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A FUNDAMENTAL MATRIX EQUATION OF PRODUCTION THEORY WITH APPLICATIONS TO THE THEORY OFINTERNATIONAL TRADE

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ABSTRACT

The paper develops the comparative statics of a multisectoral model of the private production sector of an economy with an arbitrary number M of constant (or decreasing) returns to scale sectors, an arbitrary number N of primary inputs or domestically traded goods, and an arbitrary number K of internationally traded goods. The main endogenous variables are: (1) M industry scale variables, (2) N domestic prices, and (3) K net outputs of internationally traded goods. The main exogenous variables are: (1) M industry specific tax or subsidy rates, (2) N endowments of factors (or demands for domestic goods) and (3) K prices for internationally traded goods. Our fundamental matrix equation of production theory relates the various endogenous and exogenous variables.

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1. INTRODUCTION

Our goal in this paper is to develop a producer counterpart to Barten's (1964, p. 2) fundamental matrix equation of consumer demand.[1]

We develop the comparative statics of the production sector of a competitive economy that engages in international trade. We assume that: (i) there are K internationally traded goods that can be bought or sold at the positive world prices $(p_1,\ldots,p_K)^T \equiv p \gg 0_K$,[2] (ii) there are M constant returns to scale[3] firms or industries in the economy, and (iii) there are N domestic goods which may be allocated between industries. The economy's total "endowment" vector" of the N goods is $v \equiv (v_1,\ldots,v_N)^T$.[4] We also assume that the mth industry's scale is $z_m \geq 0$ and that $z \equiv (z_1,\ldots z_M)^T \geq 0_M$ is a vector of industry scales. Usually, the scale z_m will be the amount of a major output produced by industry m. Finally, we assume that collectively, the production sector will behave as if it maximizes the value of the net output of internationally traded goods subject to the aggregate domestic resource constraints of the economy.

The comparative statics of the above model of producer behaviour were largely developed in Diewert and Woodland (1977)[5]. However, in the present paper, a new class of exogenous variables is introduced, namely $s \equiv (s_1,\ldots s_M)^T$, a vector of industry subsidies (or taxes if negative); i.e., if industry m runs at scale $z_m \geq 0$, then total subsidies paid to industry m are $s_m z_m$. If $s_m < 0$, then $-s_m z_m$ is the amount of industry specific taxes paid by industry m. Since most western governments do provide industry subsidies or impose industry specific taxes, it is of some interest to work out how a competitive production sector will respond to changes in these tax-subsidy variables.

In section 2 below, we set up the economy's primal profit maximization and use duality theory in order to derive a more convenient dual problem. In section 3, the main comparative statics equation is derived, which relates the responses of three sets of endogenous variables (w, z, and y) to changes in the three sets of exogenous variables (v, s and p), where $w \equiv (w_1,\ldots w_N)^T \geq 0_N$ is a vector of shadow prices for the N domestic goods (we will generally call w a factor price vector in what follows) and $y \equiv (y_1,\ldots,y_K)^T$ is the vector of net outputs of internationally traded goods produced by the entire production sector (if $y_k < 0$, then the k^{th} good is imported into the production sector). In section 4, the nine matrices of responses of the endogenous variables to changes in the exogenous variables are analyzed briefly.

We note that our model may be given an interpretation in the context of a multiplant or multidivisional firm: the sectoral scales z_m become plant or divisional scales; the world prices p_k become prices of goods that are traded on well established markets; the endowment vector v becomes a vector of capital stocks or firm resources that can be allocated between plants but well established market prices do not exist for these resources; finally, w becomes a vector of interdivisional transfer prices.

2. THE ECONOMY'S PROFIT MAXIMIZATION PROBLEM

We assume that there are K + N goods in the economy. The first K are the internationally traded goods, while the last N are the domestic goods. We assume that there are M sectors or industries and that the technology of sector m (m = 1, 2,...,M) can be represented by a feasible set of net outputs denoted by $T^m \equiv \{z_m C^m : z_m \geq 0\}$ where C^m is a nonempty, closed, convex set of feasible net outputs[6] that can be produced by industry m when it runs at unit scale. We assume that C^m has the following property: for $p \gg 0_K$, $w \gg 0_N$, the following maximum exists and is finite (m = 1, 2,...,M):

$$(1) \quad \max_{a,b}\{p\cdot b - w\cdot a : (b, -a)\in C^m\} \equiv \pi^m(p,w)$$

