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Functional Form Problems in Modeling Insurance and Gambling

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Abstract

Defining the outputs of the property insurance and gambling sectors of an economy has proved to be a difficult problem for national income accountants. It is well known that the traditional expected-utility model is not consistent with economic agents fully insuring their property. Thus the present paper adapts existing non-expected-utility theories to yield useful measures of output for the property insurance and gambling sectors.

Key words: non-expected utility, gambling, insurance, functional form problems, risky activities in the national accounts, state contingent commodities

1. Introduction

It is generally recognized that the traditional expected-utility model is not sufficiently flexible to provide an adequate empirical description of many aspects of economic behavior in a risky environment (see Allais [1953], Machina [1982], Epstein [1992]). Using the traditional choice over lotteries approach to modeling uncertainty, Epstein and Zin [1989, 1990, 1991a, 1991b] have provided empirically tractable generalizations of the expected-utility model that achieve a greater degree of flexibility.[1]

Instead of taking the choice over lotteries approach, it is possible to use the contingent commodity approach of Arrow [1951, 1953] and Debreu [1959, p. 101] to model choice under uncertainty. Using the latter approach, Diewert [1993] derived a non-expected-utility model that is a counterpart to the implicit linear utility model of Chew [1989] and Dekel [1986].[2] Diewert [1993, pp. 415–432] applied his model to some simple uncertain choice problems involving insurance, gambling, and investing. The present paper extends the earlier analysis by explicitly considering functional-form problems and the related problems of econometric estimation. A secondary purpose of the present paper is to indicate how the outputs of the property insurance and gambling sectors could be measured in a national income accounting framework.

A brief outline of the paper follows. Section 2 below summarizes the theoretical framework from Diewert [1993]. Sections 3 and 4 deal with the problems involved in modeling the demand for property insurance. A modified CES functional form that exhibits first order risk aversion[3] is introduced in this section. This modified CES functional form turns out to be a special case of classes of functional forms considered by Gul [1991, pp. 677–678] and by Epstein and Zin [1991b, pp. 6–9]. Using this model, it can be rational for a risk-averse consumer to fully insure. The same modified CES functional form is applied in Section 5 to model the gambling decision. Section 6 concludes.

2. A state contingent commodity approach to modeling uncertainty

Assume that a decision maker has a *certainty utility function* $f(y)$ where y is a choice vector belonging to some set S. There are N states of nature, and each state is equally likely and hence has probability $1/N$. Denote the consumption vector of the decision maker if state i occurs by y^i for $i = 1, 2, \ldots, N$. Then following Arrow [1953, p. 41] and Debreu [1959, p. 101], it is natural to assume that the decision maker's preferences over state contingent commodities can be represented by a continuous utility function $F(y^1, y^2, \ldots, y^N)$. Following Samuelson [1952, p. 674, 1960, pp. 34–35], it is also natural to assume that the state contingent preference function F has the following structure:

$$F(y^1, y^2, \ldots, y^N) = M[f(y^1), f(y^2, \ldots, f(y^N)], \tag{1}$$

where f is the certainty utility function and M is a function that expresses the decision maker's attitude toward uncertainty. If the state contingent consumption vectors are all equal, so that $y^1 = y^2 = \ldots = y^N \equiv y$, then it does not matter which state of nature occurs. Under these circumstances, it is natural to require that

$$M[f(y), f(y), \ldots, f(y)] = f(y); \tag{2}$$

that is, the state contingent preferences collapse down to the certainty preferences when all of the state contingent decision vectors y^i are forced to be the same vector y. But property (2) implies that the *stochastic preference function* M is a mean. Furthermore, since the states of nature are equally probable, it is natural to require M to be a symmetric function. Thus the stochastic preference function M must be a *symmetric mean* under the above hypotheses.

The above (equally probable) Arrow-Debreu state contingent preference model was used by Blackorby, Davidson, and Donaldson [1977, pp. 352–354] in their proof of the expected-utility theorem. In addition, they placed a separability assumption on F and then applied a representation theorem due to Gorman [1968] to deduce an additively separable functional form for M—that is, under Blackorby, Davidson, and Donaldson's assumptions, M can be written as follows:

$$M(x_1, x_2, \ldots, x_N) = \phi^{-1}[\Sigma_{i=1}^{N}(1/N)\phi(x_i)], \tag{3}$$

where $x_i \equiv \phi(y^i)$ for $i = 1, 2, \ldots, N$ and ϕ is a continuous, increasing function of one variable.

The right side of (3) defines what we will call a *separable symmetric* mean. It turns out that stochastic preference functions M of the additively separable form given by (3) cannot provide an adequate empirical description of actual choices made under uncertainty, as the pioneering work of Allais [1953, pp. 527–530] made clear. However, if we simply allow M to be an arbitrary symmetric mean as in (1), the resulting class of state contingent preference functions F is too general for empirical applications. Thus we require a class of stochastic preference functions M that are more general than the class of separable symmetric means used in (3) but are less general than the class of symmetric means used in (1). An appropriate class of mean functions that will do the job is the class of *implicitly separable means*, which will be defined below. The application of this class of functions to choice under uncertainty leads to an axiomatic model of uncertain choice which is similar to the *implicit linear utility models* obtained by Dekel [1986], Chew [1989],[4] and Chew and Epstein [1989a, p. 211].

