# THE ECONOMIC THEORY OF INDEX NUMBERS AND THE MEASUREMENT OF INPUT, OUTPUT, AND PRODUCTIVITY

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# THE ECONOMIC THEORY OF INDEX NUMBERS AND THE MEASUREMENT OF INPUT, OUTPUT, AND PRODUCTIVITY

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This paper develops index number procedures for making comparisons under very general circumstances. Malmquist input, output, and productivity comparisons are defined for structures of production with arbitrary returns to scale, substitution possibilities and biases in productivity change. For translog production structures, Törnqvist output and input indexes are shown to equal the mean of two Malmquist indexes. The Törnqvist productivity index, corrected by a scale factor, is shown to equal the mean of two Malmquist productivity indexes. Similar results are given for making cost of living comparisons under general structures of consumer preferences.

## 1. COMPARISONS BASED ON DISCRETE DATA

SINCE THE PIONEERING WORK of Solow [**20**], productivity growth or technical progress has been associated with the time derivative of the production function or some associated function such as the cost function or profit function. This formulation is a useful conceptualization, but it is not convenient for actual measurement of productivity using index numbers. The reason is that index number procedures entail comparisons using discrete data points and, therefore, require a discrete approximation to the time derivative. Only under very restrictive assumptions is the resulting index invariant to the point of approximation.

It would be desirable to be able to make index number measurements of productivity, and the related measurements of input and output, without having to approximate a concept that is specified with respect to continuous time. The purpose of this paper is to propose a framework for input, output, and productivity measurement that does not proceed from a continuous time representation. We require that the framework hold for very general structures of production, yet be empirically implementable using only observed prices and quantities of outputs and inputs.[2] The key to the proposed approach is the notion of a Malmquist input, output, or productivity index.

Moorsteen [**18**], in an independent application of an idea first suggested by Malmquist [**17**] in the consumer context, suggested comparing the input of a firm at two different points in time in terms of the maximum factor by which the input in one period could be deflated such that the firm could still produce the

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[2] Our treatment in this paper draws on ideas suggested by Caves, Christensen, and Swanson [**2**], who developed alternative measures of productivity growth in the continuous time framework, and Caves, Christensen, and Diewert [**3**], who developed multilateral indexes of productivity differences in the constant returns to scale case. In a critique of Jorgenson and Nishimizu [**13**], Denny and Fuss [**7**] also discuss the problems of using derivatives in the process of making comparisons using discrete data.

output levels observed for the other time period. This results in a Malmquist input index, and there is an analogous Malmquist output index. Neither Moorsteen nor Malmquist allowed for any differences over time in the structure of production or consumer tastes. We develop the Malmquist deflation idea in the context of unrestricted structures of production for two time periods. Once the structures of production have been allowed to differ, productivity measurement becomes an issue, and we extend the Malmquist deflation idea to define a Malmquist productivity index.

Having allowed for discrete changes in inputs, outputs, and the structure of production, it is no longer necessary to think in terms of the time dimension. In fact, it is more convenient to think in terms of two firms with different levels of inputs, outputs, and productivity. They might actually be the same firm at two different points of time or two different firms at the same or different points in time. Throughout the paper we refer to a comparison of two firms without differentiating the various possible cases.

In general a comparison of two firms, say $k$ and $l$, using the Malmquist concept, yields two distinct Malmquist indexes—one based on $k$'s structure of production and one based on $l$'s. If a functional form is specified for the structure of production, then the two Malmquist indexes can be computed for any given set of parameter values. However, without knowledge of the parameters, neither index can be computed. Thus the empirical usefulness of the Malmquist indexes is limited. This limitation does not mean, however, that in the absence of known production parameters the economist must forego comparisons based on the Malmquist concept. The theorems in this paper establish that for translog structures of production (see Christensen, Jorgenson, and Lau [**5**]) an average of the two Malmquist indexes can be computed using information on prices and quantities only, i.e. without knowledge of the translog parameters.

In our approach each firm is permitted to have its own translog form, neither of which is required to have constant returns to scale. For this case we show that the geometric means of the firm $k$ and firm $l$ Malmquist input and output indexes are Törnqvist [**22**] input and output indexes, which can be computed using only observed data on prices and quantities for firms $k$ and $l$. Furthermore, the geometric mean of the firm $k$ and firm $l$ Malmquist productivity indexes is a generalization of the Törnqvist productivity index originally proposed by Christensen and Jorgenson [**4**]. This index reduces to the Törnqvist index in the case of constant returns to scale. The index is computable using only price and quantity data for decreasing returns to scale as well as for constant returns to scale. However, for the case of increasing returns to scale, its computation also requires knowledge of the degree of return to scale.

In Sections 2, 3, and 4, respectively, we define Malmquist input, output, and productivity indexes for general structures of production for firms $k$ and $l$. Following these definitions, we derive observable indexes for the case of translog functional forms. In the Appendix we present a technical lemma, which we refer to as the Translog Identity. This identity is used to prove the theorems presented throughout the paper. Section 5 contains extensions to the problem of specifying observable indexes in the consumer case. It also contains a sketch of the development of indexes that are dual to those in the preceding sections of the paper. In Section 6, we present some concluding remarks.

## 2. INDEXES OF REAL INPUT

Consider the problem of comparing input between firms $k$ and $l$, where $k$ and $l$ may represent the same firm at two different times, or two different firms at either the same time or different times. Suppose firms $k$ and $l$ utilize the input vector $x \equiv (x_1, \ldots, x_N)$ to produce the output vector $y \equiv (y_1, y_2, \ldots, y_I) \equiv (y_1, \tilde{y})$. Let the technology be represented by the *production function*

$$(1) \qquad y_1 = F^s(\tilde{y}, x), \qquad s = k, l.$$

The production function represents the maximum amount of the first output that firm $s$ can produce, using the vector of inputs $x$, given that the vector of "other" outputs $\tilde{y}$ must also be produced.[3] We assume throughout that (1) holds; i.e., that each firm is operating on its production function. Although the production function $F^s$ is a standard means of representing the technology, it is not the most convenient representation for our purposes. Instead we use the *input distance or deflation function* $D^s$:

$$(2) \qquad D^s(y,x) \equiv \max_\delta\{\delta : F^s(\tilde{y}, x/\delta) \geq y_1\}, \qquad s = k, l,$$

which contains the same information about the technology as the production function itself. The regularity conditions that apply to the production function $F^s$ determine the regularity conditions that apply to the distance function $D^s$.[4] For example, very weak regularity conditions on $F^s$ assure that $D^s(y,x)$ will be linearly homogeneous in $x$.

It is important to spell out the relationship between the partial derivatives of $F^s$ and $D^s$. To do this we assume the following:[5]

$$(3) \qquad \begin{aligned} &F^s \text{ is differentiable at the point, } (\tilde{y}^s, x^s), \qquad s = k, l, \\ &x^s \gg 0_N, \\ &x^s \cdot \nabla_x F^s(\tilde{y}^s, x^s) > 0.^6 \end{aligned}$$

[3] It is straightforward to use the transformation function, say $t^s(x, y) = 1$, as an alternative point of departure.

[4] For discussions of these properties and references to the literature, see Deaton [6] or Diewert [11].

[5] Using some results in Blackorby and Diewert [1], it can be shown that differentiability of the distance function $D^s$ implies differentiability of $F^s$.