(1) also serves to define π^m, the linearly homogeneous and convex unit (scale) profit function for industry m, m = 1, 2,...,M. Given π^m, the unit scale production possibilities set C^m may be recovered as follows:

$$(2) \quad C^m = \{(b, -a) : p\cdot b - w\cdot a \leq \pi^m(p,w) \text{ for every } p \gg 0_K,\ w \gg 0_N\}.$$

If π^m is differentiable, then by a result due originally to Hotelling[7], the solution $a^m(p,w)$, $b^m(p,w)$ to (1) is unique and may be obtained by differentiating the unit profit function π^m appropriately:

(3) $b^m(p,w) = \nabla_p \pi^m(p,w)$,

(4) $-a^m(p,w) = \nabla_w \pi^m(p,w)$,

where $\nabla_p \pi^m \equiv (\partial\pi^m/\partial p_1,\ldots,\partial\pi/\partial p_K)^T$ denotes the vector of first order partial derivatives of π^m with respect to the components of p, etc.

Since the components of $\nabla_w \pi^m(p,w)$ will all be nonpositive if domestic goods are only used as inputs in sector m, $a^m(p,w) \equiv -\nabla_w \pi^m(p,w)$ will be non-negative in this case.

The sets C^m can be interpreted as sets of feasible input-output coefficients for industry m, with outputs being indexed positively and inputs negatively.

Given that producers face world prices $p \gg 0_K$, industry subsidies (or taxes) s, and that the economy has the vector v of domestic resources available, the economy's aggregate profit maximization problem is

(5) $$\underset{\substack{(b^m,-a^m)\varepsilon C^m\\ m=1,\ldots,M}}{\text{maximize}} \ \max_{z\geq 0_M} \left\{ \sum_{m=1}^{M} (p\cdot b^m + s_m)z_m : \sum_{m=1}^{M} a^m z_m \leq v \right\}$$

$$= \underset{(b^m,-a^m)\varepsilon C^m}{\text{maximize}} \left[\max_z \left\{ \sum_{m=1}^{M} (p\cdot b^m + s_m)z_m : v - \sum_{m=1}^{M} a^m z_m \geq 0_N,\ z \geq 0_M \right\}\right]$$

$$= \underset{\substack{(b^m,-a^m)\varepsilon C^m\\ m=1,\ldots M}}{\text{maximize}} \left[\max_{z\geq 0_M} \ \min_{w\geq 0_N} \left\{ \sum_{m=1}^{M} (p\cdot b^m + s_m)z_m + w\cdot\left(v-\sum_{m=1}^{M} a^m z_m\right) \right\}\right]^9$$

(6) $$= \max_{z\geq 0_M} \ \min_{w\geq 0_N} \left\{ \sum_{m=1}^{M} (\pi^m(p,w) + s_m)z_m + w\cdot v \right\} \equiv \pi(p,s,v)$$

where (6) follows upon performing the maximization with respect to the b^m and a^m and using (1), the definitions of the unit profit functions π^m.

Note that the domestic resource price vector w in (6) is the vector of Lagrange (or Kuhn-Tucker) multipliers for the domestic resource constraints, $\sum_{m=1}^{M} a^m z_m \leq v$.

Note also that the objective function in (6) is linear in z and hence concave in z. It is convex in w (since the unit profit functions $\pi^m(p,w)$ are convex in p,w).

Finally, note that the primal maximization problem (5) involves (K + N + 1)M variables whereas the dual problem (6) involves only M + N variables. The dual or primal problem may be used to define the economy's GNP function, $\pi(p,s,v)$, as indicated above in (6).

Suppose $z^* \gg 0_M$ and $w^* \gg 0_N$ solve (6) when the vectors p^*, s^* and v^* of exogenous variables are given. Suppose further that the functions π^m are differentiable at p^*, w^* for m = 1,...,M. Then the following first order conditions for (6) will be satisfied:

(7) $\pi^m(p^*,w^*) + s^*_m = 0,\ m = 1,\ldots,M;$

(8) $$\sum_{m=1}^{M} \nabla_w \pi^m(p^*,w^*)z^*_m + v^* = 0_N.$$

In the following section, we totally differentiate (7) and (8) in order to obtain the comparative static responses of z and w to changes in p, s and w[10].

