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SENSITIVITY ANALYSIS IN ECONOMICS

W. E. DIEWERT†
Department of Economics, The University of British Columbia, Vancouver, Canada, V6T 1W5

Abstract—The paper studies the local dependence on a parameter of a unique solution to a general inequality–equality constrained nonlinear programming problem. A theorem due to Fiacco is extended to deal with the case where the objective and constraint functions are defined only over the nonnegative orthant. Other results are developed which extend sensitivity (or comparative statics) theorems due to Samuelson and Silberberg for equality constrained maximization problems to general nonlinear programs.

INTRODUCTION

In economics, sensitivity analysis (or stability theory or perturbation theory) is known as *comparative statics analysis* [1, 2]. Most economics problems can be formulated as problems where an objective function is maximized subject to various constraints. The latter problem is a standard nonlinear programming problem. Thus Hicks and Samuelson studied how the objective function and the solution to a smooth nonlinear programming problem subject to equality constraints changed as the objective function or the constraint functions were perturbed.

It should be noted that the hypotheses that economists impose on the nonlinear program in order to obtain their comparative statics results are often much stronger than the hypotheses employed by applied mathematicians in order to obtain their sensitivity results. However, economists are usually interested in the responses of the *solution functions* to changes in exogenous parameters, whereas mathematicians have been more interested in the response of the *value function* (the optimized objective function) to changes in parameters. Relatively strong hypotheses are required in order to obtain the uniqueness (and differentiability) of the solutions to a family of nonlinear programming problems that depend on a parameter.

The purpose of the present paper is twofold: (i) we survey and extend some of the comparative statics results that economists have obtained; and (ii) we utilize some recent results in the applied mathematics literature in order to extend some of the standard economics comparative statics results obtained for equality constrained nonlinear programs to inequality constrained programs.

Most economics problems are most naturally formulated using inequality constraints. In particular, most economics variables are nonnegative and the relevant objective and constraint functions are simply not defined outside of the nonnegative orthant. Unfortunately, the standard mathematical programming sensitivity analysis theorems [3, 4] are not immediately applicable to nonlinear programming problems where the objective and constraint functions are defined only over the nonnegative orthant. Fortunately, it is reasonably easy to modify the standard theorems to deal with the above problem. Thus in Section 1 below, we present a suitable modification of Fiacco's [4] results.

In Section 2, we use the results of Section 1 in order to extend the traditional economics comparative statics results obtained for equality constrained programs to the inequality constrained case. We also extend several of Samuelson's [3] comparative statics results obtained for unconstrained problems to the constrained case.

†Dr. Diewert is Professor of Economics at The University of British Columbia. He holds the Ph.D from The University of California at Berkeley, the M.A. from The University of British Columbia, and the B.A. from The University of British Columbia. He is currently Associate Editor of *Journal of Econometrics*, and has been Associate Editor of *American Economic Review*.

In Sections 3 and 4, we consider what happens to our basic nonlinear programming problem when we impose additional constraints on the problem such that the original solution remains feasible. Applied mathematicians may well be puzzled why it is of economic interest to add a just binding constraint to a "solved" nonlinear program. The reason has to do with short run responses versus long run responses. When an exogenous parameter changes (e.g. the price of fuel increases), both consumers and producers will attempt to decrease their consumption of fuel. However, in the short run, they will face more constraints than in the long run. For example, in the long run, new purchases of fuel efficient capital equipment can be installed, buildings can be reinsulated, timers can be installed, etc. Thus the consumer's or producer's long run (utility or profit) maximization problem will generally contain far fewer constraints than the corresponding short run maximization problems. Thus the long run response of a producer to an increase in energy prices can be contrasted to the short run response by solving two parametric nonlinear programming problems, where the short run problem has many additional just binding constraints. Section 3 considers the case of extra inequality constraints while Section 4 considers the case of extra equality constraints.

Theorem 4 in Section 3 can also be viewed as a "new" approach to establishing the existence of one sided directional derivatives for the value function and the solution functions with respect to perturbations of the objective and constraint functions for a nonlinear programming problem of the form (1). Unless the assumption of linearly independent inequality and equality constraint gradients is made, it appears to be very difficult to establish the existence of these one sided derivatives [5–8].

In Section 5, we mention briefly some of the global sensitivity analysis results obtained by economists.

1. A MODIFICATION OF FIACCO'S THEOREMS ON SENSITIVITY ANALYSIS

The basic nonlinear programming problem we consider is:

$$\begin{aligned}\text{MAX}_x\{f(x,\alpha)\colon\ g_m(x,\alpha)\geq 0,\ m=1,\ldots,M;\ h_j(x,\alpha)=0,\ j=1,\ldots,J;\\ x\equiv(x_1,x_2,\ldots,x_N)^T\geq 0_N\}\end{aligned} \tag{1}$$

where f is the objective function, $g_1,\ldots,g_M$ are the M inequality constraint functions, $h_1,\ldots,h_J$ are the equality constraint functions, x is an N dimensional nonnegative vector of decision variables ($N > J$) and α is a scalar parameter which can vary in an open neighbourhood N^* around $\alpha = 0$. The functions f, g_m and h_j are assumed to be defined and twice continuously differentiable over $\{x\colon x\geq 0_N\}\times N^*\equiv R_N{}^+\times N^*$.

Notation

$x\geq 0_N$ means the N dimensional vector x has nonnegative components; $x\gg 0_N$ means x has positive components; $x > 0_N$ means $x\geq 0_N$ but $x\neq 0_N$; x^T denotes the transpose of the (column) vector x; $\nabla_x f(x^*,\alpha^*)\equiv[\partial f(x^*,\alpha^*)/\partial x_1,\ldots,\partial f(x^*,\alpha^*)/\partial x_N]^T$ is the gradient vector of f with respect to the components of x and $\nabla^2_{xx}f(x^*,\alpha^*)$ denotes the matrix of second order partial derivatives of f with respect to the components of x evaluated at (x^*,α^*).

If a unique solution to (1) exists, it is denoted by $x(\alpha)$.

The problem (1) is the same problem as that considered by Fiacco ([4], p. 289) (with a few minor notational changes) except that we have added the nonnegativity constraints $x\geq 0_N$. (Fiacco also allows α to be a vector rather than a scalar, but our results can readily be generalized to the case where α is a vector). It should be noted that merely adding the nonnegativity constraints $x\geq 0_N$ to Fiacco's formulation of the basic nonlinear programming problem would not create anything new: the nonnegativity constraints could be added to the inequality constraints $g_m(x,\alpha)\geq 0$. However, we have not only added the nonnegativity constraints $x\geq 0_N$, we have specified that the functions f, g_m and h_j are not well defined unless $x\geq 0_N$. Thus we cannot apply Fiacco's Theorem 2.1 to our problem since his theorem relies on the standard Implicit function Theorem (e.g. [9], p. 181) which requires the relevant functions to be defined over an *open* set.