[6] Notation: $x \gg 0_N$ means the $N$ dimensional vector has all components positive, $\nabla_x f(x, y) \equiv [\partial f(x, y)/\partial x_1, \ldots, \partial f(x, y)/\partial x_N]^T$ is a column vector ($T$ denoting transposition) of the partial derivative of $f$ with respect to the vector of variables $x$,

$$\nabla_{\ln x} f(x^k, y^k) \equiv [x_1^k \partial f(x^k, y^k)/\partial x_1, \ldots, x_N^k \partial f(x^k, y^k)/\partial x_N]^T,$$

$$x^T z = x \cdot z \equiv \sum x_i z_i$$

denotes the inner product of two vectors $x$ and $z$, and

$$\ln x^s \equiv [\ln x_1^s, \ldots, \ln x_N^s]^T.$$

Now we apply the Implicit Function Theorem to the equation $F^s(\tilde{y}^s, x/\delta) = y_1$ to solve for $\delta = D^s(y,x)$ around $(y^s, x^s)$. In this case, $D^s(y,x)$ is differentiable around the point $(y^s, x^s)$ with:

$$\nabla_x D^s(y^s,x^s) = \nabla_x F^s(\tilde{y}^s,x^s)/x^s \cdot \nabla_x F^s(\tilde{y}^s,x^s), \qquad s=k,l, \tag{4}$$

$$\nabla_y D^s(y^s,x^s) = \frac{1}{x^s \cdot \nabla_x F^s(\tilde{y}^s,x^s)} \begin{bmatrix} -1 \\ \nabla_{\tilde{y}} F^s(\tilde{y}^s,x^s) \end{bmatrix}. \tag{5}$$

We can now consider the problem of comparing the input of firms $k$ and $l$. We define the *firm k Malmquist* [17] *input index* as:[7]

$$Q^k(x^l,x^k) \equiv D^k(y^k,x^l)/D^k(y^k,x^k). \tag{6}$$

It is straightforward to interpret this definition in terms of the firm $k$ production function. By definition (2), $D^k(y^k,x^k)=1$, provided that (1) is satisfied for $s=k$. Thus,

$$Q^k(x^l,x^k) = D^k(y^k,x^l) = \max_\delta\{\delta : F^k(\tilde{y}^k, x^l/\delta) \geq y_1^k\} \tag{7}$$

(using (2)).

The input index $Q^k(x^l,x^k)$ is therefore the maximum $\delta$ required to deflate the input vector of firm $l$, $x^l$, onto the production surface of firm $k$, given that the output vector is that of firm $k$. Note that $Q^k(x^k,x^k)=1$; thus $Q^k(x^l,x^k)>1$ implies that the input vector of firm $l, x^l$, is "bigger" than the input vector of firm $k, x^k$, from the perspective of firm $k$'s technology.

We can reverse the roles of the firms and define the *firm l Malmquist input index* as:

$$Q^l(x^l,x^k) \equiv D^l(y^l,x^l)/D^l(y^l,x^k). \tag{8}$$

As above we note that

$$\begin{aligned} Q^l(x^l,x^k) &= 1/D^l(y^l,x^k) && \text{(if (1) holds for firm } l) \\ &= 1/\max_\delta\{\delta : F^l(\tilde{y}^l, x^k/\delta) \geq y_1^l\} \\ &= \min_\rho\{\rho : F^l(\tilde{y}^l, \rho x^k) \geq y_1^l\} && (\text{letting } \rho = 1/\delta). \end{aligned} \tag{9}$$

Thus $Q^l(x^l,x^k)$ is the minimum factor $\rho$ required to inflate the input vector of firm $k, x^k$, onto the production surface of firm $l$, given that the output vector is that of firm $l$. Note that $Q^l(x^l,x^l)=1$; thus $Q^l(x^l,x^k)>1$ implies that the input vector of firm $k, x^k$, is "smaller" than the input vector of firm $l, x^l$, from the perspective of firm $l$'s technology.

[7] See also Fisher and Shell [**12**, p. 51] and Diewert [**10**, p. 462]. Note that we have chosen the reference vector of quantities to be the $k$th firm's quantity vector $y^k$.

Each of the two Malmquist input indexes, $Q^k(x^l,x^k)$ and $Q^l(x^l,x^k)$, is defined with respect to the technology and output levels of a particular firm. But without knowledge of the parameters of the technologies, it is not possible to compute either index. This is because they require information that is not observed. However, we now show that by making use of a specific functional form and the assumption of cost minimizing behavior, it is possible to compute a geometric average of the two Malmquist indexes $Q^k(x^l,x^k)$ and $Q^l(x^l,x^k)$, using only observed information on input prices and quantities. We demonstrate this fact for the case in which each firm has a translog distance function, but the properties of the two translog functions are allowed to differ substantially. In this case the geometric average of the two Malmquist indexes turns out to be a Törnqvist [22] index.

We denote the translog distance functions for firms $k$ and $l$ as

$$\text{(10)}\quad \ln h^s(y,x) \equiv \alpha_0^s + \sum_{n=1}^{N} \beta_n^s \ln x_n + \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \beta_{nm}^s \ln x_n \ln x_m + \sum_{i=1}^{I} \alpha_i^s \ln y_i + \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I} \alpha_{ij}^s \ln y_i \ln y_j + \sum_{i=1}^{I}\sum_{n=1}^{N} \gamma_{in}^s \ln y_i \ln x_n,$$

where $\beta_{nm}^s = \beta_{mn}^s$ and $\alpha_{ij}^s = \alpha_{ji}^s$ for all $n$, $m$, $i$, $j$, and $s$.

Since we perform several manipulations involving a pair of translog functional forms in this paper, it is convenient to develop a general result to which we can refer. In the Appendix we develop a Translog Identity, which is a generalization of Diewert's [8, p. 118] Quadratic Identity. The Translog Identity reduces to the Quadratic Identity when the two translog forms are the same. Throughout the paper we use the symbol $h^s$ to indicate a translog functional form for firm $s$.

Assume that $x^s \gg 0_N$ is a solution to the cost minimization problem:

$$\text{(11)}\quad \min_x\{w^s \cdot x : F^s(\tilde{y}^s, x) \geq y_1^s\}, \qquad s = k, l,$$

where $w^s \gg 0_N$ is the vector of input prices that firm $s$ faces. Under (3) there exists a Lagrange multiplier $\lambda_s$ such that the following first order conditions are satisfied:

$$\text{(12)}\quad w^s = \lambda_s \nabla_x F^s(\tilde{y}^s, x^s), \qquad s = k, l.$$

Multiplying both sides of (12) by $x^s$ and solving for $\lambda_s$ yields

$$\text{(13)}\quad \lambda_s = w^s \cdot x^s / x^s \cdot \nabla_x F^s(\tilde{y}^s, x^s), \qquad s = k, l.$$

Substituting (13) into (12) we obtain, for $s = k, l$,

$$\text{(14)}\quad w^s / w^s \cdot x^s = \nabla_x F^s(\tilde{y}^s, x^s) / x^s \cdot \nabla_x F^s(\tilde{y}^s, x^s) = \nabla_x D^s(y^s, x^s),$$

using (4).

Now we can prove the following theorem.