Note that equations (7) are zero profit conditions whereas equations (8) tell us that industry demands for domestic resources sum up to the economy's available supplies, v.

Define the N by M matrix of optimal unit scale input coefficients A by

(9) $$A \equiv [a^{1*},\ldots,a^{M*}] \equiv [-\nabla_w \pi^1(p^*,w^*),\ldots,-\nabla_w \pi^M(p^*,w^*)]$$

and the K by M matrix of optimal net output coefficients B by

(10) $$B \equiv [b^{1*},\ldots,b^{M*}] \equiv [\nabla_p \pi^1(p^*,w^*),\ldots,\nabla_p \pi^M(p^*,w^*)].$$

By the linear homogeneity in prices property of the π^m, we have $\pi^m(p^*,w^*) = p^*\cdot b^{m*} - w^*\cdot a^{m*}$ for $m = 1,\ldots,M$. Thus using (9) and (10), (7) and (8) may be rewritten as

(11) $p^{*T}B - w^{*T}A = -s^{*T}$ (zero profit relations)

(12) $Az^* = v^*$ (demand equals supply for domestic goods).

3. A FUNDAMENTAL MATRIX EQUATION OF STATIC PRODUCTION THEORY

We assume that the unit profit functions π^m are twice continuously differentiable around $p^* \gg 0_K$, $w^* \gg 0_N$ where v^*, s^*, p^* are given, and $z^* \gg 0_M$ and w^* satisfy (7) and (8). Rewrite equations (8) and (7) for v, s, p near v^*, s^*, p^* as

(13) $-\sum_{m=1}^{M} \nabla_w \pi^m(p,w(v,s,p))z_m(v,s,p) = v$ and

(14) $-\pi^m(p,w(v,s,p)) = s_m;\ m = 1,\ldots M.$

Using Hotelling's Lemma (3), for v, s, p in a neighbourhood around v^*, s^*, p^*,, the economy's system of net supply functions for internationally traded goods may be defined as

(15) $y(v,s,p) \equiv \sum_{m=1}^{M} \nabla_p \pi^m(p,w(v,s,p))z_m(v,s,p).$

Recalling (1), our definition of B when v,s, p equals v^*, s^*, p^*, (15) may be rewritten as

(16) $y^* = Bz^*$.

Equations (13) - (15) are the basic equations in our model of production behaviour. Total differentiation of (13) - (15) yields:

$$(17)\quad \begin{bmatrix} -S_{ww}, & A, & 0_{NxK} \\ A^T, & 0_{MxM}, & 0_{MxK} \\ -S_{pw}, & -B, & I_K \end{bmatrix} \begin{bmatrix} dw \\ dz \\ dy \end{bmatrix} = \begin{bmatrix} I_N, & 0_{NxM}, & S_{wp} \\ 0_{MxN}, & I_M, & B^T \\ 0_{KxN}, & 0_{KxM}, & S_{pp} \end{bmatrix} \begin{bmatrix} dv \\ ds \\ dp \end{bmatrix}$$

where A defined by (9) is an N by M matrix of <u>unit scale input requirements</u>, B defined by (10) is a K by M matrix of <u>unit scale input-output coefficients</u> (each column of B has at least one positive element b_{km} say, corresponding to industry m producing the kth internationally traded good as an output), and the economy's aggregate symmetric positive semidefinite <u>substitution matrix</u> S is defined by

$$(18)\quad S \equiv \begin{bmatrix} S_{pp}, & S_{pw} \\ S_{wp}, & S_{ww} \end{bmatrix} \equiv \begin{bmatrix} \sum_{m=1}^{M} \nabla^2_{pp} \pi^m(p^*,w^*)z^*_m, & \sum_{m=1}^{M} \nabla^2_{pw}\pi^m(p^*,w^*)z^*_m \\ \sum_{m=1}^{M} \nabla^2_{wp} \pi^m(p^*,w^*)z^*_m, & \sum_{m=1}^{M} \nabla^2_{ww}\pi^m(p^*,w^*)z^*_m \end{bmatrix}.$$

S_{pp} is the K by K aggregate internationally traded goods substitution matrix, S_{ww} is the N by N aggregate input substitution matrix, and $S_{pw} = S_{wp}^T$ is the K by N aggregated substitution matrix between traded goods and primary factors. The linear homogeneity of the unit profit functions implies that S satisfies the following restrictions:

(19) $S_{pp}p^* + S_{pw}w^* = 0_K$;

(20) $S_{wp}p^* + S_{ww}w^* = 0_N$.

