GENERALIZED CONCAVITY AND ECONOMICS

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The paper indicates where the various kinds of generalized concave functions are used in economics. The producer's cost minimization, profit maximization, and the consumer's utility maximization problems are discussed along with some applications to various areas of economics. The role of generalized concave functions is discussed in what is known in the economics literature as comparative statics analysis and in the applied mathematics literature as perturbation or stability theory.

1. OVERVIEW

The purpose of this paper is to indicate where generalized concave functions are used in economics. In section 2, we discuss the producer's cost minimization problem where the producer is subject to a production function or technological constraint and we indicate how the various types of concavity play a role in this theory. In Section 3, we undertake a similar analysis for the consumer's utility maximization problem. In both sections, we indicate applications of the basic theory to various areas of economics. In Section 4, we study briefly the producer's profit maximization and comparative statics analysis, one of the fundamental tools of economic theory pioneered by Hicks (1946) and Samuelson (1947). The latter topic is known as perturbation theory in the applied mathematics literature. Section 5 offers a few concluding remarks.

The reader may find it useful to consult the earlier paper in this volume by Avriel, Diewert, Schaible and Ziemba (1981) for definitions and alternative terminology for the various types of generalized concave functions discussed in this paper.

†The author thanks W. Eichhorn, Y. Kannai and W.T. Ziemba for helpful comments, the Social Sciences and Humanities Research Council of Canada for financial support, and the National Bureau of Economic Research at Stanford, California for providing office space.

2. THE COST OR EXPENDITURE FUNCTION

2.1 The Basic Problem

One of the fundamental paradigms in economics has a producer competitively minimizing costs subject to his technological constraints. *Competitive* means that the producer takes input prices as fixed and unchanging during the given period of time irrespective of the producer's demand for those inputs. This is not always a reasonable assumption, particularly if the producer is large relative to the size of the market, but it is often a reasonable first approximation.

Assume that only one output is produced using N inputs and that the producer's technology can be summarized by a production function $F: y = F(x)$, where $y \geq 0$ is the maximal amount of output that can be produced during a period given the nonnegative vector of inputs $x^T \equiv (x_1, x_2, \ldots, x_N) \geq 0_N$; T denotes transposition and all vectors are columns. We further assume that the cost of purchasing one unit of input i is $p_i > 0$, $i=1,2,\ldots,N$ and that the positive vector of input prices that the producer faces is $p \equiv (p_1, p_2, \ldots, p_N)^T >> 0_N$. For $y \geq 0$, $p >> 0_N$, the *producer's cost function* C is defined as the solution to the constrained minimization problem:

$$C(y,p) \equiv \min_x \{p^T x: F(x) \geq y\} \qquad (1)$$

where $p^T x \equiv \sum_{i=1}^{N} p_i x_i$.

Of course, the minimum in (1) may not exist. However, if we impose the following very weak regularity condition on the production function F, it can be shown (e.g., see McFadden (1978)) that C will be well defined as a minimum (at least for $p >> 0_N$):

Assumption 1 on F: F is a real valued nonnegative function defined for all nonnegative input vectors $x \geq 0_N$ and F is *continuous from above;* i.e., for every $y \in$ Range F, the upper level set $U(y) \equiv \{x: F(x) \geq y\}$ is closed.

Assumption 1 above is very weak from an empirical point of view, since it cannot be contradicted by a finite set of data on the inputs and outputs of a producer. Define $\bar{y} \equiv \sup_x \{F(x): x \geq 0_N\}$. If the supremum is attained, define $Y \equiv \{y: 0 \leq y \leq \bar{y}\}$; if the sup is not attained, define $Y \equiv \{y: 0 \leq y < \bar{y}\}$.

THEOREM 1. (Diewert, 1978b): If F satisfies Assumption 1, then C *is well defined by (1) for* $y \in Y$ *and* $p >> 0_N$. *Moreover,* C *has the following seven properties:*

(2) $y \in Y,\ p >> 0_N \rightarrow C(y,p) \geq 0$ *(nonnegativity)*

(3) $y \in Y,\ p >> 0_N,\ \lambda > 0 \rightarrow C(y, \lambda p) = \lambda C(y,p)$ *(linear homogeneity in prices for fixed output)*

(4) $y \in Y,\ 0_N << p^0 < p^1$ (i.e., $p^0 \leq p^1$ but $p^0 \neq p^1$) $\rightarrow C(y, p^0) \leq C(y, p^1)$ *(nondecreasing in prices for fixed output)*

(5) $y \in Y,\ p^1 >> 0_N,\ p^2 >> 0_N,\ 0 \leq \lambda \leq 1 \rightarrow C(y, \lambda p^1 + (1-\lambda)p^2) \geq \lambda C(y, p^1) + (1-\lambda)C(y, p^2)$ *(concavity in prices for fixed output)*

(6) $y \in Y \rightarrow C(y, p)$ is continuous in p for $p >> 0_N$ *(continuity in prices for fixed output)*

(7) $p >> 0_N,\ y^0 \in Y,\ y^1 \in Y,\ y^0 < y^1 \rightarrow C(y^0, p) \leq C(y^1, p)$ *(nondecreasing in output for fixed prices)*, and

(8) $p >> 0_N,\ \alpha \in R^1 \rightarrow \{y: C(y, p) \leq \alpha\}$ is a closed set *(continuity from below in output for fixed prices)*.

From an economic point of view, properties (2), (3), (4), and (7) are intuitively obvious. The most puzzling property to an economist is the concavity property (5). In order to explain why this property holds, we sketch a proof of it due to McKenzie (1956-7; 185).

Let $y \in Y$, $p^1 >> 0_N$, $p^2 >> 0_N$ and $0 \leqslant \lambda \leqslant 1$. Then

$$\begin{aligned} C(y, \lambda p^1 + (1-\lambda)p^2) &\equiv \min_x\{(\lambda p^1 + (1-\lambda)p^2)^T x: F(x) \geqslant y\} \\ &= (\lambda p^1 + (1-\lambda)p^2)^T x^* \text{ for some } x^* \text{ such that} \\ &\quad F(x^*) \geqslant y \\ &= \lambda(p^{1T}x^*) + (1-\lambda)(p^{2T}x^*) \\ &\geqslant \lambda C(y; p^1) + (1-\lambda)C(y; p^2) \end{aligned}$$

since x^* is feasible for the p^1 and p^2 cost minimization problems, but it is not necessarily optimal; i.e., $p^{iT}x^* \geqslant C(y, p^i)$ for $i = 1, 2$.

To a mathematician, the concavity property may not be so puzzling since $C(y, p)$ is the *support function* for the upper level set $U(y) \equiv \{x: F(x) \geqslant y\}$, the set of input vectors that can produce at least the output level y.

Properties (6) and (8) are somewhat technical in nature. Property (6) follows from the fact that a concave function is continuous over the interior of its domain of definition (see Fenchel (1953; 75) or Rockafellar 1970; 82)), while (8) follows from Berge's (1963; 111-112) Upper Semicontinuous Maximum Theorem.