The main theoretical result that we use is Proposition 22 in Diewert [1993, p. 412], which we restate as follows:

Proposition 1 (implicit expected-utility theorem): *Suppose that a decision maker's preferences over certain alternatives* $y \in S$ *(where* S *is a compact and connected subset of* R^K*) can be represented by the continuous utility function* f*. Suppose further that the decision maker's preferences over* n *equally likely choices with utility payoffs* $x_i \equiv f(y^i)$ *for* $i = 1, 2, \ldots, n$ *can be represented by the stochastic preference function* $M^n(x_1, \ldots, x_n)$ *where for* $n = 2, 3, \ldots,$ *the function* M^n *satisfies axioms* (A1) *through* (A6) *listed in Appendix A. Then there exists a function of two variables* $\phi_u(z)$ *defined over* $S^2 \equiv \{(x_1, x_2): a \leq x_i \leq b, i = 1, 2, \}$ *(where* $a \equiv min_u\{f(y): y \in S\}$ *and* $b \equiv max_y\{f(y): y \in S\}$*) such that the decision maker's state contingent preferences over* N *uncertain events with positive probabilities* $P_1, \ldots, P_N$ *can be represented by the function* F *where* $u = F(y^1, \ldots, y^N, P_1, \ldots, P_N)$ *is the unique* u *solution to*

$$\Sigma_{i=1}^{N} P_i \phi_u(x_i) - \phi_u(u) = 0. \tag{4}$$

The class of stochastic preferences that appear in the above proposition are a bit too general for our purposes. In order to be able to aggregate over consumers, we shall restrict ourselves to homothetic preferences—that is, the M^n are restricted to satisfy Axiom (A7) listed in Appendix A. Using the implication (A10) of Proposition 2 in Appendix A, a decision maker's homothetic stochastic preference function over N alternatives $x_1, \ldots, x_N$ can be obtained as the u solution to the following equation:

$$\Sigma_{i=1}^{N} P_i \gamma(x_i/u) - \gamma(1) = 0, \tag{5}$$

where γ is an increasing and continuous function of one variable.

In the following sections, we shall postulate a specific functional form for γ.

3. Modeling the demand for property insurance

Consider the decision to insure a dollar's worth of property of a certain class over a given time period. In the first state of nature, which occurs with probability $P_1 > 0$, the property is destroyed; in the second state of nature, which occurs with probability $P_2 = 1 - P_1 > 0$, the property remains undamaged over the time period. A dollar's worth of insurance can be purchased at the premium rate r (where $0 < r < 1$). If state 1 occurs during the period, the insurance company pays out the face value of the policy less the premium, and this payout rate is equal to $1 - r$. If state 2 occurs (so that the property is not destroyed during the period), the company pays nothing to the consumer and keeps the premium.

Thus if the consumer purchases $i \geq 0$ units of insurance, the cost to the consumer if state 2 occurs (no damage) is ri while if state 1 occurs (complete destruction), the benefit of the policy is $(1 - r)i$. We assume that the consumer is not allowed to overinsure, so we have the following restrictions on the purchase of insurance:

$$0 \leq i \leq y, \tag{6}$$

where $y > 0$ is the total value of insurable property of this class.

The consumer's certainty utility function in this simple model is $f(x) = x$ where x equals consumption of the good and the consumer's state contingent utility function is $F(x_1, x_2, P_1, P_2)$ where $P_j > 0$ is the probability that state j occurs and x_j is consumption in state $j, j = 1, 2$. We can model the consumer's contingent consumption in stage j, x_j, as a function of the amount of insurance purchased, i, as follows:

$$x_1 = (1 - r)i; \; x_2 = y - ri. \tag{7}$$

If it is costless to run an insurance company and insurance is offered at an actuarially fair price, then the amount of premium income should equal the expected value of property claims. Thus in this "no loading" case, the premium rate r satisfies $ri = P_1 i$ or $r = P_1$; that is, the premium rate r equals the probability of loss P_1 in the actuarially fair case. Of course, this is only a limiting case and in the real world, we would expect $r > P_1$ (the "loaded" case).

Suppose that the consumer's family of stochastic preference functions, $\{M^n: n = 2, 3, \ldots\}$, defined over equally likely alternatives, satisfies (A1) through (A6). Then we may apply Proposition 1, the implicit expected-utility theorem, and conclude that the consumer's state contingent preferences over two states of nature (which have positive probabilities P_1 and P_2) can be represented by the function F, where $u = F(x_1, x_2, P_1, P_2)$ is the unique u solution to (8):

$$P_1\phi_u(x_1) + P_2\phi_u(x_2) - \phi_u(u) = 0, \tag{8}$$

where $\phi_u(z)$ is (1) jointly continuous in (z, u), (2) increasing in z for each fixed u, and (3) satisfies property (181) listed in Diewert [1993, p. 410].

If we represent consumption x_j in each state of nature $j = 1, 2$ as a function of the amount of insurance i purchased as in equations (7), and we also impose the no over insurance condition (6), then the consumer's decision to purchase insurance i in the implicit expected-utility model can be formulated as the following constrained-utility maximization problem:

$$\max_{u,i}\{u : P_1\phi_u[(1 - r)i] + P_2\phi_u(y - ri) - \phi_u(u) = 0, 0 \leq i \leq y\}. \tag{9}$$

As indicated in Section 2, we assume homothetic preferences and replace the function $\phi_u(z)$, which appears in (8) and (9), by $\gamma(z/u)$ where γ is continuous and increasing. We further restrict the decison maker's preferences by defining γ to be the following specific function:[5]

$$\gamma(z) \equiv \begin{cases} \alpha + (1-\alpha)z^{\beta} & \text{for } z \geq 1 \\ 1 - \alpha + \alpha z^{\beta} & \text{for } z < 1, \end{cases} \tag{10}$$

where the parameters α and β satisfy the following restrictions:

$$1/2 \leq \alpha < 1;\ \beta \leq 1;\ \beta \neq 0. \tag{11}$$

If we substitute $\phi_u(x_1) = \gamma(x_1/u)$, $\phi_u(x_2) = \gamma(x_2/u)$ and $\phi_u(u) = \gamma(u/u) = \gamma(1)$ into (8) and make use of (10), we find that the decision maker's state contingent utility function $u = F(x_1, x_2, P_1, P_2)$ may be defined as follows for $x_1 \leq x_2$:

$$F(x_1, x_2, P_1, P_2) \equiv [\delta x_1^{\beta} + (1-\delta)x_2^{\beta}]^{1/\beta}, \tag{12}$$

where δ is defined as

$$\delta \equiv P_1\alpha/[P_1\alpha + (1-P_1)(1-\alpha)]. \tag{13}$$

From (12), we see that for $x_1 \leq x_2$, the state contingent-utility function F has a CES functional form and using (10), we can show that for $x_1 \geq x_2$, $F(x_1, x_2, P_1, P_2) \equiv [(1-\delta)x_1^{\beta} + \delta x_2^{\beta}]^{1/\beta}$, which is a similar CES functional form except that δ and $1-\delta$ have been interchanged. Thus unless $\delta = 1/2$, the (x_1, x_2) indifference curves for the state contingent-utility function $F(x_1, x_2, P_1, P_2)$ will have *kink* at the certainty line where $x_1 = x_2$. To use the language of Chew [1989, p. 287] and Segal and Spivak [1990], the decision maker's preferences will exhibit first-degree risk aversion.[6]

From (13), it can be verified that $\delta = P_1$ for all probabilities P_1 such that $0 < P_1 < 1$ if and only if $\alpha = 1/2$. Thus when $\alpha = 1/2$, the state contingent-utility function F collapses into the following (nonkinked) ordinary CES function:[7]

$$F(x_1, x_2, P_1, P_2) = [P_1x_1^{\beta} + P_2x_2^{\beta}]^{1/\beta} \text{ for } \alpha = 1/2. \tag{14}$$

We can summarize the curvature properties of the state contingent-utility function F defined by (12) and (13) as follows: (1) the closer α is to 1, the sharper is the kink in each indifference curve at the $x_1 = x_2$ point (and the greater is the degree of first-order risk aversion) and for $\alpha = 1/2$, the kink vanishes; and (2) the smaller β is, the more pronounced is the curvature of the half CES indifference curve that emanates from the $x_1 = x_2$ point in each direction (and the greater is the degree of second order risk aversion) and for $\beta = 1$, the half indifference curves are straight lines.[8]

Returning to our specific insurance problem (9), substituting (10) into (9) yields the following constrained-maximization problem:

$$\max_i \{(\delta[(1-r)i]^{\beta} + (1-\delta)[y - ri]^{\beta})^{1/\beta} : 0 \leq i \leq y\}. \tag{15}$$

The first-order necessary condition for an interior maximum for (15) yields the following condition:

$$\begin{aligned}(y - ri^*)/(1 - r)i^* &= [r(1 - \delta)/\delta(1 - r)]^{1/(1-\beta)}\\ &= \left[\frac{r}{1 - r}\,\frac{(1 - P_1)}{P_1}\,\frac{(1 - \alpha)}{\alpha}\right]^{1/(1-\beta)}\\ &\equiv c.\end{aligned} \tag{16}$$

Solving (16) for the optimal amount of insurance i^* yields the following expression:

$$i^* = y/[r + c(1 - r)]. \tag{17}$$

The candidate solution i^* defined by (17) will in fact solve (15) provided that, in addition to the inequalities (11), the following inequalities are satisfied:[9]

$$\beta < 1; \tag{18}$$

$$[r/P_1][(1 - P_1)/(1 - r)][(1 - \alpha)/\alpha] \geq 1. \tag{19}$$

If (19) is satisfied with a strict inequality, then $i^* < y$ and the decision maker does not fully insure. If (19) is not satisfied, then the solution to (15) is the corner solution $i^* = y$; that is, the decison maker will fully insure if (19) is not satisfied. In the actuarially fair case where $r = P_1$ (and hence $1 - r = 1 - P_1$), nonsatisfaction of (19) is equivalent to $(1 - \alpha)/\alpha < 1$ or $\alpha > 1/2$. Thus if we are in the actuarially fair case and $1/2 < \alpha < 1$ and $\beta \leq 1$, then we have first-order risk aversion in our model and the decision maker will always fully insure.

If we have an interior solution so that there is not full insurance coverage, then we note that i^* (the optimal insurance) will decrease if the premium rate r increases and i^* will increase if α or P_1 (the probability of loss) increase. The parameter α is a measure of first-order risk aversion: the bigger α is, the more risk averse the decison maker is and the greater will be the tendency to fully insure.

In Appendix B, we generalizze the above model to situations where property damage follows a multinomial distribution of outcomes rather than the above binomial distribution.

We turn now to a dicussion of the problems involved in measuring the real output of the property insurance sector.

4. Measuring the real output provided by property insurance

In the context of the simple model of insurance demand developed in the previous section, it can be seen that in order to measure the real output of insurance services, we need to know the parameters α and β that appear in (10). In order to estimate these parameters, we need two or more *interior* observations on

Y^t = period t (nominal) value of insurable property,
p^t = period t price index for insurable property,

$y^t \equiv Y^t/p^t$ is the period t real value of insurable property,
I^t = period t (nominal) value of insurance coverage,
$i^t \equiv I^t/p^t$ is the period t real value of insurance coverage,
r^t = period t premium rate,
P_1^t = period t expected damage rate.