THEOREM 1: *If firms k and l have translog distance functions with identical coefficients for the second order input terms, then the Törnqvist index for comparing the inputs of k and l is equal to the geometric mean of the Malmquist input indexes evaluated at k and l. That is, if, for* $s = k, l$, (i) $D^s(y,x) = h^s(y,x)$; *i.e., the distance function is translog*; (ii) (1) *and* (3) *hold*; (iii) $x^s$ *is a solution to* (11) *for* $s = k, l$; *and* (iv) $\beta^k_{nm} = \beta^l_{nm}$, *for* $n, m = 1, \ldots, N_1$, *i.e.,* $h^k$ *and* $h^l$ *have identical coefficients for the second order terms in* $\ln x$, *but are otherwise unrestricted*; *then,*

$$(15) \quad \tfrac{1}{2}\ln Q^k(x^l,x^k) + \tfrac{1}{2}\ln Q^l(x^l,x^k) = \frac{1}{2}\sum_{n=1}^{N}\left[\frac{w_n^k x_n^k}{w^k\cdot x^k} + \frac{w_n^l x_n^l}{w^l\cdot x^l}\right]\left[\ln x_n^l - \ln x_n^k\right] \equiv \ln Q(w^l,w^k,x^l,x^k)$$

*where* $Q(w^l,w^k,x^l,x^k)$ *is the Törnqvist input index.*

PROOF:

$$\begin{aligned}
&\tfrac{1}{2}\ln Q^k(x^l,x^k) + \tfrac{1}{2}\ln Q^l(x^l,x^k) \\
&= \tfrac{1}{2}\ln\left[D^k(y^k,x^l)/D^k(y^k,x^k)\right] + \tfrac{1}{2}\ln\left[D^l(y^l,x^l)/D^l(y^l,x^k)\right] \\
&\qquad\qquad \text{using definitions (6) and (8)} \\
&= \tfrac{1}{2}\left[\nabla_{\ln x}\ln D^k(y^k,x^k) + \nabla_{\ln n}\ln D^l(y^l,x^l)\right]\cdot\left[\ln x^l - \ln x^k\right] \\
&\qquad\qquad \text{(using the restriction (iv) and the Translog Identity)} \\
&= \tfrac{1}{2}\left[\nabla_{\ln x}D^k(y^k,x^k) + \nabla_{\ln x}D^l(y^l,x^l)\right]\cdot\left[\ln x^l - \ln x^k\right] \\
&\qquad\qquad \left(\text{since } D^k(y^k,x^k) = 1 = D^l(y^l,x^l) \text{ (using (3))}\right) \\
&= \frac{1}{2}\sum_{n}^{N}\left[\frac{w_n^k x_n^k}{w^k\cdot x^k} + \frac{w_n^l x_n^l}{w^l\cdot x^l}\right]\left[\ln x_n^l - \ln x_n^k\right] \qquad \text{(using 14)} \\
&\equiv \ln Q(w^l,w^k,x^l,x^k). \qquad\qquad \textit{Q.E.D.}
\end{aligned}$$

Theorem 1 does not require optimizing behavior with respect to outputs nor does it require linear homogeneity for either firm. Furthermore, the translog distance function $D^s(y,x)$ is capable of providing a second order approximation to an arbitrary distance function. Thus the technologies in the two firms can be virtually arbitrary (to the second order) except for the restrictions (iv) under Theorem 1.[8]

[8] Diewert [**10**, p. 462–463] earlier showed that if both input distance functions were identical, $D^k = D^l \equiv D$, then under the remaining assumptions of Theorem 1, $D(y^*,x^l)/D(y^*,x^k) = Q(w^l, w^k, x^l, x^k)$, where $y_i^*$, the $i$th component of $y^*$, equals $(y_i^k y_i^l)^{1/2}$, the geometric mean of the $i$th outputs for firms $k$ and $l$ for $i = 1, 2, \ldots, I$; i.e., a Malmquist input quantity index evaluation at an "average" output vector was equal to the Törnqvist input quantity index $Q(w^l, w^k, x^l, x^k)$. Our new result shows that an "average" of two Malmquist quantity indexes also equals $Q(w^l, w^k, x^l, x^k)$, even in the case in which the distance functions are not identical.

Although Theorem 1 is very useful, we should mention some limitations associated with it: (1) although we can identify a geometric average of the Malmquist indexes $Q^k(x^l,x^k)$ and $Q^l(x^l,x^k)$, we cannot identify each individual index; (ii) we must assume cost minimizing behavior; and (iii) we must assume that both firms use positive amounts of all inputs.

## 3. INDEXES OF REAL OUTPUT

The comparison of output between firms $k$ and $l$ is analogous to the comparison of inputs of Section 2, with the roles of outputs and inputs reversed and the assumption of cost minimization replaced by the assumption of revenue maximization. Despite the analogy between this section and the previous section, we present the basic steps in the derivation since we will draw on them for the derivation of productivity indexes in Section 4.

In Section 2 we used the production function $F^s$ to represent the technology. We can equally well represent the technology with the *input requirements function*

$$(16) \qquad x_1 = g^s(y,\tilde{x}), \qquad s = k,l.$$

The input requirements function gives the minimum amount of the first input required to produce the output vector $y$, given that the vector of "other" inputs, $\tilde{x}$, is available.[9] We assume throughout that (16) holds, i.e. that the firm is operating on its input requirements function. Corresponding to the input distance function of Section 3, we now utilize the *output distance or deflation function*, $d^s$:

$$(17) \qquad d^s(y,x) \equiv \min_\delta \{\delta : g^s(y/\delta,\tilde{x}) \le x_1\}, \qquad s = k,l.$$

It is important to spell out the relationships between the partial derivatives of $g^s$ and $d^s$. To do this we assume the following: for $s = k,l$,

$$g^s \text{ is differentiable at the point } (y^s,\tilde{x}^s),$$

$$(18) \qquad y^s \gg 0_I,$$

$$y^s \cdot \nabla_y g^s(y^s,\tilde{x}^s) > 0.$$

Now we can apply the Implicit Function Theorem to the equation $g^s(y/\delta,x) = x_1$ to solve for $\delta = d^s(y,x)$ around $(y^s,x^s)$. In this case, $d^s(y,x)$ is differentiable around $(y^s,x^s)$, with

$$(19) \qquad \nabla_y d^s(y^s,x^s) = \nabla_y g^s(y^s,\tilde{x}^s)/y^s \cdot \nabla_y g^s(y^s,\tilde{x}^s), \quad \text{and}$$

$$(20) \qquad \nabla_x d^s(y^s,x^s) = \frac{1}{y^s \cdot \nabla_y g^s(y^s,\tilde{x}^s)} \begin{bmatrix} -1 \\ \nabla_{\tilde{x}} g^s(y^s,\tilde{x}^s) \end{bmatrix}, \qquad \text{for} \quad s = k,l.$$

We can now compare the output of firms $k$ and $l$. We define the firm $k$

[9] If there is no $x_1$ such that $(x_1,\tilde{x})$ can produce $y$, $g^s(y,\tilde{x}) \equiv +\infty$.

*Malmquist output index* as:

(21) $q^k(y^l, y^k) \equiv d^k(y^l, x^k)/d^k(y^k, x^k).$

It is straightforward to interpret this definition in terms of the firm $k$ input requirements function. Under (16), it can be verified that $d^k(y^k, x^k) = 1$. Thus,

(22) $$q^k(y^l, y^k) = d^k(y^l, x^k)$$
$$= \min_\delta\{\delta : g^k(y^l/\delta, x^k) \leq x_1^k\} \qquad \text{(using (17))}.$$

The output index $q^k(y^l, y^k)$ is therefore the minimum factor $\delta$ required to deflate the output vector of firm $l$, $y^l$, onto the production surface of firm $k$, given that the input vector is that of firm $k$. Note that $q^k(y^k, y^k) = 1$ under (16); thus $q^k(y^l, y^k) > 1$ implies that the output vector of firm $l$ is "bigger" than the output vector of firm $k$ from the viewpoint of firm $k$.

Reversing the roles of the firms we define a *firm l Malmquist output index* as:

(23) $q^l(y^l, y^k) \equiv d^l(y^l, x^l)/d^l(y^k, x^l).$

As above we note that

(24) $$q^l(y^l, y^k) = 1/d^l(y^k, x^l) \qquad \text{(if (16) holds for firm } l\text{)}$$
$$= 1/\min_\delta\{\delta : g^l(y^k/\delta, \tilde{x}^l) \leq x_1^l\}$$
$$= \max_\rho\{\rho : g^l(\rho y^k, \tilde{x}^l) \leq x_1^l\} \qquad (\text{letting } \rho = 1/\delta).$$

Thus $q^l(y^l, y^k)$ is the maximum factor $\rho$ required to inflate the output vector of firm $k$, $y^k$, onto the production surface of firm $l$, given that the input vector is that of firm $l$.