Another reason for reworking Fiacco's results after adding the inequality constraints $x \geq 0_N$ is that the resulting computations are simpler compared to the case of general inequality constraints.

Define the Lagrangian L for problem (1) as follows:

$$L(x, u, v, w, \alpha) \equiv f(x, \alpha) + \sum_{m=1}^{M} u_m g_m(x, \alpha) + \sum_{n=1}^{N} v_n x_n + \sum_{j=1}^{J} w_j h_j(x, \alpha) \tag{2}$$

where $u \equiv (u_1, \ldots, u_M)^T$, $v \equiv (v_1, v_2, \ldots, v_N)^T$ and $w \equiv (w_1, \ldots, w_J)^T$ are vectors of Lagrange multipliers.

The following Theorem is a trivial modification of Fiacco and McCormick's [10] Second Order Local Sufficiency Theorem (see also Fiacco [4], p. 290):

THEOREM 1 (*Second Order Sufficient Conditions for a Local Isolated Maximizing Point*)

Suppose: (i) the functions defining problem (1) when $\alpha = 0$ are twice continuously differentiable with respect to x in a neighbourhood of $x^* \geq 0_N$ (intersected with $R_N{}^+$); (ii) there exist Lagrange multiplier vectors $u^* \geq 0_M$, $v^* \geq 0_N$ and $w^* \in R^J$ such that the following first order Kuhn–Tucker [11] conditions hold:

$$g_m(x^*, 0) \geq 0, m = 1, 2, \ldots, M. \tag{3}$$

$$\sum_{m=1}^{M} u_m^* g_m(x^*, 0) = 0, \; u^* \geq 0_M$$

$$v^{*T} x^* = 0, \; v^* \geq 0_N, \; x^* \geq 0_N$$

$$h_j(x^*, 0) = 0, j = 1, 2, \ldots, J, \text{ and}$$

$$\nabla_x L(x^*, u^*, v^*, w^*, 0) = \nabla_x f(x^*, 0) + \sum_{m=1}^{M} u_m^* \nabla_x g_m(x^*, 0) + v^* + \sum_{j=1}^{J} w_j^* \nabla_x h_j(x^*, 0) = 0_N;$$

(iii) for all $y \equiv (y_1, y_2, \ldots, y_N)^T \neq 0_N$ such that

$$\begin{aligned} y^T \nabla_x g_m(x^*, 0) &\geq 0 \quad \text{for all } m \text{ such that } g_m(x^*, 0) = 0, \\ y^T \nabla_x g_m(x^*, 0) &= 0 \quad \text{for all } m \text{ such that } u_m^* > 0 \\ y_n &\geq 0 \quad \text{for all } n \text{ such that } x_n^* = 0 \\ y_n &= 0 \quad \text{for all } n \text{ such that } v_n^* > 0 \\ y^T \nabla_x h_j(x^*, 0) &= 0 \quad \text{for } j = 1, 2, \ldots, J, \end{aligned} \tag{4}$$

we have $y^T \nabla_{xx}^2 L(x^*, u^*, v^*, w^*, 0) y < 0$. Then x^* is a locally unique maximizing point problem (1) when $\alpha = 0$.

Proof. Rework Fiacco and McCormick's proof, replacing (two sided) ordinary partial derivatives by one sided partial derivatives when necessary. Q.E.D.

The following theorem is a straightforward modification of Fiacco's ([4], p. 291) Theorem 2.1:

THEOREM 2 (*Sensitivity Results for a Second Order Local Maximizer x^* of* (1) *when $\alpha = 0$*)

Suppose: (i) the functions defining problem (1) are twice continuously differentiable in a neighbourhood of $(x^*, 0)$ (intersected with $R_N{}^+ \times R_1$) where $x^* \geq 0_N$; (ii) the second order sufficient conditions for x^* to be a local maximizer of (1) when $\alpha = 0$, (3) and (4), hold with associated Lagrange multiplier vectors $u^* \geq 0_M$, $v^* \geq 0_N$ and w^*; (iii) the unit vectors e_n (e_n is N dimensional with a one in component n and zeroes elsewhere) for n such that $x_n^* = 0$ (x_n^* is the nth component of x^*), the gradient vectors $\nabla_x g_m(x^*, 0)$ for m such that $g_m(x^*, 0) = 0$, and the gradient vectors $\nabla_x h_j(x^*, 0)$ for $j = 1, 2, \ldots, J$ are *linearly independent*, and (iv) $u_m^* > 0$ when $g_m(x^*, 0) = 0$ for $m = 1, 2, \ldots, M$ and $v_n^* > 0$ when $x_n^* = 0$ for $n = 1, 2, \ldots, N$ (i.e. *strict complementary slackness* holds for (1) when $\alpha = 0$), then: (a) x^* is a locally unique maximizing point for problem (1) when $\alpha = 0$ and the associated Lagrange multipliers u^*, v^* and w^* are unique; (b) for α in a neighbourhood of 0, there exists a unique once continuously differentiable vector of functions $[x(\alpha), u(\alpha), v(\alpha), w(\alpha)]$ satisfying the second order sufficient conditions (3) and (4) (with α replacing 0) for a local maximizing point for (1) such that $[x(0), u(0), v(0), w(0)] = [x^*, u^*, v^*, w^*]$; moreover, $x(\alpha)$ is locally unique local maximizer for (1) with associated unique Lagrange multipliers $u(\alpha)$, $v(\alpha)$ and $w(\alpha)$, and (c) the linear independence and strict complementary slackness conditions (see (iii) and (iv) above) continue to hold for (1) for α near 0.

Proof. The linear independence condition (iii) means that of the $M + N + J$ constraints, only N of them can be binding. After reordering constraints if necessary, we suppose that the first R inequality constraints are binding where $0 \le R \le \min\{M, N\}$ and the last $N-S$ nonnegativity constraints are binding where $0 \le S \le N$ when $\alpha = 0$.

Case (i). $S > 0$. We have

$$g_m(x^*, 0) = 0, m = 1, 2, \ldots, R, \tag{5}$$

$$g_m(x^*, 0) > 0, m = R + 1, \ldots, M,$$

$$x_n^* > 0, n = 1, 2, \ldots, S, \text{ and}$$

$$x_n^* = 0, n = S + 1, \ldots, N.$$

Define $x^* \equiv (x_1^*, x_2^*, \ldots, x_S^*;\ x_{S+1}^*, x_{S+2}^*, \ldots, x_N^*) \equiv (x^{1*}; x^{2*}) = (x^{1*}, 0_{N-S})$ and partition $v^* \equiv (v^{1*}, v^{2*})$ similarly. The complementary slackness condition $v^{*T}x^* = 0$ along with the nonnegativity conditions $v^* \ge 0_N$, $x^* \ge 0$ in (3) imply

$$v^{1*} = 0_S. \tag{6}$$

Similarly, conditions (5) along with the complementary slackness conditions in the second line of (3) imply

$$u^{2*} = 0_{M-R} \tag{7}$$

where $u^* \equiv (u_1^*, \ldots, u_R^*;\ u_{R+1}^*, \ldots, u_M^*) \equiv (u^{1*}; u^{2*})$. Now differentiate $L(x^*, u^*, v^*, w^*, 0)$ with respect to x^1 and then with respect to the components in x^2. Doing this and using (6) and (7), the last set of equations in (3) can be rewritten as follows:

$$\nabla_{x^1} f(x^*, 0) + \sum_{m=1}^{R} u_m^* \nabla_{x^1} g_m(x^*, 0) + \sum_{j=1}^{J} w_j^* \nabla_{x^1} h_j(x^*, 0) = 0_S \tag{8}$$

$$\nabla_{x^2} f(x^*, 0) + \sum_{m=1}^{R} u_m^* \nabla_{x^2} g_m(x^*, 0) + \sum_{j=1}^{J} w_j^* \nabla_{x^2} h_j(x^*, 0) = -v^{2*}. \tag{9}$$

