1958) and Karlin (1959, p. 201) Saddle Point Theorem to the concave programming problems in (1). Upon noting that the non-negativity constraints $x \geq 0_N$ are not binding since $x^t \gg 0_N$, we conclude that there exists $\lambda^t \geq 0$ such that x^t maximizes the concave function $f(x) + \lambda^t[p^t \cdot x^t - p^t \cdot x]$ for $t = 1,\dots,T$. By a result due to Rockafellar (1970, p. 270), $0_N \in [\partial f(x^t) - \lambda^t p^t]$ for $t = 1,\dots,T$, which may be rearranged to yield the following conditions:

$$z^t = \lambda^t p^t, \qquad \lambda^t \geq 0, \qquad z^t \in \partial f(x^t), \qquad t = 1,\dots,T, \tag{A.1}$$

where z^t is some supergradient to f around the point x^t. The weakly increasing hypothesis on f implies $z^t > 0_N$ and thus $\lambda^t > 0$ which is (3).

In the non-differentiable case, the inequality (4) is still valid except that $\nabla f(x^t)$ may be replaced by any member of $\partial f(x^t)$ [see Rockafellar (1970, p. 218)]; hence we replace $\nabla f(x^t)$ by the z^t which occurs in (A.1). The rest of the proof of Theorem 1 remains the same.

Proof of Theorem 6. Repeat the argument through (40). Now apply the Uzawa–Karlin Saddle Point Theorem to the concave programming problem (39). Thus there exists $\xi^t \geq 0$ such that $X_1^t,\dots,X_H^t, y^t$ solves the unconstrained concave maximization problem:

$$\max_{X_1,\dots,X_H,y} \left\{ \sum_{h=1}^{H} A_h^t X_h + a^t \cdot y + \xi^t \left[Y^t - \sum_{h=1}^{H} C_h(X_h, p_h^t) - q^t \cdot y \right] \right\}. \tag{A.2}$$

Applying Rockafellar's (1970, p. 218) optimality conditions to (A.2), we again obtain (42) but (41) becomes

$$A_h^t = \xi^t P_h^t, \qquad h = 1,\dots,H, \quad t = 1,\dots,T, \tag{A.3}$$

where $P_h^t \in \partial C_h^t(X_h^t)$, where $C_h^t(X_h^t) \equiv C_h(X_h^t, p_h^t)$ and $\partial C_h^t(X_h^t)$ is defined to be the set of subgradients to the convex function C_h^t at the point X_h^t. Our monotonicity assumptions on g_h imply that $P_h^t > 0$. The remainder of the proof goes through as before except that we replace the partial derivative

$\partial C_h(X_h^t, p_h^t)/\partial X_h$ whenever it occurs by the subgradient P_h^t that occurs in (A.3) above.

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