The endogenous variables in (13) - (15) are: (i) $w \equiv (w_1,\ldots,w_N)^T$ a nonnegative vector of <u>factor prices</u>; (ii) $z \equiv (z_1,\ldots,z_M)^T$, a nonnegative vector of <u>industry scales</u>, and (iii) $y \equiv (y_1,\ldots,y_K)^T$, a vector of <u>net outputs</u> of internationally traded goods produced by the entire economy (if $y_k < 0$, then the kth good is imported). The <u>exogenous</u> variables are: (i) $v \equiv v_1,\ldots,v_M)^T$, an <u>endowment</u> vector of domestic goods for the economy (if $v_n < 0$, then the nth domestic good is produced by the production sector); (ii) $s \equiv (s_1,\ldots,S_M)^T$, a vector of per unit scale <u>subsidies</u> paid to each industry (if $s_m < 0$, then the nth industry pays taxes equal to $-s_m z_m$) and (iii) $p \equiv (p_1,\ldots,p_K)^T$, a positive vector of <u>world prices</u>.

In order to apply the Implicit Function Theorem to (13) - (15) to solve for w, z, and y as functions of v, s and p, we must[11] assume that the N by M matrix A has rank M (and thus the number of industries M is at most equal to the number of domestic resources N) and that $S_{ww} + A^TA$ has rank N (which requires that the domestic resource substitution matrix S_{ww} has rank at least equal to N-M). Making these assumptions, define

$$(21)\quad G^{-1} \equiv \begin{bmatrix} -S_{ww}, & A \\ A^T, & 0_{M\times M} \end{bmatrix}^{-1} \equiv \begin{bmatrix} D, & E \\ E^T, & F \end{bmatrix}.$$

The matrix equation $GG^{-1} = I_{N+M}$ implies that D, E and F satisfy:

$$(22)\ -S_{ww}D + AE^T = I_N\ ; \qquad (23)\ A^TD = 0_{M\times N}$$

$$(24)\ A^TE = I_M\ ; \qquad (25)\ -S_{ww}E + AF = 0_{N\times M}.$$

Since G is symmetric, so are D and F:

$$(26)\ D = D^T\ ; \qquad (27)\ F = F^T.$$

If we premultiply (25) by E^T and use (24), we obtain:

$$(27)\ F = F^TS_{ww}E.$$

Define the K by N + M matrix H as $H \equiv [S_{pw}, B]$. Then the inverse of the matrix on the left hand side of (17) is

$$(28)\begin{bmatrix} G, & 0 \\ -H, & I_K \end{bmatrix}^{-1} = \begin{bmatrix} G^{-1}, & 0 \\ HG^{-1}, & I_K \end{bmatrix}.$$

Define $\nabla_v w$ as the N by N matrix of partial derivatives $\partial_i(v^*, s^*, p^*)/\partial v_j$, i, j = 1,...,N; define $\nabla_s w$ as the N by M matrix of partial derivatives $\partial w_n(v^*, s^*, p^*)/\partial s_m$, n = 1,...,N, m = 1,...,M; etc. Then (21), (28) and the Implicit Function Theorem allow us to rewrite (17) as the following <u>fundamental matrix equation of production theory</u>:

$$(29)\quad \begin{bmatrix} \nabla_v w, & \nabla_s w, & \nabla_p w \\ \nabla_v z, & \nabla_s z, & \nabla_p z \\ \nabla_v y, & \nabla_s y, & \nabla_p y \end{bmatrix} = \begin{bmatrix} D & E & , DS_{wp} + EB^T \\ E^T & F & , E^TS_{wp} + FB^T \\ S_{pw}D + BE^T, & S_{pw}E + BF, & S_{pp} + HG^{-1}H^T \end{bmatrix}.$$

Using (21) and the definition of H, we may rewrite $\nabla_p y$ as

$$(30)\ \nabla_p y = S_{pp} + [S_{pw}, B]\begin{bmatrix} D, & E \\ E^T, & F \end{bmatrix}\begin{bmatrix} S_{wp} \\ B^T \end{bmatrix}$$