Given the above data, define the dependent variable

$$c^t \equiv (y^t - r^t i^t)/(1 - r^t)i^t \tag{20}$$

and form the regression model

$$lnc^t = (1 - \beta)^{-1}ln[(1 - \alpha)/\alpha] + (1 - \beta)^{-1}ln[(r^t(1 - P_1^t)/(1 - r^t)P_1^t] + e^t \tag{21}$$

in order to obtain parameter estimates for α and β. Then period t estimates for δ^t can be calculated as follows:

$$\delta^t \equiv P_1^t\alpha/[P_1^t\alpha + (1 - P_1^t)(1 - \alpha)]. \tag{22}$$

Once estimates for δ^t and β have been obtained, we calculate period t utility levels with insurance (u^{*t}) and without insurance (u^{0t}):[10]

$$u^{0t} \equiv [\delta^t 0^\beta + (1 - \delta^t)(y^t)^\beta]^{1/\beta} = (1 - \delta^t)^{1/\beta}y^t; \tag{23}$$

$$u^{*t} \equiv \{\delta^t[(1 - r^t)i^t]^\beta + (1 - \delta^t)[y^t - r^t i^t]^\beta\}^{1/\beta}. \tag{24}$$

Then the period t real quantity of insurance services Q^t is defined to be the difference between the insured utility level u^{*t} and the uninsured utility level u^{0t}:

$$Q^t \equiv u^{*t} - u^{0t}. \tag{25}$$

We define the corresponding price of insurance services P^t to be the value of insurance premiums paid divided by Q^t:

$$P^t \equiv r^t i^t p^t/Q^t = r^t I^t/Q^t. \tag{26}$$

The quantity measure Q^t for insurance services is illustrated in Figure 1.[11] The quantities x_1 and x_2 represent the consumer's stock of insurable property in states of nature 1 (property is destroyed) and 2 (no damage to property). For simplicity, we assume $P_1 = P_2 = 1/2$ and so the consumer's stochastic indifference curves are symmetric around the 45 degree line (or the certain consumption line). Since we are assuming homotheticity, the consumer's indifference curves through the points B and C are radial blowups of each other. The consumer's no insurance point is at A. If insurance could be purchased at an actuarially fair price, the consumer could move along the line AD (with slope equal to $-1/2$) and fully insure at the point D. However, in the "loaded" case, the consumer's opportunity set is

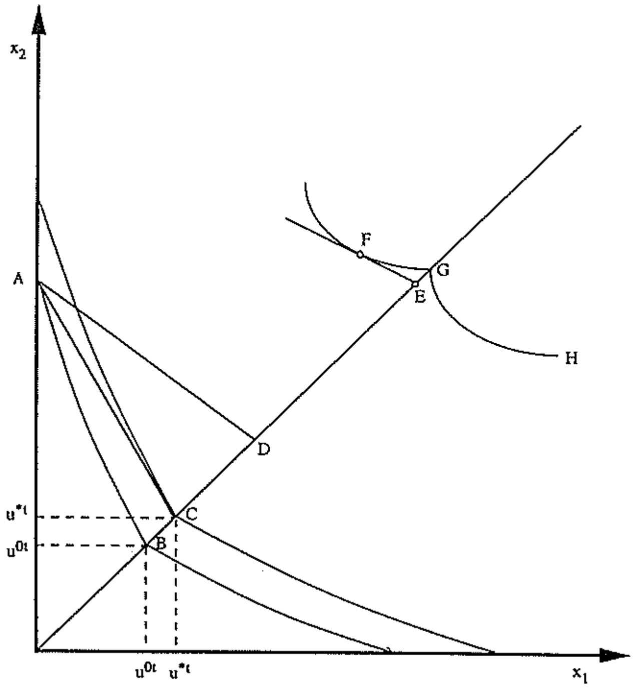

Figure 1. Quantity measure Q^t for insurance services.

the line AC (with slope $-r/(1 - r) < -1/2$ with $r > 1/2$). In Figure 1, the indifference curves are sufficiently kinked so that the consumer's optimal point is attained at C where he or she fully insures. The improvement in utility (measured in real property units) due to the availability of insurance is the difference in the post- and preinsurance utility levels, $Q^t \equiv u^{*t} - u^{0t}$, and this distance can be read off either the x_1 or x_2 axis as indicated in Figure 1.

One problem with the above model is that if the consumer is always fully insured, then the parameters α and β cannot be identified from observable data. Conversely, if the consumer is always partially insured, then we are fitting parts of indifference curves that are perhaps quite remote from the certainty equivalent line, $x_1 = x_2$. Yet in order to calculate the certainty equivalent value for insurance services as we have done in (24), it is necessary to extrapolate the indifference curves from the observed region all the way to the certainty equivalent line where $x_1 = x_2$. The same kind of extrapolation problem does not arise in ordinary certainty consumer theory.

We conclude this section by contrasting our measure of the real output of the property insurance sector with the measures that have been suggested in the national income accounting literature.

Some authors and statistical agencies[12] propose that the nominal output of the insurance sector should be measured as premium payments less claims paid out during the period. Real measures of insurance output are obtained by various methods of deflating this net nominal measure. On the other hand, Denny [1980],[13] Ruggles [1983, p. 67], and Hornstein and Prescott [1991] all suggest that gross premiums paid rather than net premiums paid is a more appropriate measure of nominal output, which is in agreement with our suggested nominal measure of output, r^tI^t. However, none of the above authors suggest the use of our certainty equivalence measure of real output, $u^{*t} - u^{0t}$.

In the following section, we adapt the theoretical model presented in Sections 2 and 3 to measure the real output of the gambling sector.

5. Measuring the real output of the gambling sector

Consider a very simple lottery. In state 1, the gambler loses his or her wager w with probability P_1. In state 2, the gambler wins Rw with probability $P_2 = 1 - P_1$ where R is the payout ratio. The gambler's state contingent consumption possibilities set as a function of the amount gambled is

$$x_1 = y - w; \; x_2 = y + Rw, \tag{27}$$

where $y > 0$ is the consumer's real "income" during the period under consideration.[14]

We again use the implicit expected-utility model with homothetic preferences that we used in the previous sections. The function $\gamma(z)$, which generates the consumer's preferences over gambles, is again defined by (10) where the parameters α and β now satisfy the following restrictions:

$$0 < \alpha < 1/2; \; \beta < 1; \; \beta \neq 0. \tag{28}$$

The gambler's state contingent utility function F is still defined by (12) where the parameter δ is defined by (13). Again we obtain a family of kinked CES indifference curves. However, now the kinks along the $x_1 = x_2$ line make obtuse angles instead of acute angles. The closer α is to 0, the more obtuse are the angles. As before, the parameter β controls the curvature of the half indifference curves that emanate from the $x_1 = x_2$ line: the smaller β is, the bigger the curvature.