Economists have been studying the cost function for a long time. Under somewhat stronger assumptions on the production function F, Shephard (1953; 14-15) obtained properties (2), (3), (4) and (5), Uzawa (1964; 217) deduced (6) and (7), and Shephard (1970; 83) obtained (8).

Theorem 1 has some important empirical implications. For example, economists often have data on cost, output and input prices for a firm during period t, C^t, y^t and p^t, respectively. A functional form for the firm's cost function, C* say, is assumed and the unknown parameters which characterize C* are estimated by minimizing the sum of squared errors $(e^t)^2$ where the error in period t is:

$$e^t \equiv C^t - C^*(y^t, p^t), \; t = 1,2,\ldots,T. \tag{9}$$

Frequently, the functional form for C* is assumed to be linear:

$$C^*(y, p) \equiv a_0 + \sum_{i=1}^{N} a_i p_i + a_{N+1}\, y \tag{10}$$

where the a_i are unknown parameters to be estimated. However, if the firm's production function satisfies the very weak regularity condition in Assumption 1 above and if the firm is competitively minimizing costs during the T periods, then property (3) of Theorem 1 implies that the firm's true cost function must be linearly homogeneous in prices p. Thus if C* is the firm's true cost function, then we must have $a_0 = 0 = a_{N+1}$. But then C* does not depend on the output level y, which is unrealistic. If $C^*(y, p)$ is a quadratic function in y, p, we encounter similar difficulties. For examples of functional forms for cost functions that are consistent with Theorem 1 but at the same time can approximate an arbitrary (differentiable) cost function to the second order, see Diewert (1974, 1978b) and Lau (1974, 1978).

We conclude this section by noting that the production function F completely determines the cost function C defined by (1). In the following section, we indicate how the process can be reversed.

2.2 Duality Between Cost and Production Functions

Given a production function F satisfying Assumption 1, for $y \in Y$ define the upper level set U(y) by

$$U(y) \equiv \{x: F(x) \geqslant y\}. \tag{11}$$

The family of level sets (or production possibilities sets) L(y) completely determines the production function F. Also, the producer's cost function C can be defined by (1) or equivalently by

$C(y, p) \equiv \min_x\{p^T x: x \in U(y)\}$ for $y \in Y$, $p >> 0_N$.

For $y \in Y$, $p >> 0_N$, define the isocost plane for output level y and input price vector p as $\{x: p^T x = C(y, p)\}$. From the definitions of

$C(y, p)$ and $L(y)$, it follows that the set $L(y)$ lies *above* this isocost plane and is tangent to it; i.e., $L(y) \subset \{x: p^T x \geq C(y, p)\}$, where $\{x: p^T x \geq C(y, p)\}$ is a supporting halfspace for the true production possibilities set $L(y)$. Thus we may use the cost function in order to form an *outer approximation* $M(y)$ to the true set $U(y)$:

$$M(y) \equiv \bigcap_{p >> 0_N} \{x: p^T x \geq C(y, p)\}. \tag{12}$$

Since $L(y) \subset \{x: p^T x \geq C(y, p)\}$ for every $p >> 0_N$, $U(y) \subset M(y)$, where by (12) $M(y)$ is the intersection of the supporting halfspaces to the set $L(y)$. $M(y)$ is often called (e.g., McFadden (1966)) the *free disposal convex hull* of $U(y)$.

Since $M(y)$ is the intersection of a family of convex sets, $M(y)$ is also a convex set. $M(y)$ also has the following (free disposal) property: $x^0 \in M(y)$, $x^0 \leq x^1 \to x^1 \in M(y)$. Thus if we want $U(y)$ to coincide with $M(y)$ for each $y \in Y$, then $U(y)$ must be a convex set with the free disposal property for every $y \in Y$. However, if the family of sets $U(y)$ is convex for every $y \in Y$, then the production function F is *quasiconcave* (see Avriel, Diewert, Schaible and Zang). Similarly, if $U(y)$ satisfies the free disposal property for every $y \in Y$, then F must be a *nondecreasing* function.

Assumption 2 on F: F is a quasiconcave function.

Assumption 3 on F: F is nondecreasing; i.e., if $x^0 < x^1$, then $F(x^0) \leq F(x^1)$.

THEOREM 2: *If the production function* F *satisfies Assumption (1) then the cost function* C *defined by (1) satisfies properties (2) - (8). If in addition,* F *satisfies Assumptions (2) and (3) and we use the cost function* C *in order to define the function* F^* *for* $x \geq 0_N$ *as*

$$F^*(x) \equiv \max_y \{y: x \in M(y)\} \tag{13}$$

where $M(y)$ *is defined in terms of* C *by (12), then* $F = F^*$; *i.e., the production function* F *is completely characterized by its cost function* C.

This duality theorem between cost and production functions (essentially due to Shephard (1953, 1970)) has been established under a variety of regularity conditions. In the economics literature, see Samuelson (1953-4; 15), Uzawa (1964), McFadden (1966, 1978), Diewert (1971, 1974, 1978), and for local duality theorems, see Epstein (1981), Diewert (1978b) and Blackorby and Diewert (1979). In the mathematics literature, see Fenchel (1953; 122-124) and Crouzeix (1977; 222).

We have seen why *concavity* plays a central role in economic theory -- from Theorem 1, the cost function $C(y, p)$ is concave in p, no matter what the functional form for the producer's production function F is (provided that F is continuous from above, an empirically harmless assumption). *Quasiconcavity* also plays a central role in economic theory: from an empirical point of view, it is harmless to assume that the producer's production function is quasiconcave (and nondecreasing), provided that the producer is competitively minimizing costs. Why is this? Assume that the true production function F satisfies only Assumption 1 and construct F's cost function C using definition (1). Then use C to construct the function F^* by definition (13). It is straightforward to show that the cost function that corresponds to F^* is also C. Thus the original production function F and the derived (from the cost function) production function F^* both generate the same cost function, and so we might as well assume that F^* is the true production function. However, F^* is quasiconcave (and nondecreasing) even though the original F need not be.

To further explain the last point, we need an additional definition. For $y \in Y$, $p >> 0_N$, let $\gamma(y, p)$ denote the solution set of x's to the cost minimization problem (1). Thus if $x \in \gamma(y, p)$, then

$p^T x = C(y, p)$. Similarly, let $\gamma^*(y, p)$ denote the solution set to the cost minimization problem: $\min_x\{p^T x: F^*(x) \geq y\}$. It can be seen that if $y \in Y$, $p >> 0_N$ and $x \in \gamma(y, p)$, then $x \in \gamma^*(y, p)$ also. However, if $x \in \gamma^*(y, p)$, then it is not necessarily true that $x \in \gamma(y, p)$, unless the original production function F was quasiconcave. Thus the net effect of assuming that F* is the true production function is to enlarge (incorrectly) the solution set to the cost minimization problem if the true F is not quasiconcave. However, this possibly enlarged solution set $\gamma^*(y, p)$ will always contain the true solution set $\gamma(y, p)$. Thus we can always assume that any observable data set on costs, output and inputs has been generated by the quasiconcave F* rather than the true F.