As with the Malmquist input indexes, the Malmquist output indexes cannot be computed without knowledge of the parameters of the firms' technologies. However, we now show that by making use of the assumption of revenue maximizing behavior, it is possible to compute the geometric average of the Malmquist indexes without knowledge of the parameters of either firm's technology.

Assume that $y^s \gg 0_I$, the observed output vector for firm $s$, is a solution to the following *revenue maximization problem*:

(25) $\max_y\{p^s \cdot y : g^s(y, \tilde{x}^s) \leq x_1^s\}, \qquad s = k, l,$

where $p^s \gg 0$ is the vector of output prices facing firm $s$. Under (18) the first order conditions for (25) are

(26) $p^s = \lambda_s \nabla_y g^s(y^s, \tilde{x}^s), \qquad s = k, l.$

Multiplying both sides of (26) by $y^s$ and solving for $\lambda_s$ yields

$$\lambda_s = p^s \cdot y^s / y^s \cdot \nabla_y g^s(y^s, \tilde{x}^s), \qquad s = k, l. \tag{27}$$

Substituting (27) into (26) yields, for $s = k, l$,

$$p^s / p^s \cdot y^s = \nabla_y g^s(y^s, \tilde{x}^s) / y^s \cdot \nabla_y g^s(y^s, \tilde{x}^s) = \nabla_y d^s(y^s, x^s) \tag{28}$$

(using (19)).

We can now prove the following theorem.

THEOREM 2: *If firms k and l have translog output distance functions with identical coefficients for the second order output terms, then the Törnqvist index for comparing the outputs of k and l is equal to the geometric mean of the Malmquist output indexes evaluated at k and l.*

*That is, if, for* $s = k, l$, (i) $d^s(y, x) = h^s(y, x)$; *i.e., the distance function is the translog function* (10); (ii) (16) *and* (18) *hold*; (iii) $y^s$ *is a solution to* (25); *and* (iv) $\alpha_{ij}^k = \alpha_{ij}^l$ *for* $i, j = 1, \ldots, I$, *i.e.* $h^k$ *and* $h^l$ *have identical coefficients for the second order terms in* $\ln y$, *but are otherwise unrestricted*; *then*

$$\begin{aligned} &\tfrac{1}{2} \ln q^k(y^l, y^k) + \tfrac{1}{2} \ln q^l(y^l, y^k) \\ &= \frac{1}{2} \sum_{i=1}^{I} \left[ \frac{p_i^k y_i^k}{p^k \cdot y^k} + \frac{p_i^l y_i^l}{p^l \cdot y^l} \right] \left[ \ln y_i^l - \ln y_i^k \right] \\ &\equiv \ln Q(p^l, p^k, y^l, y^k) \end{aligned} \tag{29}$$

*where* $Q(p^l, p^k, y^l, y^k)$ *is the Törnqvist output index.*

The proof of Theorem 2 is analogous to the proof of Theorem 1.[10] Our earlier comments on the advantages and disadvantages associated with the use of Theorem 1 apply also to Theorem 2.

## 4. PRODUCTIVITY INDEXES

There are two natural approaches to the measurement of productivity differences. One approach treats productivity differences as differences in maximum output conditional on a given level of inputs. This approach leads to *output based productivity indexes*. The alternative approach treats productivity differences as differences in minimum input requirements conditional on a given level of outputs. This view leads to *input based productivity indexes*. Output and input

[10] Theorem 2 should be contrasted with a theorem in Diewert [**10**, pp. 463–464], which showed that if both output deflation functions were identical so that $d^k = d^l = d$, then under the remaining assumptions of Theorem 2, $d(y^l, x^*)/d(y^k, x^*) = Q(p^k, p^l, y^k, y^l)$, where $x_n^*$, the $n$th component of the reference input vector $x^*$, equals $(x_n^k x_n^l)^{1/2}$ for $i = 1, 2, \ldots, N$.

based productivity indexes differ from each other by a factor that reflects the returns to scale of the production structure.[11]

We define local returns to scale for firm $s$ as follows. Consider proportionally increasing all inputs $x^s$ by a factor $\lambda$. Let $u^s(y^s,x^s,\lambda)$ be the factor of proportionality by which all outputs, $y^s$, must be increased so that the inflated input and output vectors lie on a production surface for firm $s$; i.e., $u^s(y^s,x^s,\lambda)$ is the solution to

$$\lambda x_1^s = g^s(u^s y^s, \lambda\tilde{x}^s), \qquad s = k, l. \tag{30}$$

Then the degree of local returns to scale is given by $\epsilon^s = \partial u^s(y^s,x^s,\lambda)/\partial\lambda$, evaluated at $\lambda = 1$. If returns to scale are locally constant (increasing or decreasing respectively), then $\epsilon^s = 1$ ($>1$ or $<1$, respectively). Comparing (30) and (17), we see that $u^s(y^s,x^s,\lambda)$ is the inverse of the output distance function evaluated at $(y^s,x^s,\lambda)$. Therefore, returns to scale are given by

$$\epsilon^s = \partial u^s(y^s,x^s,\lambda)/\partial\lambda = \nabla_x d(y^s,\lambda x^s)^{-1} = -x^s \cdot \nabla_x d^s(y^s,x^s) \tag{31}$$

at $\lambda = 1$. Comparing (31) and (20) we find

$$\epsilon^s = \{x_1^s - \tilde{x}^s \cdot \nabla_{\tilde{x}} g^s(y^s,\tilde{x}^s)\}/\{y^s \cdot \nabla_y g^s(y^s,\tilde{x}^s)\}. \tag{32}$$

We define the Malmquist *firm k output based productivity index*, as $m^k(x^l,x^k,y^l,y^k)$,

$$m^k(x^l,x^k,y^l,y^k) \equiv d^k(y^l,x^l)/d^k(y^k,x^k). \tag{33}$$

Under (16) $d^k(y^k,x^k) = 1$, so that

$$m^k(x^l,x^k,y^l,y^k) = d^k(y^l,x^l) \equiv \min_\delta\{\delta : g^k(y^l/\delta,\tilde{x}^l) \leq x_1^l\} \tag{34}$$

using (17).

Thus $m^k(x^l,x^k,y^l,y^k)$ is the minimal output deflation factor such that the deflated output vector for firm $l$, $y^l/m^k$, and the firm $l$ input vector, $x^l$, are just on the production surface of firm $k$. Note that $m^k(x^k,x^k,y^k,y^k) = 1$. Therefore, if firm $l$ has a higher level of productivity than firm $k$ from the perspective of firm $k$'s production structure, then $m^k > 1$.

To compare the productivity of firm $k$ with that of firm $l$ from the perspective of firm $l$, we define the *Malmquist firm l output based productivity index*, $m^l(x^l,x^k,y^l,y^k)$:

$$m^l(x^l,x^k,y^l,y^k) \equiv d^l(y^l,x^l)/d^l(y^k,x^k). \tag{35}$$

[11] The relationship between input and output based productivity measures is discussed in Caves, Christensen, and Swanson [2], in a continuous time framework.