The linear independence assumption (iii) implies that the vectors $\nabla_x g_1(x^*, 0), \ldots, \nabla_x g_R(x^*, 0)$, $\nabla_x h_1(x^*, 0), \ldots, \nabla_x h_J(x^*, 0)$, $e_{S+1}, \ldots, e_N$ are linearly independent. Using elementary column operations, it is easy to see that the following $R + J$ vectors (of dimension S), $\nabla_{x^1} g_1(x^*, 0), \ldots, \nabla_{x^1} g_R(x^*, 0)$, $\nabla_{x^1} h_1(x^*, 0), \ldots, \nabla_{x^1} h_J(x^*, 0)$, must also be linearly independent.

Now consider the following system of equations:

$$\begin{aligned}
&\nabla_{x^1} f(x^1, 0, \alpha) + \sum_{m=1}^{R} u_m \nabla_{x^1} g_m(x^1, 0, \alpha) + \sum_{j=1}^{J} w_j \nabla_{x^1} h_j(x^1, 0, \alpha) = 0_S \\
&g_1(x^1, 0, \alpha) = 0 \\
&\vdots \\
&g_R(x^1, 0, \alpha) = 0 \\
&h_1(x^1, 0, \alpha) = 0 \\
&\vdots \\
&h_J(x^1, 0, \alpha) = 0
\end{aligned} \tag{10}$$

where 0 in (10) is 0_{N-S}. We regard (10) as a system of $S + R + J$ simultaneous equations in the $S + R + J + 1$ unknowns x^1, u^1, w and α. Note that $x^1 = x^{1*} \gg 0_S$, $u^1 = u^{1*} \gg 0_R$, $w = w^*$ and $\alpha = 0$ is a solution to (10), and that the functions in (10) are all once continuously differentiable around $(x^{1*}, u^{1*}, w^*, 0)$ by assumption (i). The Jacobian matrix of (10) with respect to x^1, u^1, and w evaluated at $(x^{1*}, u^{1*}, w^*, 0)$ is

$$A \equiv \begin{bmatrix} B, & D, & E \\ D^T, & 0_{R\times R}, & 0_{R\times J} \\ E^T, & 0_{J\times J}, & 0_{J\times J} \end{bmatrix} \tag{11}$$

where

$$B \equiv \nabla^2_{x^1x^1} f(x^{1*}, 0_{N-S}, 0) + \sum_{m=1}^{R} u_m^* \nabla^2_{x^1x^1} g_m(x^{1*}, 0_{N-S}, 0) + \sum_{j=1}^{J} w_j^* \nabla^2_{x^1x^1} h_j(x^{1*}, 0_{N-S}, 0)$$

$$D \equiv [\nabla_{x^1} g_1(x^{1*}, 0_{N-S}, 0), \ldots, \nabla_{x^1} g_R(x^{1*}, 0_{N-S}, 0)]$$

and

$$E \equiv [\nabla_{x^1} h_1(x^{1*}, 0_{N-S}, 0), \ldots, \nabla_{x^1} h_J(x^{1*}, 0_{N-S}, 0)].$$

We have already noted that the $R + J$ columns of D and E are linearly independent. If $S = R + J$, then this linear independence implies that A^{-1} exists. If $R + J < S$, then it is straightforward to show that the second order conditions (4) imply the following property:

$$y^1 \neq 0_S,\ y^{1T} D = 0_R{}^T,\ y^{1T} E = 0t \rightarrow y^{1T} B y^1 < 0. \tag{12}$$

Property (12) along with the linear independence of the columns of D and E also suffices to imply the existence A^{-1}

Thus we may apply the Implicit Function Theorem to (10) and conclude that there exists a once continuously differentiable solution $[x^1(\alpha), u^1(\alpha), w(\alpha)]$ to (10) for α in a neighbourhood of 0 with $x^1(0) = x^{1*} \gg 0_S$, $u^1(0) = u^{1*} \gg 0_R$ and $w(0) = w^*$. Now use the functions $x^1(\alpha)$, $u^1(\alpha)$, $w(\alpha)$ in order to define $v^2(\alpha)$ for α close to 0:

$$\begin{aligned} v^2(\alpha) \equiv &-\nabla_{x^2} f(x^1(\alpha), 0_{N-S}, \alpha) \\ &- \sum_{m=1}^{R} u_m(\alpha) \nabla_{x^2} g_m(x^1(\alpha), 0_{N-S}, \alpha) - \sum_{j=1}^{J} w_j(\alpha) \nabla_{x^2} h_j(x^1(\alpha), 0_{N-S}, \alpha). \end{aligned} \tag{13}$$

Define also

$$v^1(\alpha) \equiv 0_S,\ u^2(\alpha) \equiv 0_{M-R},\ x^2(\alpha) \equiv 0_{N-S}, \tag{14}$$

and

$$x(\alpha)^T \equiv [x^1(\alpha)^T, x^2(\alpha)^T], v(\alpha)^T \equiv [v^1(\alpha)^T, v^2(\alpha)^T], u(\alpha)^T \equiv [u^1(\alpha)^T, u^2(\alpha)^T]. \tag{15}$$

It can be verified that the $[x(\alpha), u(\alpha), v(\alpha), w(\alpha)]$ defined above satisfies the second order sufficient conditions for a local maximum for (1) for a sufficiently close to zero. The remainder of the proof follows Fiacco's ([4], pp. 291–293) proof.

Q.E.D.

Case (ii). $S = 0$. In this case $x^* = 0_N$. The linear independence condition (iii) implies the nonbindingness of the inequality constraints (so $g_m(x^*, 0) > 0$, $m = 1, 2, \ldots, M$) and that there are no equality constraints ($J = 0$). The Kuhn–Tucker conditions (10) become ($u^* \equiv 0_M$)

$$\nabla_x f(x^*, 0) = -v^*$$

where $v^* \gg 0_N$ by strict complementary slackness. The second order sufficiency conditions (4) are logically satisfied, since there is no $y \neq 0_N$ that satisfies all of the conditions in (4). For α near 0, define $x(\alpha) \equiv 0_N$, $u(\alpha) \equiv 0_M$ and $v(\alpha) \equiv -\nabla_x f(0_N, \alpha)$. It can be verified that this $[x(\alpha), u(\alpha), v(\alpha)]$ satisfies the second order sufficient conditions for a local maximum for (1) for α sufficiently close to zero. The remainder of the proof follows Fiacco's proof.