2.3 Some Applications of Cost Functions

THEOREM 3. (Hicks (1946; 331), Samuelson (1947; 68), Shephard (1953: 11), Fenchel (1953; 104), McKenzie (1956-7), and Karlin (1959; 272): Suppose that the production function F *satisfies Assumption 1 and the cost function* C *is defined by (1). Let* $y^* \in Y$, $p^* >> 0_N$ *and suppose that* x^* *is a solution to (1), i.e.,* $x^* \in \gamma(y^*, p^*)$. *Finally, suppose that the partial derivaties of* C *with respect to input prices at* y^*, p^* *exist. Then*

$$x^* = \nabla_p C(y^*, p^*) \qquad (14)$$

and thus the solution to (1) is unique when $y = y^*$, $p = p^*$.

PROOF (Karlin (1959)): For every $p >> 0_N$, since x^* is feasible for the cost minimization problem $\min_x\{p^T x: F(x) \geq y^*\}$,

$$p^T x^* \geq C(y^*, p). \qquad (15)$$

For $p >> 0_N$, define $g(p) \equiv C(y^*, p) - p^T x^*$. From (15), $g(p) \leq 0$ for all $p >> 0_N$ but since $p^{*T} x^* = C(y^*, p^*)$, $g(p^*) = 0$. Thus $g(p)$ attains a (global) maximum at $p = p^*$ and the following first order necessary conditions must be satisfied:

$$\nabla g(p^*) = \nabla_p C(y^*, p^*) - x^* = 0_N. \qquad (16)$$

Q.E.D.

Thus if $C(y^*, p^*)$ is differentiable with respect to prices, then the vector of first order partial derivatives with respect to input prices evaluated at (y^*, p^*), $\nabla_p C(y^*, p^*)$, is the unique solution to the cost minimization problem. This property is known in the economics literature (see McFadden (1978)) as *the derivative property for the cost function* or as *Shephard's Lemma*. Actually, Shephard proved a version of the following more difficult result:

THEOREM 4 (Shephard (1953; 11)): Suppose a cost function C *satisfies properties (2) - (8). If in addition the partial derivatives of* C *with respect to input prices exist at* $y^* \in Y$, $p^* >> 0_N$, *then* $x^* \equiv \nabla_p C(y^*, p^*)$ *is the unique solution to the cost minimization problem* $\min_x\{p^{*T} x: F^*(x) \geq y^*\}$ *where* F* *is defined in terms of the given cost function* C *by (12) and (13).*

In Theorem 3, the production function F is given, while in Theorem 4, the cost function C is given, and the underlying production function F* is defined in terms of the cost function.

One common application of Theorem 4 (see Diewert (1971, 1974) or Lau (1974, 1978)) is as an easy method of generating a theoretically valid system of cost minimizing input demand functions $x(y, p)$: simply postulate a functional form for the cost function C that is consistent with properties (2) - (8) and in addition is differentiable with respect to input prices and then estimate the unknown parameters occurring in the cost function by minimizing a function of the errors e^t in the equation system

$$x^t = \nabla_p C(y^t, p^t) + e^t; \quad t=1,2,\ldots,T \qquad (17)$$

where x^t, y^t and p^t are observed data on inputs, output and input prices, respectively, during period t. The alternative to the above method is to assume a functional form for the production function F and then solve the cost minimization problem (1) directly for the system of cost minimizing demand function's $x(y, p)$, assuming that the solution set is unique. The problem with this more direct method is that the unknown parameters which characterize the production function F usually occur in the demand system $x(y, p)$ in a highly nonlinear manner. On the other hand, using Theorem 4 in order to generate the system of demand functions $x(y, p)$, we can choose our functional form for C in such a way that the system of equation (17) is linear in the unknown parameters, and thus linear regression techniques can be applied to the estimation problem.

A second equally important application of Theorem 4 can be given. Suppose that C satisfies properties (2) - (8) and is twice continuously differentiable with respect to input prices at the point $y^* \in Y$, $p^* >> 0_N$. Then by Theorem 4, the input demand functions are given by $x(y^*, p) = \nabla_p C(y^*, p)$ for p close to p^*. Differentiate each demand function $x_i(y^*, p^*) = \partial C(y^*, p^*)/\partial p_i$ with respect to input prices and denote the resulting matrix of partial derivatives as $\nabla_p x(y^*, p^*)$. From Theorem 4,

$$\nabla_p x(y^*, p^*) = \nabla^2_{pp} C(y^*, p^*) \tag{18}$$

where $\nabla^2_{pp} C(y^*, p^*) \equiv [\partial^2 C(y^*, p^*)/\partial p_i \partial p_j]$ denotes the matrix of second order partial derivatives of C with respect to the components of p. By property (5), $C(y^*, p)$ is concave in p. By Proposition 3 in Avriel, Diewert, Schaible and Ziemba (1981), $\nabla^2_{pp} C(y^*, p^*)$ is a negative semi-definite matrix, so that in particular, we must have

$$\partial x_i(y^*, p^*)/\partial p_i = \partial^2 C(y^*, p^*)/\partial p_i^2 \leq 0, \quad i=1,2,\ldots,N. \tag{19}$$

The inequalities in (19) admit a very simple economic interpretation: if the price of input i increases, then the cost minimizing demand for input i needed to produce a fixed output level y^* will not increase.

If $N = 2$, so that there are only two inputs, then we may deduce that $\partial x_1(y^*, p^*)/\partial p_2 = \partial^2 C(y^*, p^*)/\partial p_1 \partial p_2 = \partial^2 C(y^*, p^*)/\partial p_2 \partial p_1$ (using the twice continuously differentiability assumption) $= \partial x_2(y^*, p^*)/\partial p_1 = -(p_2^*/p_1^*)\partial x_2(y^*, p^*)/\partial p_2 \geq 0$ since $p_1^* \partial x_2(y^*, p^*)/\partial p_1 + p_2^* \partial x_2(y^*, p^*)/\partial p_2 = 0$. The last equality follows from Euler's Theorem on homogeneous functions, since $C(y^*, p)$ is linearly homogeneous in p, $\partial C(y^*, p)/\partial p_2$ is homogeneous of degree zero in $(p_1, p_2) \equiv p^T$.

The above results were obtained (under somewhat stronger hypotheses) by Hicks (1946; 311 and 331) and Samuelson (1947; 69). The present derivation is taken from McFadden (1966, 1978) and Diewert (1978b). These results illustrate a second major application of the duality between cost and production functions: duality theory usually enables us to derive theoretical theorems about the solutions to various economic optimization problems in a comparatively effortless manner.

One question that we have not yet resolved is: under what conditions on the production function F will the cost function C be differentiable with respect to input prices? It is fairly easy to show that a sufficient condition for C to be once differentiable with respect to input prices is that F be *strictly quasiconcave*. However, the following example shows that strict quasiconcavity of F is not necessary. Let $a_1 > 0$, $a_2 > 0$, $N = 2$ and define $F(x_1, x_2) \equiv \min\{x_1/a_1, x_2/a_2\}$. This production function is known in the economics literature as a Leontief or fixed coefficient production function. It is not strictly quasiconcave nor is it directionally differentiable since the two sided directional derivatives of F, $D_v F(x) \equiv \lim_{t \to 0}[F(x + tv) - F(x)]/t$, do not exist for all directions v. However, the cost function which corresponds to F is $C(y, p_1, p_2) \equiv (a_1 p_1 + a_2 p_2)y$ which has partial derivatives of all orders. Thus simple necessary and sufficient conditions on F for the differentiability of C with respect to input prices do not seem to be known at present.