Under (16) $d^l(y^l, x^l) = 1$, so that

$$(36) \qquad m^l(x^l, x^k, y^l, y^k) = 1/d^l(y^k, x^k)$$

$$= 1/\min_\delta\{\delta : g^l(y^k/\delta, \tilde{x}^k) \leq x_1^k\} \qquad \text{using (17)}$$

$$= \max_\rho\{\rho : g^l(\rho y^k, \tilde{x}^k) \leq x_1^k\}.$$

Thus $m^l(x^l, x^k, y^l, y^k)$ is the maximum inflation factor for the firm $k$ output vector such that the resulting inflated output vector, $m^l y^k$, and the firm $k$ input vector, $x^k$, are just on the production surface of firm $l$. If firm $l$ has a higher level of productivity than firm $k$ from the perspective of firm $l$'s production structure, then $m^l > 1$.

Assumption (18) in the previous section enabled us to compute the partial derivatives of the output deflation functions $d^s(y^s, x^s)$ with respect to their arguments in terms of partial derivatives of the input requirements functions $g^s(y^s, \tilde{x}^s)$. We also used the assumption of revenue maximizing behavior to derive (28). We maintain the assumption that the firm maximizes revenues conditional on input levels, but now we also assume that the firm minimizes cost conditional on output levels; i.e. that $\tilde{x}^s$ is the solution to

$$(37) \qquad \min_x\{w^s \cdot x + w_1^s g^s(y^s, x)\}, \qquad s = k, l,$$

which yields the first order conditions:

$$(38) \qquad \tilde{w}^s = -w_1^s \nabla_{\tilde{x}} g^s(y^s, \tilde{x}^s), \qquad s = k, l.$$

Multiplying both sides of (38) by $\tilde{x}^s$ and adding $w_1^s x_1^s$, we obtain

$$(39) \qquad w^s \cdot x^s = \tilde{w}^s \cdot \tilde{x}^s + w_1^s x_1^s = -w_1^s \tilde{x}^s \cdot \nabla_{\tilde{x}} g^s(y^s, \tilde{x}^s) + w_1^s x_1^s, \qquad s = k, l.$$

Combining (38) and (39) we find, for $s = k, l$,

$$(40) \qquad \frac{1}{w^s \cdot x^s}\begin{bmatrix} w_1^s \\ \tilde{w}^s \end{bmatrix} = \frac{1}{x_1^s - \tilde{x}^s \cdot \nabla_{\tilde{x}} g^s(y^s, \tilde{x}^s)}\begin{bmatrix} 1 \\ -\nabla_{\tilde{x}} g^s(y^s, \tilde{x}^s) \end{bmatrix}$$

$$= \frac{y^s \cdot \nabla_y g^s(y^s, \tilde{x}^s)\begin{bmatrix} 1 \\ -\nabla_{\tilde{x}} g^s(y^s, \tilde{x}^s) \end{bmatrix}}{\{x_1^s - \tilde{x}^s \cdot g^s(y^s, \tilde{x}^s)\}\{y^s \cdot \nabla_y g^s(y_s, \tilde{x}^s)\}}.$$

Using (20) and (32) yields, for $s = k, l$,

$$\frac{1}{w^s \cdot x^s}\begin{bmatrix} w_1^s \\ \tilde{w}^s \end{bmatrix} = -(\epsilon^s)^{-1}\nabla_x d^s(y^s, x^s), \quad \text{or} \tag{41}$$

$$-\nabla_x d^s(y^s, x^s) = \frac{\epsilon^s}{w^s \cdot x^s}\begin{bmatrix} w_1^s \\ \tilde{w}_s \end{bmatrix}. \tag{42}$$

We can now prove the following theorem.

THEOREM 3: *If firms k and l have translog output distance functions with identical second order coefficients, then the product of a scale factor and the ratio of the Törnqvist index for comparing the outputs of k and l to the Törnqvist index for comparing inputs of k and l is equal to the geometric mean of the firm k and firm l Malmquist output based productivity indexes. That is, if for* $s = k, l$, (i) $d^s(y,x) = h^s(y,x)$: *i.e., the output distance function is translog*; (ii) (16) *and* (18) *hold*; (iii) $(y^s, x^s)$ *is a solution to* (25) *and* (37); *and*

(iv) $\alpha_{nm}^k = \alpha_{nm}^l$ *for* $n, m = 1, \ldots, N$,

$\beta_{ij}^k = \beta_{ij}^l$ *for* $i, j = 1, \ldots, I$,

$\gamma_{in}^k = \gamma_{in}^l$ *for* $i = 1, \ldots, I$ *and* $n = 1, \ldots, N$,

*then,*

$$\begin{aligned}
\ln m(x^l, x^k, y^l, y^k) &\equiv \tfrac{1}{2}\ln m^k(x^l, x^k, y^l, y^k) + \tfrac{1}{2}\ln m^l(x^l, x^k, y^l, y^k) \\
&= \frac{1}{2}\sum_i^I\left[\frac{p_i^k y_i^k}{p^k \cdot y^k} + \frac{p_i^l y_i^l}{p^l \cdot y^l}\right][\ln y_i^l - \ln y_i^k] \\
&\quad - \frac{1}{2}\sum_n^N\left[\frac{w_n^k x_n^k}{w^k \cdot x^k} + \frac{w_n x_n}{w^l \cdot x^l}\right][\ln x_n^l - \ln x_n^k] \\
&\quad + \frac{1}{2}\sum_n^N\left[\frac{w_n^k x_n^k}{w^k \cdot x^k}(1 - \epsilon^k)\right. \\
&\qquad \left. + \frac{w_n^l x_n^l}{w^l \cdot x^l}(1 - \epsilon^k)\right][\ln x_n^l - \ln x_n^k] \\
&\equiv \ln Q(p^k, p^l, y^k, y^l) - \ln Q(w^k, w^l, x^k, x^l) \\
&\quad + r(w^l, w^k, x^l, x^k, \epsilon^l, \epsilon^k)
\end{aligned}$$

*where the returns to scale term r is defined by the third and fourth lines above.*

PROOF:

$$\frac{1}{2}\ln m^k(x^l,x^k,y^l,y^k)+\frac{1}{2}\ln m^l(x^l,x^k,y^l,y^k)$$

$$=\frac{1}{2}\left[\ln d^k(y^l,x^l)-\ln d^k(y^k,x^k)\right]$$

$$+\frac{1}{2}\left[\ln d^l(y^l,x^l)-\ln d^l(x^k,y^k)\right]$$

(by definitions (33) and (35))

$$=\frac{1}{2}\left[\nabla_{\ln y}\ln d^k(y^k,x^k)+\nabla_{\ln y}\ln d^l(y^l,x^l)\right]\cdot\left[\ln y^l-\ln y^k\right]$$

$$+\frac{1}{2}\left[\nabla_{\ln x}\ln d^k(y^k,x^k)+\nabla_{\ln x}\ln d^l(y^l,x^l)\right]\cdot\left[\ln x^l-\ln x^k\right]$$

(applying Lemma 1)

$$=\frac{1}{2}\sum_i^I\left[\frac{p_i^ky_i^k}{p^k\cdot y^k}+\frac{p_i^ly_i^l}{p^l\cdot y^l}\right]\left[\ln y_i^l-\ln y_i^k\right]$$

$$-\frac{1}{2}\sum_n^N\left[\frac{w_n^kx_n^k}{w^k\cdot x^k}\epsilon^k+\frac{w_n^lx_n^l}{w^l\cdot x^l}\epsilon^l\right]\left[\ln x_n^l-\ln x_n^k\right]$$

(using (28) and (42))

$$=\frac{1}{2}\sum_i^I\left[\frac{p_i^ky_i^k}{p^k\cdot y^k}+\frac{p_i^ly_i^l}{p^l\cdot y^l}\right]\left[\ln y_i^l-\ln y_i^k\right]$$

$$-\frac{1}{2}\sum_n^N\left[\frac{w_n^kx_n^k}{w^k\cdot x^k}+\frac{w_n^lx_n^l}{w^l\cdot x^l}\right]\left[\ln x_n^l-\ln x_n^k\right]$$

$$+\frac{1}{2}\sum_n^N\left[\frac{w_nx_n}{w^k\cdot x^k}(1-\epsilon^k)+\frac{w_n^lx_n^l}{w^l\cdot x^l}(1-\epsilon^l)\right](\ln x_n^l-\ln x_n^k)$$

$$=\ln Q(p^l,p^k,y^l,y^k)-\ln Q(w^l,w^k,x^l,x^k)$$

$$+r(w^l,w^k,x^l,x^k,\epsilon^l,\epsilon^k).$$

Note that the scale factor $r$ is the Törnqvist input index with the shares multiplied by unity minus the degree of returns to scale. Given data on inputs, input prices, outputs and output prices, we can compute $m(x^l,x^k,y^l,y^k)$, provided that we know the local degree of returns to scale, $\epsilon^s$ for $s=k,l$.