Q.E.D.

The reader should consult Fiacco and Hutzler [7], Section 5, for other generalizations of Fiacco's Theorem 2.1.

The key to the proof of the above theorem is the system of equations (10) which involves only variables which are locally unrestricted, and thus the classical Implicit Function Theorem can be applied.

The Implicit Function Theorem also allows us to exhibit the structure of the partial derivatives of the solution to (1):

COROLLARY 2.1

If the first R inequality constraints and the last $N - S$ nonnegativity constraints ($0 < S \leq N$) of (1) are binding when $\alpha = 0$, then the first order derivatives of the solution function $x(\alpha)$ and the Lagrange multiplier functions $u(\alpha)$, $v(\alpha)$, $w(\alpha)$ with respect to α evaluated at $\alpha = 0$ are defined by

$$\begin{bmatrix} \nabla_\alpha x^1(0) \\ \nabla_\alpha u^1(0) \\ \nabla_\alpha w(0) \end{bmatrix} \equiv A^{-1}b; \quad \begin{matrix} \nabla_\alpha x^2(0) = 0_{N-S} \\ \nabla_\alpha u^2(0) = 0_{M-R} \\ \nabla_\alpha v^1(0) = 0_S \end{matrix} \tag{16}$$

where A is defined by (11) and b is defined by

$$b \equiv \begin{bmatrix} -\nabla^2_{x^1\alpha} f(x^*, 0) - \sum_{m=1}^{R} u_m^* \nabla^2_{x^1\alpha} g_m(x^*, 0) - \sum_{j=1}^{J} w_j^* \nabla^2_{x^1\alpha} h_j(x^*, 0) \\ -\partial g_1(x^*, 0)/\partial\alpha \\ \vdots \\ -\partial g_R(x^*, 0)/\partial\alpha \\ -\partial h_1(x^*, 0)/\partial\alpha \\ \vdots \\ -\partial h_J(x^*, 0)/\partial\alpha \end{bmatrix}. \tag{17}$$

and $\nabla_\alpha v^2(0)$ can be calculated by differentiating (13) with respect to α, using the derivatives in (16).

COROLLARY 2.2

Suppose: (i) the assumptions (i)–(iv) in Theorem 2 hold, (ii) the first R inequality constraints and the last $N-S$ nonnegativity constraints of (1) are binding when $\alpha=0$, and (iii) the functions defining problem (1) are thrice continuously differentiable in a neighbourhood of $(x^*, 0)$ (intersected with $R_N^+ \times R_1$). Then the second order derivatives of the solution function $x(\alpha)$ and the Lagrange multiplier functions $u(\alpha)$, $v(\alpha)$, $w(\alpha)$ exist and are continuous functions of α for α close to zero. The second order derivatives of $x^1(\alpha)$, $u^1(\alpha)$, and $w(\alpha)$ evaluated at $\alpha=0$ are

$$\begin{bmatrix} \nabla^2_{\alpha\alpha} x^1(0) \\ \nabla^2_{\alpha\alpha} u^1(0) \\ \nabla^2_{\alpha\alpha} w(0) \end{bmatrix} = A^{-1}\frac{[\mathrm{d}b(0)]}{\mathrm{d}\alpha} - A^{-1}\frac{[\mathrm{d}A(0)]}{\mathrm{d}\alpha}A^{-1}b \tag{18}$$

where A is defined by (11), b is defined by (17), and $A(\alpha)$ and $b(\alpha)$ are defined by (11) and (17) except that we replace $[x^*, 0]$ by $[x(\alpha), \alpha]$ and u^*, w^* are replaced by $u(\alpha)$, $w(\alpha)$. We also have

$$\nabla^2_{\alpha\alpha} x^2(0) = 0_{N-S},\ \nabla^2_{\alpha\alpha} u^2(0) = 0_{M-R},\ \nabla^2_{\alpha\alpha} v^1(0) = 0_S \tag{19}$$

and $\nabla^2_{\alpha\alpha} v^2(0)$ can be calculated by differentiating (13) twice with respect to α, using the derivatives in (16), (18) and (19).

The above Theorem and its corollaries accomplish three things: (i) a rigorous justification is provided for the economist's casual approach to the problem of nonnegativity constraints, which is to assume that the binding nonnegativity constraints will remain binding under perturbation (i.e. as α varies) and thus the corresponding x_n will remain fixed at 0; (ii) we show how the computations of the first and second order derivatives of $x(\alpha)$, $u(\alpha)$, $v(\alpha)$, $w(\alpha)$ at $\alpha=0$ "simplify" compared to the computations that would be required if we treated the nonnegativity constraints as general inequality constraints (see Armacost and Fiacco [12] for a summary of the computations involved in this latter general case); and (iii) we show how Fiacco's [4] Sensitivity Theorem can be extended to the case where the objective and constraint functions for (1) are defined only over the nonnegative orthant rather than over an open set.

Theorem 2 can readily be extended to cover the case where only a subset of the x_n variables are subject to nonnegativity constraints. The Theorem can also be extended to cover the case where the objective and constraint functions are defined over a general polyhedral set with a nonempty interior: if the feasible x region is defined as $\{x : a^{kT}x \le b^k;\ k=1,2,\ldots,K\}$ where the a^k are vectors and the b^k are scalars, we need only define $x_{N+k} \equiv b^k - a^{kT}x$ for $k=1,2,\ldots,K$ and add the nonnegativity constraints $x_{N+k} \ge 0$, $k=1,2,\ldots,K$ to our original problem. The nonempty interior assumption is required in order to define partial derivatives of the objective and constraint functions. Unfortunately, this technique does not work for a general convex (nonpolyhedral) domain of definition set.

In the following two sections, we study in more detail the derivatives of $x^1(\alpha)$, $u^1(\alpha)$ and $w(\alpha)$ with respect to α given by (16), and we also derive expressions for the first and second order derivatives of the objective function with respect to α.