The gambler's utility maximization problem may be written as follows:[15]

$$\max_x\{(\delta[y - w]^\beta + (1 - \delta)[y + Rw]^\beta)^{1/\beta} : 0 \leq w \leq y\}. \tag{29}$$

The first-order necessary conditions for an interior maximum for (29) yield the following equation that characterizes the optimal wager, w^*:

$$\begin{aligned} [y + Rw^*]/[y - w^*] &= [(1 - \delta)\delta^{-1}R]^{1/(1-\beta)} \\ &= [(1 - P_1)(1 - \alpha)R/P_1\alpha]^{1/(1-\beta)} \text{ using (13)} \\ &\equiv b. \end{aligned} \tag{30}$$

Solving (30) for w^* yields the following equation:

$$w^* = y(b - 1)/(b + R). \tag{31}$$

In the actuarially fair case, we have $R = P_1/(1 - P_1)$ and in the unfair case, we have $R < P_1/(1 - P_1)$.

In order to obtain $w^* \geq 0$, we require $b \geq 1$. Thus in the actuarially fair case, we require $(1 - \alpha)/\alpha \geq 1$ or $\alpha \leq 1/2$.

In the unfair case, if $b \leq 1$, then the solution to (29) under the restrictions (28) is the corner solution $w^* = 0$; that is, the gambler will wager nothing if the degree of unfairness becomes too great.

Given a series of observations on income in period t, Y^t; on amounts wagered, T^t; and the relevant period t consumer price index, p^t; then we can calculate period t real income y^t and real wagers w^t by

$$y^t \equiv Y^t/p^t;\ w^t \equiv W^t/p^t. \tag{32}$$

Furthermore, given information on payout ratios, R^t; and probabilities of winning, $P_2^t = 1 - P_1^t$, we can use (30) to calculate a b^t for each period and then a regression model counterpart (21) can be set up in order to estimate α and β.

Once estimates for α and β have been obtained, we calculate period t utility levels with gambling (u^*) and without gambling (u^{0t}):

$$u^{0t} \equiv [\delta^t[y^t]^\beta + (1 - \delta^t)[y^t]^\beta]^{1/\beta} = y^t; \tag{33}$$

$$u^{*t} \equiv [\delta^t[y^t - w^t]^\beta + (1 + \delta^t)[y^t + R^t w^t]^\beta]^{1/\beta}. \tag{34}$$

The period t real output of gambling services Q^t can again be defined by $u^{*t} - u^{0t}$ (recall (25)) and the corresponding period t price can be defined as deflated (by Q^t) money gambled in period t:

$$P^t \equiv w^t/Q^t = W^t/p^tQ^t. \tag{35}$$

The above model of gambling is illustrated in the upper portion of Figure 1 for the case $P_1 = P_2 = 1/2$. In this case, the indifference curves are symmetric around the certainty line. The consumer's initial no gambling point is on the certainty line at the point E. The gambler's opportunity set is the line segment emanating from E in the direction F; its slope is $-R$ where $R < 1/2$ in the unfair case. This opportunity set is tangent to the highest attainable indifference curve FGH at the point F. The certainty level of utility that is equivalent to the uncertain point F is G. Thus the point G has the coordinates (u^{*t}, u^{*t}) while the initial point E has the coordinates (u^{0t}, u^{0t}). Thus the ex ante increase in utility that the consumer gets from gambling is $u^{*t} - u^{0t}$.

Of course, most real-life lotteries do not offer an equal chance of winning and losing; rather they offer a large probability of losing (P_1 tends to 1) along with a large prize (R becomes large). In this case, both the gambling line FE and the half indifference curve HG will have steeper slopes than shown in the diagram. In the actuarially fair case where $R = P_1/(1 - P_1)$, b defined by (30) turns out to equal the constant $[(1 - \alpha)/\alpha]^{1/(1-\beta)}$. Thus in this case, the optimal wager w^* defined by (31) will tend to 0 as P_1 tends to 1. In the unfair case, the optimal amount of wagering will be small as P_1 becomes large.

6. Conclusion

Two of the more difficult problems in national income accounting are the measurement of the real outputs of the insurance and gambling sectors. In this paper, we have tried to adapt existing choice under uncertainty theories to yield relatively simple measures of real outputs for these industries. The kinked CES model that we used in this paper is a special case of Chew's [1989] scalar invariant weighted-utility model, of Gul's [1991] disappointment-aversion model, of Epstein and Zin's [1991b] semiweighted utility model, and of Diewert's [1993] implicit expected-utility model. Thus it appears that the new non-expected-utility theories that have been developed in the past ten years can be very useful in providing possible solutions to very difficult measurement problems.

Appendix A: Axioms for stochastic preference functions

Let the stochastic preference function M^n be a function of n variables defined over the set $S^n \equiv \{x : a1_n \leq x \leq b1_n\}$ where 1_n is a vector of ones of dimension n and the scalars a and b satisfy $a < b$. $M^n(x_1, \ldots, x_n)$ represents the utility level of the decision maker if the outcome x_i occurs in state of nature i for $i = 1, 2, \ldots, n$ where the n states of nature are equally probable.