Another interesting question is: under what conditions on F will C be twice differentiable with respect to prices with a negative definite Hessian matrix $\nabla^2_{pp}C(y^*, p^*)$ (which would imply that the inequalities in (19) are strict)? Unfortunately, $\nabla^2_{pp}C(y^*, p^*)$ cannot be negative definite for any F, since property (3) on C (linear homogeneity in prices) implies, using Euler's Theorem on homogeneous functions, that

$$\nabla^2_{pp}C(y^*, p^*)p^* = 0_N \tag{20}$$

so that p^* is an eigenvector of $\nabla^2_{pp}C(y^*, p^*)$ that corresponds to a zero eigenvalue. However, we can ask under what conditions will $\nabla^2_{pp}C(y^*, p^*)$ be negative definite in the subspace orthogonal to $p^* >> 0_N$, and what does the last condition on C imply about F?

Suppose:

C satisfies (2) - (8), $y^* \in$ Interior Y, $p^* >> 0_N$, C is twice continuously differentiable in a neighborhood around y^*, p^* with $x^* \equiv \nabla_p C(y^*, p^*) >> 0_N$ and $\nabla^2_{pp}C(y^*, p^*)$ satisifes the following property: (21)

$$z^T p^* = 0,\ z \neq 0_N \rightarrow z^T\nabla^2_{pp}C(y^*, p^*)z < 0. \tag{22}$$

First we show that (21) and (22) imply that the inequalities in (19) hold strictly provided that $N \geqslant 2$. By (5), $\nabla^2_{pp}C(y^*, p^*)$ is negative semidefinite while (3) implies (20); i.e., that p^* is an eigenvalue of $\nabla^2_{pp}C(y^*, p^*)$ that corresponds to a zero eigenvalue. (22) implies that the other N-1 eigenvalues of $\nabla^2_{pp}C(y^*, p^*)$ are negative. Thus $\nabla^2_{pp}C(y^*, p^*)$ satisifies:

$$k \in R^1,\ z \neq kp^* \rightarrow z^T\nabla^2_{pp}C(y^*, p^*)z < 0. \tag{23}$$

From (19), $\partial^2C(y^*, p^*)/\partial p_i^2 \leqslant 0$. Suppose $\partial^2C(y^*, p^*)/\partial p_i^2 = 0$. Then $e_i^T\nabla^2_{pp}C(y^*, p^*)e_i = \partial^2C(y^*, p^*)/\partial p_i^2 = 0$ where e_i is the i^{th} unit vector. Since $p^* >> 0_N$, $e_i \neq kp^*$ for any scalar k, and we have contradicted (23). Thus our supposition is false and

$$\partial x_i(y^*, p^*)/\partial p_i = \partial^2C(y^*, p^*)/\partial p_i^2 < 0 \quad \text{for } i=1,2,\ldots,N. \tag{24}$$

Thus assumptions (21) and (22) on C are sufficient to imply that the i^{th} input demand function $x_i(y, p)$ has a negative slope with respect to the i^{th} price p_i for all y, p in a neighborhood of y^*, p^*.

The meaning of (22) will become clearer if we note that (21) and (22) are equivalent to (21) and (25):

$$z^T\nabla_p C(y^*, p^*) = 0,\ z \neq 0_N \rightarrow z^T\nabla^2_{pp}C(y^*, p^*)z < 0. \tag{25}$$

Suppose (21) and (22) hold. Then (23) also holds. Let $z^T\nabla_p C(y^*, p^*) = 0$, $z \neq 0_N$ where $\nabla_p C(y^*, p^*) \equiv x^* >> 0_N$ by (21). If $z = kp^*$ for some $k \neq 0$, then $z^T\nabla_p C(y^*, p^*) < 0$ if $k < 0$ or $z^T\nabla_p C(y^*, p^*) > 0$ if $k > 0$ since $p^* >> 0_N$, which contradicts $z^T\nabla_p C(y^*, p^*) = 0$. Thus $z \neq kp^*$ for any scalar k and (23) implies $z^T\nabla^2_{pp}C(y^*, p^*)z < 0$. Thus (25) holds. The proof that (21) and (25) imply (22) is similar.

From Proposition 17 in Avriel, Diewert, Schaible and Ziemba (1981) (21) and (25) imply that C(y, p) is *strongly pseudoconcave* (quasiconcave) with respect to p for y, p close to y^*, p^*. We now ask: what does local strong pseudoconcavity of the cost function C imply about the corresponding dual production function F^* defined by (12) and (13)? Using the material in Blackorby and Diewert (1979; 589-591), it can be shown that if C satisfies (21) and (25), then F^* satisfies Assumptions 1 - 3 and the following additional conditions (where $x^* \equiv \nabla_p C(y^*, p^*) >> 0_N$):

F^* is twice continuously differentiable in a neighborhood around x^* with $\nabla F^*(x^*) >> 0_N$; and (26)

$$z^T \nabla F^*(x^*) = 0,\ z \neq 0_N \rightarrow z^T \nabla^2 F^*(x^*) z < 0. \tag{27}$$

Using Proposition 17 in Avriel, Diewert, Schaible and Ziemba (1981), it can be seen that the differentiability properties of F^* and (27) imply that F^* is *strongly pseudoconcave* (quasiconcave) in a neighborhood of x^*.

It can also be shown that if a production function F satisfies Assumptions 1 - 3, (26) and (27), then the corresponding cost function C defined by (1) satisfies (2) - (8), (21) and (25). Thus local strong pseudoconcavity (quasiconcavity) of the production function is equivalent to local strong pseudoconcavity (quasiconcavity) of the cost function in the twice continuously differentiable case.

We have shown how concavity, quasiconcavity, strict quasiconcavity and strong quasiconcavity arise in the context of the producer's cost minimization problem. We conclude this section by indicating a few additional economic applications of cost functions.

First, the producer's cost minimization problem can be given an interpretation in the context of consumer theory: interpret F as the consumer's *utility function* (if $F(x^0) < F(x^1)$, then the consumer prefers the commodity vector x^1 to x^0), interpret $p >> 0_N$ as a vector of commodity prices and $y \in Y$ as a utility level rather than as an output level. Then the cost minimization problem (1), $\min_x\{p^T x: F(x) \geqslant y\}$ can be interpreted as the problem of minimizing the cost or expenditure needed to achieve a certain utility level $y \in Y$ given that the consumer faces the vector of commodity prices $p >> 0_N$. In the economics literature, the resulting cost function C(y, p) is often called an *expenditure function* (e.g., Blackorby and Diewert (1979)).