2. COMPARATIVE STATICS THEOREMS

We first define the value function V as the optimal value of the objective function for (1) for α close to 0:

$$V(\alpha) \equiv f(x(\alpha), \alpha). \tag{20}$$

Using (14) and the strict complementary slackness conditions, it can be seen that

$$V(\alpha) = f(x(\alpha), \alpha) + \sum_{m=1}^{M} u_m(\alpha) g_m(x(\alpha), \alpha) + \sum_{n=1}^{N} v_n(\alpha) x_n(\alpha) + \sum_{j=1}^{J} w_j(\alpha) h_j(x(\alpha), \alpha)$$

$$= L(x(\alpha), u(\alpha), v(\alpha), w(\alpha)) \tag{21}$$

$$= f(x^1(\alpha), 0_{N-S}, \alpha) + \sum_{m=1}^{R} u_m(\alpha) g_m(x^1(\alpha), 0_{N-S}, \alpha) \tag{22}$$

$$+ \sum_{j=1}^{J} w_j(\alpha) h_j(x^1(\alpha), 0_{N-S}, \alpha).$$

Thus we have 3 alternative expressions for the value function V. Differentiating (22) with respect to α for α close to 0, we find, using (10), that

$$\frac{\mathrm{d}V(\alpha)}{\mathrm{d}\alpha} = \frac{\partial f}{\partial \alpha}(x^1(\alpha), 0_{N-S}, \alpha) + \sum_{m=1}^{R} u_m(\alpha) \frac{\partial g_m}{\partial \alpha}(x^1(\alpha), 0_{N-S}, \alpha)$$

$$+ \sum_{j=1}^{J} w_j(\alpha) \frac{\partial h_j}{\partial \alpha}(x^1(\alpha), 0_{N-S}, \alpha), \tag{23}$$

where $\partial f(x^1, 0_{N-S}, \alpha)/\partial \alpha$ means differentiate f with respect to its last argument, etc. Evaluate (23) at $\alpha = 0$:

$$\nabla_\alpha V(0) = \nabla_\alpha f(x^*, 0) + \sum_{m=1}^{R} u_m^* \nabla_\alpha g_m(x^*, 0) + \sum_{j=1}^{J} w_j^* \nabla_\alpha h_j(x^*, 0). \tag{24}$$

Now differentiate (23) with respect to α and evaluate the resulting derivatives at $\alpha = 0$:

$$\nabla_{\alpha\alpha}^2 V(0) = \nabla_{\alpha\alpha}^2 f^* + \sum_{m=1}^{R} u_m^* \nabla_{\alpha\alpha}^2 g_m^* + \sum_{j=1}^{J} w_j^* \nabla_{\alpha\alpha}^2 h_j^* + \Bigg[\nabla_{\alpha x^1}^2 f^* + \sum_{m=1}^{R} u_m^* \nabla_{\alpha x^1}^2 g_m^*$$

$$+ \sum_{j=1}^{J} w_j^* \nabla_{\alpha x^1}^2 h_j^* \Bigg] \nabla_\alpha x^1(0) + \sum_{m=1}^{R} \nabla_\alpha u_m(0) \nabla_\alpha g_m^* + \sum_{j=1}^{J} \nabla_\alpha w_j(0) \nabla_\alpha h_j^*$$

$$= \nabla_{\alpha\alpha}^2 f^* + \sum_{m=1}^{R} u_m^* \nabla_{\alpha\alpha}^2 g_m^* + \sum_{j=1}^{J} w_j^* \nabla_{\alpha\alpha}^2 h_j^* - b^T A^{-1} b \tag{25}$$

where $f^* \equiv f(x^{1*}, 0_{N-S}, 0)$, $g_m^* \equiv g_m(x^*, 0)$, $h_j^* \equiv h_j(x^*, 0)$, A is defined by (11), and b is defined by (17).

Suppose α appears only in the function g_m for some $m = 1, 2, \ldots, R$ and in particular, $g_m(x, \alpha) \equiv g_m(x, 0) + \alpha$ for α close to zero. Then the effect of increasing α from an initial zero level will be to relax the mth binding inequality constraint, and from (24), we have

$$\nabla_\alpha V(0) = u_m^*, \quad m \in \{1, 2, \ldots, R\} \tag{26}$$

i.e. we now have the standard interpretation for the mth binding inequality Lagrange multiplier (as being the marginal increase in the value of the program due to a marginal relaxation of the mth inequality constraint). For the nonbinding inequality constraints, we have a similar interpretation; i.e., if $g_m(x, \alpha) \equiv g_m(x, 0) + \alpha$ for some $m = R + 1, \ldots, M$ and α appears nowhere else, then

$$\nabla_\alpha V(0) = 0 \equiv u_m^*, \quad m \in \{R+1, R+2, \ldots, M\}. \tag{27}$$

Similarly, if α appears only in the jth equality constraint function for some $j = 1, 2, \ldots, J$ with $h_j(x, \alpha) \equiv h_j(x, 0) + \alpha$, then from (24),

$$\nabla_\alpha V(0) = w_j^*, \quad j \in \{1, 2, \ldots, J\}. \tag{28}$$

Finally, suppose α appears only in the objective function f. Then again from (24),

$$\nabla_\alpha V(0) = \nabla_\alpha f(x^*, 0), \tag{29}$$

i.e. the first order change in the value function V with respect to α is exactly equal to the first order change in the objective function f, holding x^* constant ([2], p. 34). This is Samuelson's famous Envelope Theorem.

The relations (24)–(29) were somewhat sketchily and nonrigorously obtained by Samuelson [2] for the case of equality constraints and rigorously obtained by Armacost and Fiacco ([12], Theorem 3] under Fiacco's ([4], Theorem 2-1) regularity conditions on the problem (1). We have simply adapted the above author's results to our particular regularity conditions on problem (1).

To make further progress, it is necessary to consider the structure of A^{-1} in some detail. Below, we survey what seems to be known about the structure of A^{-1}.

First, recall that the matrices D and E defined below (11) were the gradient vectors of the binding inequality constraints and the equality constraints with respect to the S components of x^1, evaluated at the initial equilibrium. Define the S by $R+J$ matrix C by

$$C \equiv [D, E]. \tag{30}$$

Define the vector of nontrivial Lagrange multiplier or shadow price functions $p(\alpha) \equiv [p_1(\alpha), \ldots, p_{R+J}(\alpha)]^T$ by

$$p(\alpha)^T \equiv [u^1(\alpha)^T, w(\alpha)^T]. \tag{31}$$

We assume that S, the number of locally unrestricted x_n variables is equal to or greater than one. Our linear independence assumptions (the $R+J$ columns of C are linearly independent) also imply that $S \geq R+J$ so that the number of locally unrestricted x_n variables is equal to or greater than the number of binding inequality and equality constraints.

Recall that A was defined by (11). Using (30),

$$A^{-1} = \begin{bmatrix} B, & C \\ C^T, & 0 \end{bmatrix}^{-1} \equiv \begin{bmatrix} C, & H \\ H^T, & K \end{bmatrix} \tag{32}$$

where it can be shown that the matrices G, H and K satisfy the following matrix equations:

$$\begin{aligned} &BG + CH^T = I_S;\ C^TG = 0_{R+JXS};\ C^TH = I_{R+J}; \\ &BH + CK = 0_{SXR+J};\ G = G^T;\ K = K^T;\ K = -H^TBH \end{aligned} \tag{33}$$

where the last equation in (33) follows by premultiplying the fourth equation by H^T and using the third equation in (33).