4. COMPARATIVE STATICS THEOREMS

Note that the matrix on the right hand side of (29) is symmetric. Thus we obtain the following extensions of Samuelson's (1953, p. 10) reciprocity relations:

$$(31)\ \nabla_s w = [\nabla_v z]^T\ ; \qquad (32)\ \nabla_p w = [\nabla_v y]^T\ ; \qquad (33)\ \nabla_p z = [\nabla_s y]^T$$

Thus the response of the nth factor price to a marginal increase in the mth industry subsidy, $\partial w_n/\partial s_m$, is equal to the change in the scale of industry m due to a marginal increase in the economy's endowment of input n, $\partial z_m/\partial v_n$. Similarly, (32) says that the response of factor price n to a marginal increase in the price of the kth traded good, $\partial w_n/\partial p_k$, is equal to the change in the net output of the kth internationally traded good induced by a marginal increase in the endowment of domestic good n, $\partial y_k/\partial v_n$. Finally, (33) says that the change in industry m's scale induced by a marginal increase in the price of the kth traded good induced by a marginal increase in subsidies paid to industry m, $\partial y_k/\partial s_m$[12].

We turn now to the properties of the three matrices which occur along the main diagonal of (29).

The N by N matrix $D = \nabla_v w$ is known to be negative semidefinite of rank N-M (recall (23)) while the M by M matrix $F = \nabla_s z$ is known to be positive semidefinite, and each zero eigenvalue of S_{ww} (if any exist) reduces the rank of F by one. For proofs of these statements, see Diewert and Woodland (1977). What is new is that the (potential) introduction of industry subsidies into the model provides an economic interpretation of the F matrix: its ijth element, $\partial z_i/\partial s_j$, is the response of the scale of industry i to a marginal increase in the subsidy rate paid to industry j. The positive semidefiniteness of F implies that

$$(32)\ \partial z_m/\partial s_m \geq 0,\ m = 1, 2,\ldots,M.$$

The inequalities in (32) will all be strict if the aggregate input substitution matrix S_{ww} is of full rank N. In this case, an increase in the subsidy rate for industry m will definitely expand the scale of that industry.

It is difficult to deduce the rank and definiteness properties of the K by K matrix $\nabla_p y$ defined by (3) since D is negative semidefinite while F is positive semidefinite. However, the following alternative formula for $\nabla_p y$ is valid:

$$(33)\ \nabla_p y = [I_K, \nabla_p{}^T w]\ S[I_K, \nabla_p^T w]^T$$

where S is the aggregate K + N by K + N positive semidefinite substitution matrix defined by (18), and (using the symmetry of the matrices in (29)) $\nabla_p^T w = S_{pw}D + BE^T$. In order to establish (33), we need to show that

$$(34)\ -DS_{ww}D = D \quad \text{and} \quad (35)\ -DS_{ww}E = 0_{NxM}.$$

However, (34) and (35) follow by premultiplying (22) and (25) by D and then using (23) and (26). Using (27), (34) and (35), the reader may verify that the right hand sides of (30) and (33) are equal.

It follows from (33) that $\nabla_p y$ is a positive semidefinite matrix. If S is of maximal rank K + N − 1, then $\nabla_p y$ is either of rank K or K − 1[13].

The properties of the Rybczynski matrix $E^T = \nabla_v z$ are identical to those properties already developed in the literature; see the analysis and references in Jones (1979, ch. 8) or Woodland (1982, ch. 3). Thus we now have economic interpretations for all of the matrices that occur in the fundamental matrix equation (29).

By looking at the primal problem (5) or the dual problem (6), we can readily deduce the homogeneity properties of our solution functions w(v, s, p), z(v, s, p) and y(v, s, p): w is homogenous of degree 0 in v and homogenous of degree 1 in s, p, while z and y are homogenous of degree 1 in v and homogenous of degree 0 in s, p. Thus by Euler's Theorem on homogenous functions, the matrices in (29) must satisfy the following restrictions:

$$(36)\ [\nabla_v w]v^* = 0_N \quad ; \quad (37)\ [\nabla_s w]s^* + [\nabla_p w]p^* = w^*;$$

$$(38)\ [\nabla_v z]v^* = z^* \quad ; \quad (39)\ [\nabla_s z]s^* + [\nabla_p z]p^* = 0_M;$$

$$(40)\ [\nabla_v y]v^* = y^* \quad ; \quad (41)\ [\nabla_s y]s^* + [\nabla_p y]p^* = 0_K.$$