For $n = 2, 3, 4, \ldots$, the function M^n satisfies the following six axioms:

Mean value property: $\gamma 1_n \in S^n$ implies $M^n(\gamma 1_n) = \gamma$. (A1)

Symmetry: $M^n(Px) = M^n(x)$ where Px is a permutation of the components of the vector $x \equiv (x_1, x_2, \ldots, x_n)$. (A2)

Continuous function: $M^n(x)$ is a continuous function for $x \in S^n$. (A3)

Strictly increasing function: $M^n(x)$ is a strictly increasing function over S^n—that is, if $x^1 \in S^n$, $x^2 \in S^n$ and $x_1 < x_2$, then $M^n(x^1) < M^n(x^2)$. (A4)

Conditional consistency in aggregation: There exists a function of three variables, $M_2^n(x_1, x_2, u)$, defined for $(x_1, x_2, u) \in S^3$ and such that $x \in S^n$ and $u = M^n(x)$ implies $u = M^n[M_2^n(x_1, x_2, u), M_2^n(x_1, x_2, u), x_3, \ldots, x_n]$. The function M_2^n has the following properties: (A5)

(1) *Symmetric mean*: For every $u \in S^1$, $M_2^n(x_1, x_2, u)$ is a symmetric mean in $(x_1, x_2) \in S^2$;
(2) *Jointly continuous*: $M_2^n(x_1, x_2, u)$ is jointly continuous in (x_1, x_2, u);
(3) *Bisymmetry*: $M_2^n(x_1, x_2, u)$ is bisymmetric in x_1, x_2 for each $u \in S^1$; that is, for every $(x_1, x_2, x_3, x_4, u) \in S^5$, $M_2^n[M_2^n(x_1, x_2, u), M_2^n(x_3, x_4, u), u] = M_2^n[M_2^n(x_1, x_3, u), M_2^n(x_2, x_4, u), u]$.

Consistency of stochastic preferences in the implicity separable model:

$M_2^n(x_1, x_2, u) = M_2(x_1, x_2, u)$ for $(x_1, x_2, u) \in S^3$ and for $n = 2, 3, \ldots$ (A6)

Axiom (A6) says that the conditional certainty equivalence functions M_2^n are always the same no matter how many equally likely alternatives n the decision maker faces.

To obtain homothetic stochastic preference functions, we need to add the following axiom:

Positive linear homogeneity: $x \in S^n$, $\lambda \geq 0$, $\lambda x^n \in S^n$ implies $M^n(\lambda x) = \lambda M^n(x)$ (A7)

The linear homogeneity axiom (A7) implies that each indifference surface of $M_n(x)$ is a radial blowup of a single indifference surface.

A way of imposing homothetic preferences is to use Proposition 17 in Diewert [1993, p. 398], which we restate as Proposition 2 below.

Consider the following property for $\phi_u(z)$:

for $(z, u) \in S^2$, $\phi_u(z) = \gamma(z/u)$ where $\gamma(z)$ is a continuous increasing function of z for $a/b \leq z \leq b/a$, where $0 < a < b$. (A8)

Proposition 2: *Let* $\phi_u(z)$ *be defined for* $(z, u) \in S^2$ *and satisfy* (A8). *For* $x \equiv (x_1, \ldots, x_n) \in S^n$, *define* $u = M(x)$ *as the* u *solution to*

$$\Sigma_{i=1}^n (1/n)\gamma(x_i/u) - \gamma(1) = 0. \tag{A9}$$

Then M^n *satisfies* (A1) *through* (A7).

The function of one variable γ that appears in (A8) and (A9) may be used to generate a decision maker's homothetic stochastic preference function M over N alternatives with probabilities $P_1, \ldots, P_N$ as follows: $u = M(x_1, \ldots, x_n; P_1, \ldots, P_n)$ is the unique u solution to

$$\Sigma_{i=1}^N P_i \gamma(x_i/u) - \gamma(1) = 0 \tag{A10}$$

where x_i is the outcome in state i, $i = 1, \ldots, N$.

Appendix B: More general insurance models

In this Appendix, we indicate how the simple 2 state model of insurance demand can be extended to an N state model.

Consider a homogeneous class of assets that could be insured against property damage. For a given time period, there are $N \geq 2$ states of nature: the probability that state n occurs is P_n for $n = 1, 2, \ldots, N$. Let $0 \leq f_n \leq 1$ be the fraction of the asset class that is destroyed if state n occurs where

$$1 \geq f_1 > f_2 > \ldots > f_{N-1} > f_N = 0. \tag{B1}$$

Thus in state N, no damage occurs, while in state 1, maximum damage occurs (complete destruction if $f_1 = 1$).

As in Section 3, let $y > 0$ be the amount of property that could be insured, let i be the amount of insurance coverage where $0 \le i \le y$ (no over insurance allowed) and let $r > 0$ be the premium rate.

If state n occurs, the loss of property is $f_n y$. If $i \ge f_n y$, then the insurer pays for this loss. However, if the damage in state n exceeds the insurance coverage ($i < f_n y$), then the insurer pays out only i and the consumer loses the amount $f_n y - i$ in addition to the premium paid, ri. Define the function of one variable $\psi(x)$ as follows:

$$\psi(x) \equiv \begin{cases} x & \text{for } x \ge 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{B2}$$

Let x_n be the consumer's consumption of the asset if state n occurs for $n = 1, \ldots, N$. For each n, we can express x_n as a function of the insurance coverage i as follows:

$$x_n = y - ri - \psi(f_n y - i), \qquad n = 1, 2, \ldots, N. \tag{B3}$$

Thus each x_n is a piecewise linear (and concave) function of the amount of insurance purchased, i.