The following theorem summarizes some of the properties of G and K.

THEOREM 3

Suppose the $R+J$ columns of C are linearly independent and the symmetric matrix B satisfies: $z \neq 0_S$, $z^TC = 0^T_{R+J} \rightarrow z^TBz < 0$; i.e. if $S > R+J$, the matrix B is negative definite in the subspace orthogonal to the columns of C. Then: (i) A^{-1} defined by (32) exists; (ii) G is a negative semidefinite symmetric matrix of rank $S-(R+J)$; (iii) if $S > R+J$, then G is negative definite in the subspace orthogonal to the columns of C; (iv) let $R+J > 0$ and let L denote the number of zero eigenvalues of B where $0 \leq L \leq R+J \leq S$; then the rank of K is $R+J-L$ so that each 0 (unrestricted) eigenvalue of B reduces the rank of K by one; (v) if B is negative semidefinite, then K is positive semidefinite, and (vi) if B is negative definite, then K is positive definite.

Proof. (i) and (ii) date back to Carathéodory[13] at least; see also Samuelson ([2], p. 379) and Fiacco and McCormick ([10], Theorem 14). (iii) This result follows from the following expression for G obtained by Diewert and Woodland ([14], p. 395):

$$G = \sum_{i=1}^{S-R-J} \lambda_i^{-1} z^i z^{iT} \tag{34}$$

where the $\lambda_i < 0$ are the eigenvalues of B restricted to the subspace orthogonal to the columns of C and the z^i are the corresponding restricted eigenvectors; i.e. the λ_i are the $S-(R+J)$ roots of the determinantal equation

$$\begin{bmatrix} B-\lambda I_S, & C \\ C^T & , & 0 \end{bmatrix} = 0$$

and the corresponding restricted eigenvectors satisfy for $i = 1, 2, \ldots, S$; $z^{iT}C = 0^T_{R+J}$; $z^{iT}z^i = 1$; $Bz^i + Cy^i = \lambda_i z^i$ for some y^i and $z^{iT}z^j = 0$ if $i \neq j$. The result also follows from the analysis in Fiacco[15]. (iv). See Diewert and Woodland ([14], p. 396). (v) Follows from the last equation in (33) when B is negative semidefinite. (vi) Follows from (iv) and (v). Armacost and Fiacco ([16], pp. 18–24) provide proofs of results equivalent to (v) and (vi).
Q.E.D.

For numerical evaluation, it is useful to give explicit formulas for G, H and K. Armacost and Fiacco [16] do this for three cases: case 1 where $B = \nabla^2_{x^1x^1}L(x^*, u^*, v^*, w^*)$ has an inverse; case 2 where the number of binding constraints $R + J$ equals S, the number of locally unrestricted x_n, and case 3, a general case. Below, we provide an alternative set of formulae which treats all three cases simultaneously.

Since B is negative definite in the subspace orthogonal to the linearly independent columns of C, by Finsler's Theorem (see Debreu [17] for example) there exists a scalar k large enough so that $\hat{B} \equiv B - kCC^T$ is a negative definite symmetric matrix, and hence $\hat{B}^{-1}$ exists. Now define G, H and K in terms of $\hat{B}$ and C as follows:

$$\begin{aligned} G &\equiv \hat{B}^{-1} - \hat{B}^{-1}C[C^T\hat{B}^{-1}C]^{-1}C^T\hat{B}^{-1}, \\ H &\equiv \hat{B}^{-1}C[C^T\hat{B}^{-1}C]^{-1} \end{aligned} \tag{35}$$

and

$$K \equiv -kI_{R+J} - [C^T\hat{B}^{-1}C]^{-1}$$

It can be verified directly that G, H and K defined by (35) satisfy the equations in (33) and hence they are the desired elements of A^{-1}. Formulae closely related to (25) were also obtained by Buys and Gonin [18] and Armacost and Fiacco [26].

Having surveyed what is known in general about the structure of A^{-1}, we may return to our main concern which is to determine if possible the signs of the derivatives $\nabla_\alpha V(0)$, $\nabla^2_{\alpha\alpha}V(0)$, $\nabla_\alpha x^1(0)$ and $\nabla_\alpha p(0)$.

Let us first rewrite (16) using (30)–(32):

$$\begin{bmatrix} \nabla_\alpha x^1(0) \\ \nabla_\alpha p(0) \end{bmatrix} = A^{-1}b = \begin{bmatrix} G, & H \\ H^T, & K \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix} \tag{36}$$

where

$$b^1 \equiv -\nabla^2_{x^1\alpha}f^* - \sum_{m=1}^{R} u^*_m \nabla^2_{x^1\alpha}g^*_m - \sum_{j=1}^{J} w^*_j \nabla^2_{x^1\alpha}h^*_j \tag{37}$$

and

$$b^{2T} \equiv -[\nabla_\alpha g^*_1, \ldots, \nabla_\alpha g^*_R; \nabla_\alpha h^*_1, \ldots, \nabla_\alpha h^*_J].$$

Now suppose α occurs only in the objective function f. We have already established (29) under this hypothesis; i.e. that $\nabla_\alpha V(0) = \nabla_\alpha f(x^*, 0) \equiv \nabla_\alpha f^*$. Now we may evaluate the second derivative of V using (25), (36) and (37):

$$\begin{aligned} \nabla^2_{\alpha\alpha} V(0) &= \nabla^2_{\alpha\alpha} f^* - [\nabla^2_{\alpha x^1} f^*] G [\nabla^2_{x^1\alpha} f^*] \\ &\geq \nabla^2_{\alpha\alpha} f^* \end{aligned} \tag{38}$$

where the inequality in (38) follows from the negative semidefiniteness of G and $[\nabla^2_{\alpha x^1} f^*]^T = [\nabla^2_{x^1\alpha} f^*]$. Thus paraphrasing Samuelson ([2], p. 35), if a marginal change in a parameter of the objective function holding x^* fixed leads to a higher value of the objective function, it will lead to an even higher value when x is allowed to adjust optimally.

Note that a strict inequality will hold in (38) if the columns of $[\nabla^2_{x^1\alpha} f^*; \nabla_{x^1} g_1^*, \ldots, \nabla_{x^1} g_R^*; \nabla_{x^1} h_1^*, \ldots, \nabla_{x^1} h_J^*] = [\nabla^2_{x^1\alpha} f^*, C]$ are linearly independent since by (34), G is spanned by the subspace orthogonal to C and is negative definite there.

Premultiplying the first set of equations in (36) by $\nabla^2_{\alpha x^1} f^*$ yields

$$\begin{aligned} [\nabla^2_{\alpha x^1} f^*] \nabla_\alpha x^1(0) &= -[\nabla^2_{\alpha x^1} f^*] G [\nabla^2_{x^1\alpha} f^*] \\ &\geq 0 \end{aligned} \tag{39}$$

where the inequality (39) again follows from the negative semidefiniteness of G. The inequality will be strict iff the $1 + R + J$ columns of $[\nabla^2_{x^1\alpha} f^*, C]$ are linearly independent. In the case of no constraints, this inequality was first obtained by Samuelson ([2], p. 32).