This leads us to consider the evaluation of $m$ under constant, increasing, and decreasing returns to scale.

CASE (i)—Constant Returns to Scale: Under constant returns to scale $\epsilon^s = 1$ for $s = k$ and $l$; hence the scale factor, $r$, vanishes and $m$ is given by the ratio of the Törnqvist output index to the Törnqvist input index.

CASE (ii)—Decreasing Returns to Scale: Under the cost minimization and revenue maximization hypotheses, we know, from (26) and (39), that

$$\frac{p^s \cdot y^s}{w^s \cdot x^s} = \frac{\lambda_s y^s \cdot \nabla_y g^s(y^s, \tilde{x}^s)}{-w_1 \tilde{x}^s \cdot \nabla_{\tilde{x}} g^s(y^s, \tilde{x}^s) + w_1^s x_1^s}. \tag{43}$$

Substituting (32) into (43) we find

$$\frac{p^s \cdot y^s}{w^s \cdot x^s} = \frac{\lambda_s}{w_1}(\epsilon^s)^{-1}. \tag{44}$$

We now assume that $(y^s, x^s)$ is the solution, not only to the cost minimization (37) and revenue maximization (25) problems, but also to the profit maximization problem:

$$\max_{y,\tilde{x}}\{p^s \cdot y^s - \tilde{w}^s \cdot \tilde{x} - w_1^s g^s(y, \tilde{x})\}. \tag{45}$$

The partial derivative of (45) with respect to $y$ is

$$p^s = w_1^s \nabla_y g^s(y^s, \tilde{x}^s), \tag{46}$$

which reveals that the Lagrangian multiplier $\lambda_s$, in (26), is equal to $w_1$ if $(y^s, x^s)$ is a solution to (45). Hence, under profit maximization, returns to scale, $\epsilon^s$, can be computed from (44) as

$$\epsilon^s = (p^s \cdot y^s / w^s \cdot x^s)^{-1} \quad s = k, l. \tag{47}$$

In this case it is straightforward to use observed data on cost and revenue to compute $m$ using Theorem 3.

CASE (iii)—Increasing Returns to Scale: In this case there is no profit maximizing solution. Thus we cannot use observed costs and revenues to compute returns to scale. Thus $m$ can only be computed if there is knowledge of $\epsilon^s$. Note that the importance of the scale factor $r$ depends not only on the degree of returns to scale, $\epsilon^s$, but on the input differences between the two firms. If the firms employ very nearly the same input levels, then precise knowledge of, $\epsilon^s$, is not crucial to the measurement of their relative productivity levels.

Analogous to the output based productivity indexes $m^k(x^l, x^k, y^l, y^k)$ and $m^l(x^l, x^k, y^l, y^k)$, we can define input based productivity indexes. The

*Malmquist firm k input based productivity index* $M^k$ is defined as

(48) $$M^k(x^l,x^k,y^l,y^k) \equiv D^k(y^k,x^k)/D^k(y^l,x^l).$$

Under (1) $D^k(y^k,x^k)=1$ so that

(49) $$\begin{aligned} M^k(x^l,x^k,y^l,y^k) &= 1/D^k(y^l,x^l) \\ &= 1/\max_\delta\{\delta : f^k(\tilde{y}^l,x^l/\delta) \geq y_1^l\} \\ &= \min_\rho\{\rho : f^k(\tilde{y}^l,\rho x^l) \geq y_1^l\} \qquad (\text{where} \quad \rho = 1/\delta). \end{aligned}$$

Thus $M^k(x^l,x^k,y^l,y^k)$ is the minimal input inflation factor such that the inflated input for firm $l, M^k x^l$, and the firm $l$ output vector $y^l \equiv (y_1^l, \tilde{y}^l)$ lie on the production surface of firm $k$. Note that $M^k(x^k,x^l,y^k,y^k)=1$; therefore if firm $l$ has a higher productivity level than firm $k$, $M^k > 1$.

*The Malmquist firm l input based productivity index* $M^l$ is given by

(50) $$M^l(x^l,x^k,y^l,y^k) \equiv D^l(y^k,x^k)/D^l(y^l,x^l).$$

Under (1) $D^l(y^l,x^l)=1$, so that

(51) $$M^l(x^l,x^k,y^l,y^k) = D^l(y^k,x^k) = \max_\delta\{\delta : f^l(y^k,x^k/\delta) \geq y_1^k\}.$$

Thus $M^l$ is the maximum input deflation factor such that deflated input for firm $k$, $x^k/M^l$, and the firm $k$ output vector lie on the production surface of firm $l$. Note that $M^l(x^l,x^l,y^l,y^l)=1$; therefore if firm $l$ has a higher productivity level than firm $k$, $M^l > 1$.

The derivative of input based measures of productivity differences is analogous to the derivation of the ouput based measures: the only difference is that we begin with the production function $y_1 = f^s(\tilde{y},x)$ and its associated input distance function $D^s(y^s,x^s)$ instead of the input requirements function and its associated output distance function. We do not carry out all the required steps but simply state the relevant theorem.

THEOREM 4. *If firms k and l have translog input distance functions, with identical second order coefficients, then the product of a scale factor and the ratio of the Törnqvist index for comparing the outputs of k and l to the Törnqvist index for comparing the inputs of k and l is equal to the geometric mean of the firm k and firm l Malmquist input based productivity indexes.*

*That is, if for* $s=k,l$, (i) $D^s(y^s,x^s) = h^s(y,x)$, *i.e., the input distance function is translog*; (ii) (1) *and* (3) *hold*; (iii) $x^s$ *is a solution to* (11) *and* $y^s$ *is a solution to*

$\max_y\{p_1 f^s(y,x^s)+\tilde{p}_s\cdot\tilde{y}\}$; *and* (iv) *restrictions* (iv) *under Theorem* 3 *hold; then,*

$$\begin{aligned}
\ln M(x^l,x^k,y^l,y^k) &\equiv \tfrac{1}{2}\ln M^k(x^l,x^k,y^l,y^k)+\tfrac{1}{2}\ln M^l(x^l,x^k,y^l,y^k)\\
&= \frac{1}{2}\sum_i^I\left[\frac{p_i^k y_i^k}{p^k\cdot y^k}+\frac{p_i^l y_i^l}{p^l\cdot y^l}\right]\left[\ln y_i^l-\ln y_i^k\right]\\
&\quad -\frac{1}{2}\sum_n^N\left[\frac{w_n^k x_n^k}{w^k\cdot x^k}+\frac{w_n^l x_n^l}{w^l\cdot x^l}\right]\left[\ln x_n^l-\ln x_n^k\right]\\
&\quad +\frac{1}{2}\sum_i^I\left[\left((\epsilon^k)^{-1}-1\right)\frac{p_i^k y_i^k}{p^k\cdot y^k}\right.\\
&\qquad\left.+\left((\epsilon^l)^{-1}\right)\frac{p_i^l y_i^l}{p^l\cdot y^l}\right]\left[\ln y_i^l-\ln y_i^k\right]\\
&= \ln Q(p^l,p^k,y^l,y^k)-\ln Q(w^l,w^k,x^l,x^k)\\
&\quad +R(x^l,x^k,y^l,y^k,\epsilon^l,\epsilon^k),
\end{aligned}$$