Two applications of expenditure functions can readily be given. The expenditure function plays a central role in the economic theory of the cost of living index. Let the consumer face the commodity price vectors $p^0 >> 0_N$ and $p^1 >> 0_N$ during periods 0 and 1, let F be the consumer's utility function and C be the corresponding expenditure function defined by (1). Then the Könus (1939) *cost of living index* P corresponding to $x > 0_N$ (the reference vector of quantities), is defined as

$$P(p^0, p^1, x) \equiv C[F(x), p^1]/C[F(x), p^0]. \tag{28}$$

P is interpreted as follows: pick a reference indifference surface indexed by the quantity vector $x > 0_N$. Then $P(p^0, p^1, x)$ is the minimum cost of achieving the standard of living indexed by x when the consumer faces commodity prices p^1 relative to the minimum cost of achieving the same standard of living when the consumer faces period 0 prices p^0. Thus P can be interpreted as a level of prices in period 1 relative to a level of prices in period 0. The mathematical properties of P are completely determined by the mathematical properties of F and C.

The expenditure function also plays a central role in the *cost benefit analysis* which makes use of the *consumer surplus* concept. Let F, C, p^0 and p^1 be defined as in the previous paragraph and let x^0 and x^1 be the consumer's observed consumption vectors in periods 0 and 1, respectively. Then Hicks (1941-42; 128) has used the expenditure function in order to define two closely related measures of consumer surplus or welfare change: Hick's *compensating variation in income* is defined as $C[F(x^1), p^1] - C[F(x^0), p^1]$ while Hick's *equivalent variation in income* is defined as $C[F(x^1), p^0] - C[F(x^0), p^0]$. If the sign of either of these consumer surplus measures is positive (negative), then the consumer's utility or welfare has increased (decreased) going from period 0 to period 1. The mathematical properties of these consumer surplus measures are again completely determined by the mathematical properties of F and C.

3. CONSUMER THEORY

3.1 The Basic Problem

Another fundamental paradigm in economics is where a consumer's preferences over alternative (nonnegative) consumption vectors $x \geqslant 0_N$ are represented by a continuous *utility function* F and the consumer chooses his optimal consumption vector by maximizing utility F(x) with respect to the consumption vector x, subject to a budget constraint of the form $p^Tx \leqslant m$ where $p >> 0_N$ is a vector of positive commodity prices that he faces and $m > 0$ is the amount of money that he can spend on the commodities during the period under consideration. Formally, the consumer's utility maximization problem is defined by (29) below for $p >> 0_N$:

$$G(p) \equiv \max_x\{F(x): p^Tx \leqslant 1, x \geqslant 0_N\} \qquad (29)$$

where we have replaced the consumer's original budget constraint, $\bar{p}^Tx \leqslant m$ say, by the equivalent budget constraint $p^Tx \leqslant 1$ where $p = (p_1, p_2,\ldots,p_N) \equiv (\bar{p}_1/m, \bar{p}_2/m,\ldots,\bar{p}_N/m)$ denotes a vector of *normalized prices* (the original commodity price vector divided by the consumer's positive income). In what follows, the price vector p is always interpreted as a vector of normalized commodity prices.

Note that we have not only defined the consumer's constrained utility maximization problem in the right hand side of (29), but we have also defined the function G as the maximized utility level as a function of the normalized prices that the consumer faces. The function G is known in the economics literature as the *indirect utility function*.

It is technically more convenient to assume that the utility function F is continuous rather than just being continuous from above as in Section 2.

Assumption 4 on F: F is a real valued continuous function defined for all nonnegative consumption vectors $x \geqslant 0_N$.

THEOREM 5 (Diewert (1974; 121-124)): Let F satisfy Assumption 4. Then G is well defined by (29) for $p >> 0_N$ and satisfies:

(30) G is a continuous (finite) function,

(31) G is a nonincreasing (if $0 << p^0 < p^1$, then $G(p^0) \geqslant G(p^1)$),

(32) G is a quasiconvex function ($\alpha \in R^1 \to \{p: G(p) \leqslant \alpha, p >> 0_N\}$ is a convex set), and

(33) if for $x >> 0_N$, we define $F^*(x) \equiv \min_p\{G(p): p^Tx \leqslant 1, p \geqslant 0_N\}$, then F is continuous over $x >> 0_N$ and has a continuous extension to the nonnegative orthant $x \geqslant 0_N$.*

A few technical comments on the above theorem are in order. Property (33) requires G to be defined over the nonnegative orthant. We extend G from the positive orthant to the nonnegative orthant using the *Fenchel* (1953; 78) *closure operation;* i.e., define the epigraph of G over the positive orthant as $E \equiv \{(u, p): p >> 0_N, u \geqslant G(p)\}$, let $\bar{E}$ denote the closure of E and define $\bar{G}$ over the nonnegative orthant by $\bar{G}(p) \equiv \inf_u\{u: (u, p) \in \bar{E}\}$. For $p >> 0_N$, $\bar{G}(p) = G(p)$. Although $\bar{G}$ need not be finite on the boundary of the nonnegative orthant, $\bar{G}$ will be continuous from below and thus the minimum in (33) will exist using a result in Berge (1963; 76) (we have abused notation in (33) by letting $G = \bar{G}$).

From the perspective of the role of generalized concavity in economics, the interesting point to notice about the above theorem is that the indirect utility function G is *quasiconvex* (i.e., -G is *quasiconcave*) irrespective of the properties of the direct utility function F (provided only that F is continuous).

From the viewpoint of economics, properties (30) and (33) are technical in nature while property (31) is intuitively obvious: if the prices that a consumer faces increase, then his feasible consumption set $\{x: p^Tx \leqslant 1, x \geqslant 0_N\}$ shrinks, and thus maximal utility must decrease or remain constant. However, property (32) is not intuitive and it may be useful to indicate a proof of it (see Diewert, (1974; 160)).

Let $0 \leqslant \lambda \leqslant 1$, $p^i >> 0_N$, and $G(p^i) \leqslant u$ for $i = 1, 2$. Define the sets $H^i \equiv \{x: p^{iT}x \leqslant 1, x \geqslant 0_N\}$, $i = 1, 2$ and $H^\lambda \equiv \{x: (\lambda p^1 + (1-\lambda)p^2)^T x \leqslant 1, x \geqslant 0_N\}$. Then

$$\begin{aligned} G(\lambda p^1 + (1-\lambda)p^2) &\equiv \max_x\{F(x): x \in H^\lambda\} \\ &\leqslant \max_x\{F(x): x \in H^1 \cup H^2\} \\ &\qquad \text{since } H^\lambda \subset H^1 \cup H^2 \\ &\leqslant u \end{aligned}$$

since $\max_x\{F(x): x \in H^i\} = G(p^i) \leqslant u$ for $i = 1, 2$.