Now let us further specify that all of the S partial derivatives in $\nabla^2_{\alpha x^1} f^* = \nabla^2_{\alpha x^1} f(x^{1*}, 0_{N-S}, 0)$ are zero except the ith, so that

$$\nabla^2_{\alpha x^1} f^* = e_i \partial^2 f(x^*, 0)/\partial\alpha\partial x_i, \; i \in \{1, 2, \ldots, S\} \tag{40}$$

where e_i is a unit vector of dimension S with a one in component i. Under these conditions, (39) becomes

$$\frac{\partial^2 f}{\partial\alpha\partial x_i}(x^*, 0)\frac{\mathrm{d}x_i(0)}{\mathrm{d}\alpha} = -\left[\frac{\partial^2 f}{\partial\alpha\partial x_i}(x^*, 0)\right]^2 e_i^T G e_i \geq 0, \; i \in \{1, \ldots, S\} \tag{41}$$

with a strict inequality in (41) iff $[e_i, C]$ has linearly independent columns. This inequality was first obtained by Samuelson ([2], p. 32) for the unconstrained maximization case.

If we further specify that f be linear in x and α so that $f(x, \alpha) \equiv (c + e_i\alpha)^T x$ where e_i is a unit vector of dimension N, then $\partial^2 f(x^*, 0)/\partial\alpha\partial x_i = 1$ and (41) becomes $\mathrm{d}x_i(0)/\mathrm{d}\alpha \geq 0$. In this case, $\alpha > 0$ can be interpreted as an increase in the price of good x_i, and the inequality $\mathrm{d}x_i(0)/\mathrm{d}\alpha \geq 0$ can be interpreted as saying that the optimal net supply of good x_i will not decrease as its price increases. This type of inequality can be found in Samuelson ([2], p. 37).

We now turn our attention to the case where α appears only in g_m for some $m = 1, 2, \ldots, R$ and in particular, $g_m(x, \alpha) \equiv g_m(x, 0) + \alpha$. In this case, from (36) we obtain the following valid equations:

$$\begin{bmatrix} \nabla_\alpha x^1(0) \\ \nabla_\alpha p(0) \end{bmatrix} = -\begin{bmatrix} H \\ K \end{bmatrix} e_m, \; m \in \{1, 2, \ldots, R\} \tag{42}$$

where e_m is a unit vector of dimension $R + J$ with a one in component m. If B is negative semidefinite, then from part (v) of Theorem 3, K is positive semidefinite so that from (41),

$$e_m^T \nabla_\alpha p(0) = \frac{\mathrm{d}p_m(0)}{\mathrm{d}\alpha} = \frac{\mathrm{d}u_m(0)}{\mathrm{d}\alpha} = -e_m^T K e_m \leq 0. \tag{43}$$

If B is negative definite, then from part (vi) of Theorem 3, K is positive definite and the inequality in (43) will be strict. These results are entirely analogous to similar results obtained by Armacost and Fiacco [16] for their nonlinear programming problem (which did not involve nonnegativity constraints).

The properties of A^{-1} play a large role in modern microeconomic theory. Additional restrictions on the derivatives $\nabla_\alpha x^1(0)$ and $\nabla_\alpha p(0)$ can be obtained as we impose additional structure on the matrices B and C. For some examples of additional comparative statics theorems, see Diewert and Woodland [14].

3. ADDITIONAL INEQUALITY CONSTRAINTS MAINTAINING FEASIBILITY OF x^*

Suppose we add some additional inequality constraints (44) to the nonlinear programming problem (1):

$$g_{M+i}(x, \alpha) \geq 0; \quad i = 1, 2, \ldots, I. \tag{44}$$

Suppose further that the functions g_{M+i} are twice continuously differentiable in a neighbourhood of $(x^*, 0)$ intersected with $R_N{}^+ XR_1$, where x^* solves the original problem (1) and the additional constraints are just binding, so that

$$g_{M+i}(x^*, 0) = 0; \; i = 1, 2, \ldots, I. \tag{45}$$

There are two cases to be considered. In this section, we consider case (i): each gradient vector $\nabla_{x^1} g_{M+i}(x^{1*}, 0_{N-S}, 0)$ for $i = 1, 2, \ldots, I$ is *linearly dependent* on the following gradient vectors,

$$\nabla_{x^1} g_1(x^*, 0), \ldots, \; \nabla_{x^1} g_R(x^*, 0), \; \nabla_{x^1} h_1(x^*, 0), \ldots, \nabla_{x^1} h_J(x^*, 0),$$

which occurred in the original problem. Case (ii) is the case where the gradient vectors $\nabla_{x^1} g_{M+i}(x^{1*}, o_{N-S}, 0)$, $i = 1, \ldots, I$ and the original binding constraint gradient vectors in the matrix C are all linearly independent. The analysis in the following section (which deals with the case of additional linearly independent just binding equality constraints) can be modified to deal with special cases of case (ii). For example, the case where $[\nabla_{x^1} g^*_{M+1}, C]$ has linearly independent columns and $\partial u_{M+1}(0)/\partial\alpha > 0$ can be treated in the same manner as our treatment of an additional binding equality constraint which follows in the next section. If $\partial u_{M+1}(0)/\partial\alpha < 0$, then the new inequality constraint can be ignored, but if $\partial u_{M+1}(0)/\partial\alpha = 0$, then we cannot apply the analysis presented in the next section.

Recall that Corollories 2.1 and 2.2 enabled us to compute how the original solution function $x(\alpha)$ changes as α changes near 0. In particular, recall that (16) or (36) defined $\nabla_\alpha x^1(0) = Gb^1 + Hb^2$ and (18) defined $\nabla^2_{\alpha\alpha} x^1(0)$ under the additional hypothesis of thrice differentiability of the original objective and constraint functions. We use these derivatives in our last assumption, (46) and (47): for $i = 1, 2, \ldots, I$, either

$$\nabla_\alpha g_{M+i}(x^*, 0) + [\nabla_{x^1} g_{M+i}(x^*, 0)]^T \nabla_\alpha x^1(0) > 0 \tag{46}$$

or

$$\nabla_\alpha g^*_{M+i} + [\nabla_{x^1} g^*_{M+i}]^T \nabla_\alpha x^1(0) = 0 \tag{47}$$

but

$$\nabla^2_{\alpha\alpha} g^*_{M+i} + [\nabla_{x^1} g^*_{M+i}]^T \nabla^2_{\alpha\alpha} x^1(0) + 2[\nabla^2_{x^1\alpha} g^*_{M+i}]^T \nabla_\alpha x^1(0) + [\nabla_\alpha x^1(0)]^T [\nabla^2_{x^1x^1} g^*_{M+i}][\nabla_\alpha x^1(0)] > 0.$$

Thus if (46) or (47) hold, the ith extra inequality constraint, $g_{M+i}(x(\alpha), \alpha) \geq 0$, will hold (strictly) for α small and positive. This observation leads to the following Theorem.