*where the output returns to scale factor R is defined by the third and fourth lines above.*

As with the output based index $m$, we can compute the input based index, $M$, using observed data, provided we have knowledge of $\epsilon^s$ for $s=k,l$. For the case of constant returns to scale, $R$ vanishes and $M$ can be computed from the ratio of the Törnqvist output to the Törnqvist input index. For decreasing returns to scale, profit maximization implies $(\epsilon^s)^{-1}=p^s\cdot y^s/w^s\cdot x^s$; hence $M$ can be computed from observed data. For increasing returns to scale, knowledge of $\epsilon^s$ is required to compute the productivity indexes.

The output and input based productivity indexes are identical except for the scale terms $r$ and $R$. Under constant returns to scale, the terms $r$ and $R$ vanish so that the input and output based measures are identical; otherwise the two measures will in general differ. For example, consider the case in which the firms $k$ and $l$, both have increasing returns to scale and employ identical inputs, but $l$ produces more of every output. The output based productivity index $m$ will be identical to the Törnqvist output indexes (since $r=0$) and will show $m>1$. The input based index $M$ will equal the Törnqvist output index plus $R$, and $R$ is negative under increasing returns to scale. Hence $m>M>1$; the input based index will still show $l$ more productive than $k$ but by a lesser amount than the output based index. This is because increasing returns to scale implies that the percentage input change required to equalize the firms' output is less than the percentage difference in output.

## 5. EXTENSIONS

It is straightforward to adapt Theorem 1 to the consumer context for the purpose of constructing indexes of quantities consumed. In this case $F^s$ is the utility function for consumer $s$, $y_1^s$ is the level of utility, $x^s \gg 0_N$ is the consumption vector for consumer $s$, $w^s \gg 0_N$ is the vector of commodity prices that consumer $s$ faces, and $\tilde{y}^s$ is a vector of planned purchases or demographic variables for consumer $s$. The input distance function is now interpreted as a commodity distance function, and the assumption (11) that the firm minimizes the cost of producing a given output vector subject to (1) is now replaced by the assumption that the consumer minimizes the expenditure required to obtain a given level of utility subject to (1). Under these specifications, Theorem 1 can be applied directly to justify the Törnqvist index for comparing the consumption levels of consumers $k$ and $l$.

The analysis presented above in Sections 2, 3, and 4 was developed in terms of quantities of inputs and outputs. Indexes that are dual to the quantity indexes can be derived by substituting prices of inputs and outputs for the quantities. This leads to input and output price indexes and a productivity index specified in terms of the relationship between input and output prices. The dual to the consumption index, discussed in the preceding paragraph, is the consumption price index or cost of living index.

To illustrate the use of our approach in the consumer context and in the derivation of dual, or price, indexes, we derive the consumer cost of living index implied by our general framework. Let $u^s = F^s(z,x)$ denote the utility, or preference, function for consumer $s$, where $x$ is a consumption vector and $z$ is a vector of demographic variables or public goods. We assume that $x^s$ is a solution to the following expenditure minimization problem:

$$\text{(53)} \qquad \min_x\{w^s \cdot x : F^s(z^s,x) \geq u^s\} \equiv C^s(u^s,z^s,w^s),$$

where $w^s \gg 0_N$ is the vector of prices of the commodities in $x$. Equation (53) defines the expenditure function of consumer $s$, $C^s$. Shephard [19] showed that, if $C^s(u^s,z^s,w^s)$ is differentiable with respect to the components of $w^s$, then the solution, $x^s$, to (53) is given by

$$\text{(54)} \qquad x^s = \nabla_w C^s(u^s,z^s,w^s).$$

The *consumer k Konüs* [15] *cost of living or price index* can be defined as

$$\text{(55)} \qquad K^k(w^l,w^k) \equiv C^k(u^k,z^k,w^l)/C^k(u^k,z^k,w^k)$$

$$= \min_x\{w^l \cdot x : f^k(z^k,x) \geq u^k\}/w^k \cdot x^k \qquad \text{using (53).}$$

Thus $K^k(w^l,w^k)$ is the ratio of the cost of achieving utility level $u^k$ with prices $w^l$ to the cost of achieving $u^k$ with prices $w^k$, when the demographic variables and preference function are those of consumer $k$.

Similarly, the *consumer l Konüs cost of living or price index* is

$$K^l(w^l,w^k) \equiv C^l(u^l,z^l,w^l)/C^l(u^l,z^l,w^k) = w^l\cdot x^l/\min_x\{w^k\cdot x : f^l(z^l,x)\geq u^l\}. \tag{56}$$

Without knowledge of the coefficients of the $C$ functions we cannot compute either $K^k(w^l,w^k)$ or $K^l(w^l,w^k)$; however, under the assumptions of Theorem 5 below, we can identify the geometric mean of $K^k(w^l,w^k)$ and $K^l(w^l,w^k)$.

THEOREM 5: *If the cost functions for k and l are translog with identical coefficients on the second order terms in commodity prices, then the Törnqvist index for comparing the cost of living for k and l is equal to the geometric mean of the consumer k and consumer l Konüs cost of living indexes. That is, if, for* $s = k, l$, (i) *the cost functions* $C^s$ *are translog: i.e.,* $C^s(u,z,w) = h^s(y,x)$ *for* $y = (u,z)$ *and* $w = x$; (ii) $x^s$ *is given by* (53);

$$\text{(iii)}\qquad \alpha^k_{nm} = \alpha^l_{mn} \qquad \textit{for}\quad m,n = 1,2,\ldots,N, \tag{57}$$

*then,*

$$\ln K(w^l,w^k) \equiv \frac{1}{2}\ln K^k(w^l,w^k) + \frac{1}{2}\ln K^l(w^l,w^k)$$
$$= \frac{1}{2}\sum_{n=1}^{N}\left[\frac{w_n^k x_n^k}{w^k\cdot x^k} + \frac{w_n^l x_n^l}{w^l\cdot x^l}\right]\left[\ln w_n^l - \ln w_n^k\right]$$
$$\equiv \ln P(w^l,w^k,x^l,x^k)$$

*where* $P(w^l,w^k,x^l,x^k)$ *is the Törnqvist cost of living index.*

PROOF:

$$\frac{1}{2}\ln K^k(w^l,w^k) + \frac{1}{2}\ln K^l(w^l,w^k)$$
$$= \frac{1}{2}\left[\ln C^k(u^k,z^k,w^l) - \ln C^k(u^k,z^k,w^k)\right]$$
$$+ \frac{1}{2}\left[\ln C^l(u^l,z^l,w^l) - \ln C^l(u^l,z^l,w^k)\right]$$

(using definitions (55) and (56))

$$= \frac{1}{2}\left[\nabla_{\ln w}\ln C^k(u^k,z^k,w^k) + \nabla_{\ln w}\ln C^l(u^l,z^l,w^l)\right]\left[\ln w^l - \ln w^k\right]$$

(using (57) and applying the translog identity with $(u,z)\equiv y$ and $w \equiv x$)

$$= \frac{1}{2}\sum_{n=1}^{N}\left[\frac{w_n^k x_n^k}{w^k \cdot x^k} + \frac{w_n^l x_n^l}{w^l \cdot x^l}\right]\left[\ln w_n^l - \ln w_n^k\right] \qquad \text{(using (54))}$$

$$= \ln P(w^k, w^l, x^k, x^l). \qquad \textit{Q.E.D.}$$

Theorem 5 allows for considerable variation in tastes between the two individuals, and it is more powerful than Theorem 1 in that Theorem 5 does not require positive levels of consumption for all commodities; however, Theorem 5 does share some limitations with Theorem 1: (i) expenditure minimizing behavior must be assumed, and (ii) we can identify only a geometric average of the Konüs indexes rather than each individual index.