The N state counterpart to the insurance choice problem (9) is:

$$\max_{u,i}\{u : \Sigma_{n=1}^{N} P_n \gamma([y - ri - \psi(f_n y - i)]/u) - \gamma(1) = 0; 0 \le i \le y\}, \tag{B4}$$

where γ is defined by (10) and (11). Substituting (10) into (B4) yields

$$\max_i\{(\Sigma_{n=1}^{N} \delta_n(i)[y - ri - \psi(f_n y - i)]^{\beta})^{1/\beta} : 0 \le i \le y\}, \tag{B5}$$

where the coefficients $\delta_n(i)$ are defined by

$$\delta_n(i) \equiv \begin{cases} P_n\alpha/[\Sigma_{j=1}^{n^*(i)} \alpha P_j + \Sigma_{k=n^*(i)+1}^{N}(1 - \alpha)P_k], & n = 1, \ldots, n^*(i); \\ P_n(1 - \alpha)/[\Sigma_{j=1}^{n^*(i)} \alpha P_j + \Sigma_{k=n^*(i)+1}^{N}(1 - \alpha)P_k], & n = n^*(i) + 1, \ldots, N. \end{cases} \tag{B6}$$

Define the optimized objective function in (B5) as $u^*(i)$. The integer $n^*(i)$, which appears in (B6), is chosen to satisfy the following inequalities:

$$y - ri - \psi(f_n y - i) \ge u^*(i) \qquad \text{for } n = n^*(i) + 1, \ldots, N;$$

$$y - ri - \psi(f_n y - i) < u^*(i) \qquad \text{for } n = 1, 2, \ldots, n^*(i). \tag{B7}$$

The model defined by (B5) collapses to our old two state insurance model (15) if $N = 2$, $f_1 = 1$ and $f_2 = 0$. If the first-order risk aversion parameter α equals 1/2, then the $\delta_n(i)$ defined by (B6) collapse down to $\delta_n(i) = P_n$ for $n = 1, \ldots, N$ and the objective function in (B6) becomes an ordinary CES function (or a mean of order β), which is consistent with homothetic expected utility maximization.

If $N > 2$, then the first-order condition for (B5) does not yield an equation that can be solved explicitly for the optimal insurance i^* as a function of the exogenous variables. Hence econometric estimation of the parameters α and β is much more difficult in the multiple state case compared to the two state case.

An explicit solution to the problem (B4) can be obtained if we replace our old generating function $\gamma(z)$ defined by (10) by the following kinked quadratic generating function:

$$\gamma(z) \equiv \begin{cases} z + \alpha(z-1) + \beta(z-1)^2 & \text{for } z \geq 1; \\ z & \text{for } z < 1; \end{cases} \tag{B8}$$

where the first-order risk aversion parameter α must satisfy $-1 < \alpha \leq 0$ for risk-averting behavior and the second-order risk aversion parameter β must satisfy $\beta < 0$ for strict risk aversion.

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Notes

1. Epstein and Zin [1989, 1991a] provide an infinite horizon extension of a model due to Kreps and Porteus [1978, 1979]. The non-expected-utility models of Weil [1990] and Farmer [1990] are closely related to this Epstein-Zin model. Other empirically useful generalizations of the expected-utility model have been provided by Chew [1983, 1989] and Chew and Epstein [1989a, 1989b, 1990], Epstein and Zin [1991b] and Gul [1991].
2. See also Chew and Epstein [1989a, p. 211] for an alternative development of this model using a separability approach. Chew's [1983] weighted utility, Gul's [1991] disappointment averse utility, and Epstein and Zin's [1991b] semiweighted utility models are all special cases of the implicit linear-utility model.
3. The term appears in Segal and Spivak [1990], but the concept appears to be due to Chew [1989, p. 287]. See also Epstein and Zin [1990] and Epstein [1992, pp. 9–11].
4. Chew [1989, pp. 286–287] used the term *implicit weighted utility* to describe his model. His function $\eta(z, u)$ corresponds to our function of two variables $\phi_u(z)$, which appears in (4).
5. Chew [1989, p. 287] obtained a special case of his general model by setting his $\eta(z, u) = \sigma(z/u)$ where σ was an increasing continuous function of one variable. Thus if we ignore the different axiomatic justifications for the uncertainty models (preferences over lotteries versus preferences over state contingent commodities), our model (10) is a special case of Chew's scalar invariant implicit weighted utility model. Our model (10) is also a special case of Gul's [1991, p. 674] disappointment aversion model; in place of our z^β, Gul has the general utility function $u(z)$. Similarly, (10) is a special case of Epstein and Zin's [1991b, p. 9] semiweighted utility model (set their $\delta = 0$).
6. For further expositions on this concept, see Epstein and Zin [1990], Epstein [1992, pp. 9–11] and Diewert [1993, pp. 415–423]. Note that the inequality restrictions (11) imply that $F(x_1, x_2, P_1, P_2)$ is a concave function of (x_1, x_2). To obtain strict concavity and hence strict risk aversion, we require the additional restriction $\beta < 1$.
7. The preferences defined by (14) are the only homothetic preferences consistent with the expected-utility theorem (see Diewert [1993, p. 399]).
8. For a diagram of these preferences in the case where $P_1 = P_2$, see Figure 2 in Epstein and Zin [1991b]. Another useful diagram can be found in Diewert [1993, p. 419].

9. If the inequalities (18) and (19) are satisfied, then $c \geq 1$ and $i^* \leq y$ and hence the inequality constraint in (15) will be satisfied.
10. In order to evaluate (23), we require $0 < \beta \leq 1$.
11. Machina [1995] calls diagrams of this type Hirshleifer [1965]–Yaari [1965] diagrams.
12. See Hirshhorn and Geehan [1977] and the references listed in Hornstein and Prescott [1991, pp. 917–918].
13. Denny's theoretical position is very close to ours as the following quotation indicates: "Consumers purchase life insurance to avoid risks and any output measure must reflect this fact" (Denny [1980, p. 151]).
14. Thus nominal wagers and income are W and Y, respectively, and real wagers and income are $w \equiv W/p$ and $y \equiv Y/p$ where p is the relevant consumer price index.
15. This is a counterpart to (15) in Section 3. For more details on this gambling problem, see Diewert [1993, pp. 424–427].