Suppose that the utility function F satisfies Assumption 4. Then we may proceed as in the previous section and define the function F* using (1), (12) and (13). The resulting F* is the free disposal, quasiconcave hull of the original function F. It can be verified that F and F* generate the same indirect utility function G by definition (29) and furthermore, that the F* defined in (33) coincides with the F* defined using (1), (12) and (13). Again, it is costless from an empirical point of view to assume that F satisfies Assumptions 2 and 3 in addition to Assumption 4; i.e., given Assumption 4 on F and the assumption of utility maximizing behavior on the part of the consumer, denote the solution set to the original utility maximization problem (29) as $\beta(p)$, denote the solution set to $\max_x\{F^*(x): p^T x \leqslant 1, x \geqslant 0_N\}$ as $\beta^*(p)$ where F* is the free disposal quasiconcave hull of F. Then for every $p >> 0_N$, $\beta(p) \subset \beta^*(p)$. Thus if p and x are observed price and quantity vectors that correspond to the original utility maximization problem (29) (i.e., $x \in \beta(p)$), then x is also a solution to $\max_x\{F^*(x): p^T x \leqslant 1, x \geqslant 0_N\}$; (i.e., $x \in \beta^*(p)$ also).

Hence in both the producer's competitive cost minimization problem and the consumer's maximization problem, it is completely harmless from an empirical point of view to assume that the producer's production function or the consumer's utility function is *quasiconcave*.

There is a complete *duality* between utility functions F satisfying Assumptions 2, 3 and 4 and indirect utility functions G satisfying (30) - (33). For references to such duality theorems in the economics literature, see Diewert (1974; 132) or Blackorby and Diewert (1979). For a similar duality theorem in the mathematics literature, see Crouzeix (1977, 222; 1981).

3.2 Semistrict Quasiconcavity and Consumer Theory

Suppose F satisfies Assumption 1 and we define the consumer's utility maximization problem by (29) for every (normalized) commodity price vector $p >> 0_N$. By a theorem due to Berge (1963; 76), the indirect utility function G(p) is well defined as a maximum in (29). Suppose $G(p) < \sup_x\{F(x): x \geqslant 0_N\}$ and denote the solution set of optimal consumption vectors for (29) as $\beta(p)$. If F is constant over a neighborhood (in economics terminology, some indifference surfaces are thick), then $\beta(p)$ can contain points x such that $p^T x < 1$; i.e., the consumer's entire budget is not spent. From the viewpoint of economic applications, it is useful to be able to rule out this kind of behavior. The following theorem gives sufficient conditions for this.

THEOREM 5: Suppose F *satisfies Assumption 1 and is semistrictly quasiconcave. Then* F *satisfies the following property:*

$$p >> 0_N,\ G(p) < \sup_x\{F(x): x \geqslant 0_N\},\ x \in \beta(p) \rightarrow p^T x = 1. \qquad (34)$$

PROOF: Suppose there exists p* such that

$$p^* >> 0_N,\ G(p^*) < \sup_x\{F(x): x \geqslant 0_N\},\ x^* \in \beta(p^*) \text{ and } p^{*T}x^* < 1. \qquad (35)$$

Then there exists $x^2 \geqslant 0_N$ such that $G(p^*) = F(x^*) < F(x^2)$ with $p^{*T}x^2 > 1$. By the semistrict quasiconcavity of F (see Definition 7 in Avriel, Diewert, Schaible and Ziemba):

$$F(x^*) < F(\lambda x^* + (1-\lambda)x^2) \quad \text{for } 0 \leqslant \lambda < 1. \tag{36}$$

Since $p^{*T}x^* < 1$ and $p^{*T}x^2 > 1$, there exists $0 < \lambda^* < 1$ such that $p^{*T}(\lambda^* x^* + (1-\lambda^*)x^2) = 1$. Thus $\lambda^* x^* + (1-\lambda^*)x^2$ is feasible for the $G(p^*)$ maximization problem, $\max_x\{F(x): p^{*T}x \leqslant 1,\ x \geqslant 0_N\}$. But (36) shows that this feasible solution gives a higher level of utility than $F(x^*) = G(p^*)$, a contradiction to the definition of $G(p^*)$. Thus (35) is false and (34) follows.

Q.E.D.

Semistrict quasiconcavity of F is not a necessary condition for (34); the following function of two variables satisfies Assumptions 1 through 4 as well as property (34), but it is not semistrictly quasiconcave (consider the behavior of the function along the x_1 axis):

$$F(x_1,x_2) \equiv \begin{cases} 0 \text{ if } 0 \leqslant x_1 \leqslant 1,\ x_2 = 0 \\ t \text{ if } x_1 \geqslant 0,\ x_2 \geqslant 0,\ t > 0,\ tx_1 + (1+t)x_2 = t(1+t) \end{cases}$$

3.3 Strict Quasiconcavity and Consumer Theory

In most applications of consumer theory in economics, we require not only that property (34) hold, but also that the set of optimal consumption vectors $\beta(p)$ is a singleton. Thus we are interested in finding conditions on the direct utility function F or the indirect utility function G that will ensure that the consumer's demand correspondence, the set of maximizers $\beta(p)$ to (29), is an ordinary function, at least for a region of prices.

THEOREM 6: Suppose the direct utility function F *satisfies Assumptions 2 and 4 and the following (local) condition:*

$$p^* >> 0_N,\ G(p^*) < \sup_x\{F(x): x \geqslant 0_N\};\ x^* \in \beta(p^*) \textit{ and } F \textit{ is strictly quasiconcave over } N_\delta(x^*) \equiv \{x: (x - x^*)^T(x - x^*) < \delta^2,\ x \geqslant 0_N\}, \tag{37}$$

the open ball of radius δ *around* x^* *for some* $\delta > 0$.

Then (38) holds:

$$p^* >> 0_N,\ \beta(p) \textit{ is a continuous function in a neighborhood of normalized prices around } p^*,\ P \textit{ say, with } p^T\beta(p) = 1 \textit{ for } p \in P. \tag{38}$$

PROOF: Let p^* and x^* be defined in (37) and let x^2 be such that $G(p^*) = F(x^*) < F(x^2)$. Suppose $p^{*T}x^* < 1$. Then since F is quasiconcave, for $0 \leqslant \lambda \leqslant 1$, $F(\lambda x^* + (1-\lambda)x^2) \geqslant F(x^*)$. By the local strict quasiconcavity of F, there exists $0 < \lambda^* < 1$ such that

$$F(\lambda x^* + (1-\lambda)x^2) > F(x^*) \quad \text{for } \lambda^* \leqslant \lambda < 1. \tag{39}$$

For λ^1 sufficiently close to 1, $p^{*T}(\lambda^1 x^* + (1-\lambda^1)x^2) \leqslant 1$, which implies along with (39) that $G(p^*) \geqslant F(\lambda^1 x^* + (1-\lambda^1)x^2) > F(x^*)$, a contradiction. Thus our supposition is false, and $p^{*T}x^* = 1$.

Now suppose there exists $x^1 \neq x^*$ such that $x^1 \in \beta(p^*)$. Then the quasiconcavity and local strict quasiconcavity of F again imply a contradiction. Thus $\beta(p^*)$ is the singleton x^* with $p^{*T}x^* = 1$.

By the Debreu (1952; 889-890, 1959; 19), Berge (1963; 116) Maximum Theorem, $\beta(p)$ is an upper semicontinuous correspondence for $p >> 0_N$. Hence for p sufficiently close to p^*, $\beta(p) \cap N_\delta(x^*) \neq \phi$. Strict quasiconcavity of F in $N_\delta(x^*)$ will imply as before that $\beta(p)$ is single valued and $p^T\beta(p) = 1$ for p close to p^*. When β is single valued, upper semicontinuity reduces to continuity.