The above approach to the problem of constructing consumer price indexes from observable price and quantity data can readily be adapted to the problem of constructing producer price indexes.

## 6. CONCLUDING REMARKS

Diewert [**8**] recommended the use of superlative index numbers, i.e. those which are exact for particular flexible functional forms. For example, he showed that the Törnqvist input index is exact for the homogeneous translog form, and thus is a superlative index. We have shown that the Törnqvist input index is also exact for the geometric mean of the two Malmquist input indexes when the two underlying functions are both translog (not necessarily homogeneous) but with different parameters. This result implies that the Törnqvist index is superlative in a considerably more general sense than shown by Diewert. We are not aware of other indexes that can be shown to be superlative in this more general sense. Diewert [**9**] has shown that superlative indexes will numerically approximate each other; thus any superlative index (in the sense of Diewert [**8**]) will be approximately equal to the geometric mean of two Malmquist indexes based on the translog form.

Comparisons based on econometric estimates of the structure of production have often been viewed as being more desirable than index number comparisons; this view is based on the belief that index numbers are consistent only with restricted structures of production. Our results show that this belief is erroneous; in fact, the structures of production which we have considered in this paper are so general that they would be difficult to estimate econometrically. Of course, econometric estimation is still required to obtain knowledge of the parameters of the structure of production and the implied substitution and scale properties. Estimation also permits the individual Malmquist indexes to be identified, in addition to their mean. But estimation is not required simply to obtain comparisons based on a general structure of production.

Any data set in which econometric estimation is feasible will have multiple observations, and thus, the issue of multilateral vs. bilateral comparisons will arise. In our previous paper, Caves, Christensen, and Diewert [**3**], we discussed procedures for making multilateral comparisons of input, output, and productivity for the case of constant return to scale structures of production. The index numbers that we have proposed in this paper, for arbitrary scale economies, can be extended to multilateral comparisons following the approach recommended in our previous paper.

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## APPENDIX

TECHNICAL LEMMA—A TRANSLOG IDENTITY: *Let* $y \equiv (y_1, y_2, \ldots, y_I)$ *and* $x \equiv (x_1, x_2, \ldots, x_N)$ *be vectors of positive variables. Let the translog functions* $h^s(y,x)$ *be defined by equation* (10) *in the text. The translog functions can provide second order approximations to arbitrary twice continuously differentiable functions. In addition it is straightforward to make the functions linearly homogeneous in the x or y variables;* $h^s$ *will be linearly homogeneous in x if and only if*

$$\sum_{n=1}^{N} \beta_n^s = 1, \sum_{m=1}^{N} \beta_{nm}^s = 0 \quad \text{for} \quad n = 1, \ldots, N \quad \text{and} \quad \sum_{i=1}^{I} \gamma_{in}^s = 0 \quad \text{for} \quad n = 1, \ldots, N \tag{1A}$$

*and linearly homogeneous in y if and only if*

$$\sum_{i=1}^{I} \alpha_i^s = 1, \sum_{j=1}^{I} \alpha_{ij}^s = 0 \quad \text{for} \quad i = 1, \ldots, I \quad \text{and} \quad \sum_{n=1}^{N} \gamma_{in}^s = 0 \quad \text{for} \quad i = 1, \ldots, I. \tag{2A}$$

LEMMA—A Translog Identity: *Let the translog functional forms* $h^k$ *and* $h^l$ *be defined by* (10) *for* $s = k, l$. *If* $\beta_{nm}^k = \beta_{nm}^l$ *for* $n, m = 1, \ldots, N$, *then the following identity holds for all* $x^k \gg 0_N$, $x^l \gg 0_N$, $y^k \gg 0_I$, *and* $y^l \gg 0_I$:

$$\begin{aligned} &\ln\{h^k(y^k,x^l)/h^k(y^k,x^k)\} + \ln\{h^l(y^l,x^l)/h^l(y^l,x^k)\} \\ &= \sum_{n=1}^{N} \left\{ x_n^k \frac{\partial \ln h^k}{\partial x_n}(y^k,x^k) + x_n^l \frac{\partial \ln h^l}{\partial x_n}(y^l,x^l) \right\} \{\ln x_n^l - \ln x_n^k\} \\ &\equiv \{\nabla_{\ln x} \ln h^k(y^k,x^k) + \nabla_{\ln x} \ln h^l(y^l,x^l)\} \cdot \{\ln x^l - \ln x^k\}. \end{aligned} \tag{3A}$$

PROOF: Since the functional forms $h^k$ and $h^l$ are quadratic in the logarithms of their variables, we can apply Diewert's [**8**, p. 118] Quadratic Identity[12] to obtain the following equalities:

$$\begin{aligned} &\ln h^k(y^k,x^l) - \ln h^k(y^k,x^k) \\ &= \tfrac{1}{2}[\nabla_{\ln x} \ln h^k(y^k,x^l) + \nabla_{\ln x} \ln h^k(y^k,x^k)] \cdot [\ln x^l - \ln x^k], \quad \text{and} \end{aligned} \tag{4A}$$

$$\begin{aligned} &\ln h^l(y^l,x^l) - \ln h^l(y^l,x^k) \\ &= \tfrac{1}{2}[\nabla_{\ln x} \ln h^l(y^l,x^l) + \nabla_{\ln x} \ln h^l(y^l,x^k)] \cdot [\ln x^l - \ln x^k]. \end{aligned} \tag{5A}$$

[12] Diewert's result is a global version of Kloek [**14**] and Theil's [**21**, pp. 222–223] local result. See also Lau [**16**].

Substitute (4A) and (5A) into (3A) and rearrange terms to obtain

$$\text{(6A)} \qquad [\nabla_{\ln x}\ln h^k(y^k,x^k)+\nabla_{\ln x}\ln h^l(y^l,x^l)]\cdot[\ln x^l-\ln x^k]$$

$$+\frac{1}{2}[\nabla_{\ln x}\ln h^k(y^k,x^l)-\nabla_{\ln x}\ln h^k(y^k,x^k)$$

$$+\nabla_{\ln x}\ln h^l(y^l,x^k)-\nabla_{\ln x}\ln h^l(y^l,x^l)]\cdot[\ln x^l-\ln x^k]$$

$$=[\nabla_{\ln x}\ln h^k(y^k,x^k)+\nabla_{\ln x}\ln h^l(y^l,x^l)]\cdot[\ln x^l-\ln x^k]$$

since the second and third lines of (6A) equal

$$\frac{1}{2}\sum_{n=1}^{N}\left\{\sum_{m=1}^{N}\beta_{nm}^k(\ln x_m^l-\ln x_m^k)-\sum_{m=1}^{N}\beta_{nm}^l(\ln x_m^l-\ln x_m^k)\right\}\{\ln x_n^l-\ln x_n^k\}$$

$$=0 \quad \text{if} \quad \beta_{nm}^k=\beta_{nm}^l \quad \text{for all } n,m.$$

*Q.E.D.*

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