References

ALLAIS, M. [1953]: "Le comportement de l'homme rationnel devant le risque: critique de postulats et axiomes de l'école américains," *Econometrica*, 21, 503–546.

ARROW, K.J. [1951]: "Alternative Approaches to the Theory of Choice in Risk-Taking Situations," *Econometrica*, 19, 404–437.

ARROW, K.J. [1953]: "Le role des valeurs boursières pour le répartition la meilleure des risques." In *Econométrie*, Centre National de la Recherche Scientifique, Paris (41–48). Reprinted in English translation in 1964 as "The Role of Securities in the Optimal Allocation of Risk-Bearing," *Review of Economic Studies*, 31, 91–96.

BLACKORBY, C., DAVIDSON, R., and DONALDSON, D. [1977]: "A Homiletic Exposition of the Expected Utility Hypothesis," *Economica*, 44, 351–358.

CHEW, S.H. [1983]: "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox," *Econometrica*, 51, 1065–1092.

CHEW, S.H. [1989]: "Axiomatic Utility Theories with the Betweenness Property," *Annals of Operations Research*, 19, 273–298.

CHEW, S.H., and EPSTEIN, L.G. [1989a]: "A Unifying Approach to Axiomatic Non-Expected Utility Theories," *Journal of Economic Theory*, 49, 207–240.

CHEW, S.H., and EPSTEIN, L.G. [1989b]: "The Structure of Preferences and Attitudes Towards the Timing of the Resolution of Uncertainty," *International Economic Review*, 30, 103–117.

CHEW, S.H., and EPSTEIN, L.G. [1990]: "Nonexpected Utility Preferences in a Temporal Framework with an Application to Consumption-Savings Behavior," *Journal of Economic Theory*, 50, 54–81.

DEBREU, G. [1959]: *Theory of Value*. Wiley, New York.

DEKEL, E. [1986]: "An Axiomatic Characterization of Preferences Under Uncertainty: Weakening the Independence Axiom," *Journal of Economic Theory*, 40, 304–318.

DENNY, M. [1980]: "Measuring the Real Output of the Life Insurance Industry: A Comment," *Review of Economics and Statistics*, 62, 150–152.

DIEWERT, W.E. [1993]: "Symmetric Means and Choice Under Uncertainty," in *Essays in Index Number Theory* (Vol. 1), W.E. Diewert and A.O. Nakamura (Eds.), North-Holland, Amsterdam (355–433).

EPSTEIN, L.G. [1992]: "Behavior Under Risk: Recent Developments in Theory and Applications," in *Advances in Economic Theory*, J.J. Laffont (Ed.), Cambridge University Press, New York.

EPSTEIN, L.G., and ZIN, S.E. [1989]: "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

EPSTEIN, L.G., and ZIN, S.E. [1990]: "First Order Risk Aversion and the Equity Premium Puzzle," *Journal of Monetary Economics*, 26, 387–407.

EPSTEIN, L.G., and ZIN, S.E. [1991a]: "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis," *Journal of Political Economy*, 99, 263–286.

EPSTEIN, L.G., and ZIN, S.E. [1991b]: "The Independence Axiom and Asset Returns," Technical Working Paper No. 109, National Bureau of Economic Research, Cambridge (Mass.).

FARMER, R.E.A. [1990]: "Rince Preferences," *Quarterly Journal of Economics*, 105, 43–60.

GORMAN, W.M. [1968]: "The Structure of Utility Functions," *Review of Economic Studies*, 35, 367–390.

GUL, F. [1991]: "A Theory of Disappointment Aversion," *Econometrica*, 59, 667–686.

HIRSHHORN, R., and GEEHAN, R. [1977]: "Measuring the Real Output of the Life Insurance Industry," *Review of Economics and Statistics*, 59, 211–219.

HIRSHLEIFER, J. [1965]: "Investment Decision Under Uncertainty: Choice-Theoretic Approaches," *Quarterly Journal of Economics*, 79, 509–536.

HORNSTEIN, A., and PRESCOTT, E.C. [1991]: "Insurance Contracts as Commodities: A Note," *Review of Economics Studies*, 58, 917–928.

KREPS, D., and PORTEUS, E. [1978]: "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200.

KREPS, E., and PORTEUS, E. [1979]: "Temporal Von Neumann-Morgenstern and Induced Preferences," *Journal of Economic Theory*, 20, 81–109.

MACHINA, M. [1982]: " 'Expected Utility' Analysis Without the Independence Axiom," *Econometrica*, 50, 277–323.

MACHINA, M. [1995]: "Non-Expected Utility and the Robustness of the Classical Insurance Paradigm," *The Geneva Papers on Risk and Insurance Theory*, 20, 9–50.

RUGGLES, R. [1983]: "The United States National Income Account, 1947–1977: Their Conceptual Basis and Evolution," in *The U.S. National Income and Product Accounts: Selected Topics*, M. Foss (Ed.), NBER Studies in Income and Wealth, Vol. 47, University of Chicago Press, Chicago (15–96).

SAMUELSON, P.A. [1952]: "Probability Utility and the Independence Axiom," *Econometrica*, 20, 670–678.

SAMUELSON, P.A. [1960]: "The St. Petersburg Paradox as a Divergent Double Limit," *International Economic Review*, 1, 31–37.

SEGAL, W., and SPIVAK, A. [1990]: "First Order versus Second Order Risk Aversion," *Journal of Economic Theory*, 51, 11–125.

WEIL, P. [1990]: "Nonexpected Utility in Macroeconomics," *Quarterly Journal of Economics*, 105, 29–42.

YAARI, M. [1965]: "Convexity in the Theory of Choice Under Risk," *Quarterly Journal of Economics*, 79, 278–290.