Q.E.D.

Versions of the above theorem are common in the economics literature (e.g., Debreu (1959)) except that strict quasiconcavity is generally assumed to hold globally rather than locally.

Another set of conditions that are sufficient to ensure that the consumer's demand correspondence $\beta(p)$ is single valued and continuous around $p^* >> 0_N$ can be phrased in terms of the indirect utility function

G. Sufficient conditions are that G satisfy (30) - (33) and the following condition:

G is once continuously differentiable in neighborhood around p^* with $\nabla G(p^*) \neq 0_N$. (40)

If the above conditions on G are satisfied, then it can be shown (see Roy (1947; 222) or Diewert (1974; 125)) that $\beta(p) = (p^T\nabla G(p))^{-1}\nabla G(p)$ for p close to p^*.

Neither set of sufficient conditions for the single valuedness and continuity of $\beta(p)$ implies the other. It appears to be difficult to find necessary and sufficient conditions for this problem, just as it is difficult to find necessary and sufficient conditions for the differentiability of the cost function with respect to input prices.

3.4 Strong Quasiconcavity and Consumer Theory

Suppose that we not only want the consumer's demand correspondence $\beta(p)$ to be single valued and continuous for a neighborhood of prices p, but also we want the demand functions to be once continuously differentiable.

As usual, it is difficult to find necessary and sufficient conditions on F for the above property to hold. However, the following theorem gives sufficient conditions.

THEOREM 7: Suppose the direct utility function F satisfies:

Assumptions 2,3, and 4, (41)

it is twice continuously differentiable in a neighborhood around $x^* >> 0_N$ *with* $\nabla F(x^*) >> 0$, *and* (42)

$z^Tz = 1,\ z^T\nabla F(x^*) = 0 \rightarrow z^T\nabla^2F(x^*)z < 0.$ (43)

Then the indirect utility function G defined by (29) satisifes (30) - (33). Moreover, if we define $p^* \equiv (x^{*T}\nabla F(x^*))^{-1}\nabla F(x^*)$, *then G*

satisfies

G is twice continuously differentiable in a neighborhood around $p^* >> 0_N$ *with* $\nabla G(p^*) << 0_N$, *and* (44)

$z^Tz = 1,\ z^T\nabla G(p^*) = 0 \rightarrow z^T\nabla G(p^*)z > 0.$ (45)

Moreover, the solution to the utility maximization problem (29) reduces to $\beta(p) = (p^T\nabla G(p))^{-1}\nabla G(p)$ *for p close to* p^* *and is once continuously differentiable there.*

Conversely, given an indirect utility function G satisfying (30) - (33), (44) and (45), then F^* *defined in (33) satisfies (41) - (43) if we define* $x^* = (p^{*T}\nabla G(p^*))^{-1}\nabla G(p^*)$.

The above theorem can readily be proven using the techniques in Blackorby and Diewert (1979).

Our differentiability assumptions and (43) imply that F is *strongly quasiconcave* in a neighborhood around x^* while (45) implies that G is *strongly quasiconvex* around p^*.

Thus sufficient conditions for the local continuous differentiability of a consumer's system of demand functions can be obtained by assuming that the direct utility function F is strongly quasiconcave locally or equivalently, by assuming that the indirect function G is strongly quasiconvex locally.

3.5 Pseudoconcavity and Consumer Theory

To obtain a system of demand functions consistent with utility maximizing behavior, economists frequently postulate a functional form for direct utility function F and then they attempt to solve algebraically the utility maximization problem (29) for an explicit solution. As part of this exercise, economists often assume that necessary conditions for x^0 to solve (29) are also sufficient.

When F is not necessarily differentiable, if x^0 is a solution to

(29) when normalized prices $p >> 0_N$ prevail, it can be verified that the following necessary conditions must be satisfied:

$$D_v^{+u}F(x^0) \leq 0 \quad \text{for every direction } v \in S(x^0, p) \qquad (46)$$

where $D_v^{+u}F(x^0) \equiv \limsup_{t \to 0^+} \{F(x^0 + tv) - F(x^0)\}/t$ is the upper Dini derivative in the direction v (see Diewert (1981)) and the set of feasible directions is defined as $S(x^0, p) \equiv \{v: v^Tv = 1$ and there exists at $t > 0$ such that $x^0 + tv \geq 0_N$ and $p^T(x^0 + tv) \leq 1\}$. The meaning of (46) is that for all feasible directions, F cannot increase locally.

If F is once continuously differentiable over the nonnegative orthant, (46) reduces to the following more familiar Kuhn-Tucker (1951) conditions: if x^0 solves (29), then there exists a multiplier λ^0 such that

$$\nabla F(x^0) - \lambda^0 p \leq 0_N,\ \lambda^0 \geq 0,\ x^0 \geq 0_N,\ x^{0T}[\nabla F(x^0) - \lambda^0 p] = 0, \quad (47)$$
$$p^Tx^0 - 1 \leq 0,\ \lambda^0[p^Tx^0 - 1] = 0.$$

Since the constraint set in (29) is polyhedral, constraint qualification conditions are satisfied.

From the definition of pseudoconcavity (see Diewert (1981), definition B), the *necessary* conditions (46) or (47) are also *sufficient* if F is *pseudoconcave;* i.e., if F is pseudoconcave and x^0 satisfies (46), then x^0 is a solution to the consumer's utility maximization problem (29). Similarly, if F is *strictly pseudoconcave* and x^0 satisfies (46), then x^0 is the *unique* solution to (29).

There are several other similar applications of the concept of strict pseudoconcavity to economics; e.g., see Donaldson and Eaton (1978; 1981).

Thus far, we have shown how seven different kinds of generalized concavity occur naturally in economics. In the following section, we indicate how an additional two types occur.

4. PROFIT MAXIMIZATION AND COMPARATIVE STATICS ANALYSIS

Consider the model of producer behavior presented in section 2.1 where the producer minimized costs subject to an output constraint. We now make a stronger assumption about producer behavior; namely, that he competitively maximizes profits. Denote the production function by F, and input vector by $x \geq 0_N$, an input price vector by $\bar{p} >> 0_N$ and a scalar price of output by $\bar{p}_0 > 0$. A competitive producer takes all prices as fixed and chooses x in order to maximize profits, $\bar{p}_0F(x) - \bar{p}^Tx$. Thus the producer's competitive profit maximization problem is

$$\sup_x\{F(x) - p^Tx: x \geq 0_N\} \equiv \pi(p) \qquad (48)$$

where the vector of normalized input prices is $p \equiv (\bar{p}_0)^{-1}\bar{p} \equiv (\bar{p}_1/\bar{p}_0, \bar{p}_2/\bar{p}_0, \ldots, \bar{p}_N/\bar{p}_0)$. It is obvious that maximizing profits is equivalent to solving (48), since the objective function in (48) is simply profits divided by the positive price of output $\bar{p}_0$. The maximized value in (48) is defined to be the producer's (normalized) *profit function* π (see Lau (1978) or Jorgenson and Lau (1974a, 1974b) for additional details on this model).