Finally, we note that the formulae (29) and (33) simplify under various hypotheses. We list some hypotheses of economic interest and note their consequences: (i) the number of sectors M equals the number of domestic goods N; in this case $D = 0_{NxN}$ and $E = (A^T)^{-1}$; (ii) fixed coefficients in production with respect to traded goods; i.e., $S_{pp} \equiv 0_{KxK}$ and $S_{pw} = S_{wp}^T \equiv 0_{KxN}$; (iii) no joint production; each column of B has but one positive element and the elements of A are nonnegative; (iv) no joint production and no intermediate inputs; $K = M$, $B \equiv I_M$, $z = y$, $S_{pp} = 0_{KxK}$ and $S_{pw} = 0_{KxK}$ in this case; (v) no joint production, no intermediate inputs and the number of traded goods equals the number of primary inputs; in this case, $K = M = N$, $B = I_N$, $A \geq 0_{NxN}$, $z = y$, $S_{pp} = 0$ and $S_{pw} = 0$. This last case is the one treated most frequently in the international trade literature. The reader is invited to work out the consequences of the above special cases on the fundamental equation (29).

1. See also Barten and Böhm (1982, p. 410).

2. Notation: the vectors p and y are column vectors, p^T denotes the transpose of p and $p \cdot y$ or $p^T y \equiv \sum_{k=1}^{K} p_k y_k$ denotes the inner product of p and y. $p >> 0_K$ means each component of p is positive while $p \geq 0_K$ means each component is nonnegative. If x and y are vectors and f is a differentiable function of x and y, then $\nabla_x f(x^*,y^*)$ denotes the vector of first order partial derivatives of f with respect to the components of x and $\nabla^2_{xx} f(x^*,y^*)$ denotes the matrix of second order partial derivatives of f with respect to the components of x evaluated at x^*, y^*.

3. If a firm exhibits diminishing returns to scale, we may follow the approach of McKenzie (1959, p. 66) and introduce a firm specific fixed factor to which the pure profits of the firm will be imputed.

4. If $v_n > 0$ (the normal case), then the nth domestic good is an input into the production sector. If $v_n < 0$, then at least $-v_n$ of the nth good must be produced by the economy.

5. See also Samuelson (1953) (1967), Jones and Scheinkman (1977), Chang (1979), Jones (1979, ch. 8) and Woodland (1982, ch. 3).

6. If $x \in C^m$ and a component of x is negative, then the corresponding good is used as an input in sector m. We also assume that C^m is subject to free disposal; i.e. if $x'' \in C^m$ and $x' \leq x''$, then $x' \in C^m$ also.

7. For additional material on unit profit functions and their duality and differentiability properties, see the analysis and references in Diewert and Woodland (1977).

8. This GNP maximization problem (without the subsidy variables) was originally considered by Samuelson (1953) (1967) for neoclassical technologies, by Samuelson (1958) for linear technologies and by McKenzie (1955) for general technologies.

9. This equality follows by applying the Uzawa (1958, p. 34) - Karlin (1959, p. 201) Saddle Point Theorem, which requires that: (i) the sets C^m be convex, (ii) the objective and constraint functions be concave in the composite variables $b^m z_m$ and $-a^m z_m$, $m = 1,\ldots,M$ (which is true in our present application since objective and constraint functions are linear) and (iii) that an appropriate constraint qualification condition be satisfied (such as the Slater constraint qualification: there exist $(\tilde{b}^m, -\tilde{a}^m) \epsilon C^m$ and $\tilde{z}_m \geq 0$ such that $v - \sum_{m=1}^{M} \tilde{a}^m \tilde{z}_m >> 0_N$). If the sets C^m were singletons (this is the linear technology, no substitutability case), then we could readily obtain a dual to the primal problem (5) by applying the duality theorem for linear programs, as was originally done by Samuelson (1958).

10. Comparative statics analysis (which is known as sensitivity analysis in the applied mathematics literature) is explained in Samuelson (1947).

11. See Diewert and Woodland (1977, p. 391) for necessary and sufficient conditions for G^{-1} to exist where G is defined in (21).

12. (31) and (33) are new results while (32) provides a modest genralization to the case of a subsidy ridden economy of Diewert and Woodland's (1977, p. 390) generalization of the original Samuelson (1953) result.

13. If S has maximal rank K + N - 1 and $s^* = 0_M$, then it can be shown that $[\nabla_p y] p^* = 0_K$ and hence $\nabla_p y$ has rank K-1.

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