Mathematicians will recognize π as the (negative of the) *conjugate function* to F (see Fenchel (1953; 90) or Rockafellar (1970)).

Recall how we used the cost function C to define outer approximations to the true level of the production function F. Now we show how the profit function π can be used in order to provide another outer approximation to the true production function.

For $p \geq 0_N$, define the following halfspace in R^{N+1}: $H(p) \equiv \{(y, x): y - p^Tx \leq \pi(p)\}$. Then an outer approximation to the true production possibilities set $S \equiv \{(y, x): y \leq F(x), x \geq 0_N\}$ can be defined as $\hat{S} \equiv \bigcap_{p \geq 0_N} H(p)$. Assuming that $\pi(p) < +\alpha$ for at least one $p >> 0_N$, an outer approximation to F can be defined for $x \geq 0_N$ as

$$\hat{F}(x) \equiv \max_y\{y: (y, x) \in \hat{S}\}. \quad (49)$$

Since $\hat{S}$ is a closed convex set, the approximating production function $\hat{F}$ is a *concave function*. Moreover, for any input price vector $p^* >> 0_N$ such that there exists an $x^* \geqslant 0_N$ which attains the sup in (48), we have $\max_x\{F^*(x) - p^{*T}x: x \geqslant 0_N\} = \max_x\{F(x) - p^{*T}x: x \geqslant 0_N\} = p^{*T}x^* = \pi(p^*)$. Hence from an empirical point of view, it is harmless to assume that the producer's production function is concave, provided that the producer is competitively maximizing profits. The present situation is analogous to the result we obtained in Section 2.2 where we indicated that the production function could be assumed to be quasiconcave provided that the producer was competitively minimizing costs. We now obtain the stronger concavity result because we assume the stronger hypothesis of competitive profit maximizing behavior.

For references to duality theorems between production functions, production possibilities sets and profit functions, see McFadden (1978), Lau (1978) and Diewert (1974).

However, our main concern in this section is not to show how concavity arises naturally in the context of production theory, but to show how strong concavity arises naturally in the content of comparative statics theory.

A typical application of the Hicks (1946) Samuelson (1947) comparative statics analysis can be illustrated by using the profit maximization model (48). The analysis given below follows Lau (1978; 147-149). Given a positive vector of (normalized) input prices $p^* >> 0_N$, it is assumed that $x^* >> 0_N$ is the unique solution to the profit maximization problem $\max_x\{F(x) - p^{*T}x: x \geqslant 0_N\} \equiv \pi(p^*)$. (A sufficient condition for this is that F be globally concave and locally *strictly concave* around a point $x^* >> 0_N$ such that $\nabla F(x^*) = p^*$.) It is also assumed that F is twice continuously differentiable in a neighborhood of x^*. Since

$x^* >> 0_N$ solves the profit maximization problem, the nonnegativity constraints $x \geqslant 0_N$ are not binding and hence the following *first order necessary conditions* must be satisifed:

$$\nabla F(x^*) - p^* = 0_N. \quad (50)$$

Since F is assumed to be twice continuously differentiable, the following *second order necessary conditions* must also be satisfied:

$$z \neq 0_N \rightarrow z^T\nabla^2F(x^*)z \leqslant 0; \quad (51)$$

i.e., the matrix of second order partial derivatives of F evaluated at x^* must be negative semidefinite. Comparative statics analysis proceeds by assuming that F satisfies the following stronger conditions at x^* *(second order sufficient conditions).*

$$z \neq 0_N \rightarrow z^T\nabla^2F(x^*)z < 0; \quad (52)$$

i.e., $\nabla^2F(x^*)$ is a negative definite symmetric matrix, and hence $[\nabla^2F(x^*)]^{-1}$ exists. Note that (51) implies that F is *strongly concave* in a neighborhood of x^* (see Proposition 5 in Avriel, Diewert, Schaible and Ziemba (1981)).

The reason for assuming (51) is that we may now apply the Implicit Function Theorem to (49) to deduce the existence of a unique once continuously differentiable solution to the profit maximization problem (48), $x(p)$ say, for p close to p^*, and the matrix of partial derivatives of the functions $[x_1(p),\ldots,x_N(p)]^T \equiv x(p)$ evaluated at p^* is

$$\nabla x(p^*) \equiv [\partial x_i(p^*)/\partial p_j] = [\nabla^2F(x^*)]^{-1}. \quad (53)$$

Since $\nabla^2F(x^*)$ is a negative definite matrix, so is its inverse $[\nabla^2F(x^*)]^{-1}$. Hence the input demand functions satisfy the following restrictions (the diagonal elements of a negative definite matrix are negative):

$$\partial x_i(p^*)/\partial p_i < 0, \quad i = 1,2,\ldots,N. \quad (54)$$

Thus if the price of the i^{th} input increases, the demand for the i^{th} input decreases. Hence the assumption of local strong concavity leads to economically meaningful theorems.

In the mathematical programming literature, the terms "stability theory" or "perturbation theory" are used in place of the economist's term "comparative statics analysis".

5. CONCLUSION

We have considered three models of economic behavior in this paper (the producer's cost minimization problem, the consumer's utility maximization problem, and the producer's profit maximization problem) and have shown how nine different kinds of generalized concavity and convexity arise in the context of these models. However, we have by no means exhausted the applications of generalized concavity in economics.

In many economic problems (e.g., see Kannai (1977)), it is extremely useful to be able to represent a consumer's preferences by means of a concave utility function rather than a quasiconcave function. Thus given a quasiconcave utility function $F(x)$ defined for $x \geqslant 0_N$, we ask under what conditions does there exist an increasing function of one variable, f say, such that $f[F]$ is a concave function? Under various regularity conditions, answers have been given by Fenchel (1953), Kannai (1977; 1981), Crouzeix (1977; 1980), Schaible and Zang (1979), and Zang (1981). In a related vein, Diewert (1973) using the work of Afriat (1967) shows how any finite set of price-quantity data generated by a consumer maximizing a continuous from above utility function can be generated by maximizing a concave utility function, while Kannai (1974) shows how arbitrary quasiconcave preferences can be approximated by concave preferences.

A problem which is related to the issue of concavifiability is: given that the direct utility function is concave, what does this imply about the corresponding cost and indirect utility functions C and G? On the other hand, if G is convex, what does this imply about the corresponding F and C? Answers to these questions can be found in Fenchel (1953; 122-123), Crouzeix (1977; 110, 1980) and Diewert (1978a; 33).

Another interesting problem is: given that $F(x^1, x^2,\ldots,x^M) = \sum_{i=1}^{M} f^i(x^i)$ where $x^i \in S^i$, a convex subset of finite dimensional Euclidean space for each i and F is quasiconcave, what does this imply about the functions f^i? For answers to this question and some economic applications (particularly to consumer choice under uncertainty), see Yaari (1977; 1184), Blackorby, Davidson and Donaldson (1977) and Debreu and Koopmans (1978).

Finally, another area which is too large to be reviewed here is economic dynamics, where strong concavity plays a particulary important role (see, e.g.,Rockafellar (1976; 75)).

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