THEOREM 4

Suppose the nonlinear programming problem (1) satisfies the hypotheses given in Theorem 2 and the additional hypotheses given in this section. Then the solution functions

$[x(\alpha), u(\alpha), v(\alpha), w(\alpha)]$ which were valid for the original problem (1) for α in a neighbourhood around 0 are also valid for the new problem, except that now α must be nonnegative. The formulae for the first and second order derivatives of the value function, $\nabla_\alpha V(0)$ and $\nabla^2_{\alpha\alpha} V(0)$, defined by (24)–(29) remain valid, except that the derivatives must now be interpreted as one sided derivatives (in the positive direction). Similarly, the formulae for the derivatives $\nabla_\alpha x^1(0)$ and $\nabla_\alpha p(0)$ defined by (36) and all of the other formulae developed in the previous section remain valid if the derivatives are interpreted as one sided derivatives.

Our "new" approach to sufficiency conditions for the existence of one sided derivatives appears to be a straightforward modification of Wilde's [24, 25] approach.

We turn now to the study of nonlinear programming problems of the form (1) where an additional binding equality constraint is added.

4. ADDITIONAL EQUALITY CONSTRAINTS MAINTAINING FEASIBILITY OF x^*

In this section, we return to problem (1) and make the assumptions listed in Theorem 2. In addition, we add an extra equality constraint of the form

$$h_{J+1}(x, \alpha) = 0 \tag{48}$$

which is consistent with the solution to the original problem (1); i.e. we assume that

$$h_{J+1}(x^*, 0) = 0 \tag{49}$$

In addition, we assume that h_{J+1} is twice continuously differentiable in a neighbourhood of $(x^*, 0)$ intersected with $R_N^+ \times R_1$ and its gradient vector is not linearly dependent on the original set of binding inequality and equality constraint gradient vectors; i.e. we assume that the vectors

$$[\nabla_{x^1} g_1^*, \ldots, \nabla_{x^1} g_R^*; \nabla_{x^1} h_1^*, \ldots, \nabla_{x^1} h_J^*; \nabla_{x^1} h_{J+1}^*] \equiv [C, c] \tag{50}$$

are linearly independent vectors, so that $S > R + J$.

Note that our new problem has the same structure as the original problem (1), so the reader may be puzzled as to why we have introduced it. Our purpose in introducing the new problem is to derive some inequalities between the first and second derivatives of the new value function $V_{(J+1)}(\alpha)$ and the first derivatives of the new solution function $x_{(J+1)}(\alpha)$ and the old functions $V(\alpha)$ and $x(\alpha)$. These inequalities play an important role in economic theory.

We now indicate how our earlier computations have to be modified in order to take account of the additional constraint (48).

Let w_{J+1} be the Lagrange multiplier that corresponds to the new constraint (48). The system of equations (10) is modified as follows: add (48) as an additional equation to (10) and add the term $w_{J+1}\nabla_{x^1} h_{J+1}(x^1, 0_{N-S}, \alpha)$ to the l.h.s. of the first equation in (10). Note that our linear independence assumption (50) implies that our old initial solution to (10), x^{1*}, u^{1*}, w^* will also be the unique solution to the new system (10) when $\alpha = 0$ if we define

$$w_{J+1}^* = 0. \tag{51}$$

The Jacobian matrix of the modified system of equations (10) around $(x^{1*}, u^{1*}, w^*, w_{J+1}^*)$ is, recalling (50):

$$A_{(J+1)} \equiv \begin{bmatrix} B, & C, & c \\ C^T, & 0, & 0 \\ c^T, & 0, & 0 \end{bmatrix} \tag{52}$$

Note that the B matrix in (52) is the same as the B in (11) since $w_{J+1}^* = 0$.

Our regularity conditions imply the existence of $A_{(J+1)}^{-1}$ and we write this inverse in terms of partitioned matrices as

$$A_{(J+1)}^{-1} \equiv \begin{bmatrix} G_{(J+1)}, & H_{(J+1)} \\ H_{(J+1)}^T, & K_{(J+1)} \end{bmatrix} \tag{53}$$

where $G_{(J+1)}$ is an S by S negative semidefinite symmetric matrix. Recall our old definition of A, (11), and our old definition for the elements of A^{-1}, (32). If we define $\tilde{c}^T \equiv [c^T, 0_{R+J}^T]$, it can be verified that the following expression defines $A_{(J+1)}^{-1}$ in terms of A^{-1} and $\tilde{c}$:

$$A_{(J+1)}^{-1} = \begin{bmatrix} A^{-1} - (\tilde{c}^T A^{-1}\tilde{c})^{-1} A^{-1}\tilde{c}\tilde{c}^T A^{-1}, & (\tilde{c}^T A^{-1}\tilde{c})^{-1} A^{-1}\tilde{c} \\ (\tilde{c}^T A^{-1}\tilde{c})^{-1}\tilde{c}^T A^{-1}, & -(\tilde{c}^T A^{-1}\tilde{c})^{-1} \end{bmatrix}. \tag{54}$$

Comparing (54) with (53), we find that

$$G_{(J+1)} = G - (c^T G c)^{-1} G c c^T G \tag{55}$$

where G is the S by S matrix defined in (32). Since G is negative definite in the subspace orthogonal to the columns of C (recall (34)), and since by (50) the columns of $[C, c]$ are linearly independent,

$$Gc \neq 0_S \tag{56}$$

and

$$c^T G c < 0. \tag{57}$$

Denote the solution function to (1) with (48) added by $x_{(J+1)}(\alpha)$. Suppose that α appears only in the objective function f. Then from the analogue to (36),

$$\nabla_\alpha x_{(J+1)}^1(0) = G_{(J+1)} b^1 = -[G - (c^T G c)^{-1} G c c^T G][\nabla_{x^1\alpha}^2 f^*] \tag{58}$$

where (58) follows using (37) and (55).

Define the new optimal value function $V_{(J+1)}$ by $V_{(J+1)}(\alpha) \equiv f(x_{J+1}^1(\alpha), 0_{N-S}, \alpha)$. As in section 2,

$$\nabla_\alpha V_{(J+1)}(0) = \nabla_\alpha f(x^*, 0) \equiv \nabla_\alpha f^* = \nabla_\alpha V(0) \tag{59}$$

but the second derivative is (recall (38))

$$\nabla_{\alpha\alpha}^2 V_{(J+1)}(0) = \nabla_{\alpha\alpha}^2 f^* - [\nabla_{\alpha x^1}^2 f^*] G_{(J+1)} [\nabla_{x^1\alpha}^2 f^*] = \nabla_{\alpha\alpha}^2 V(0) + [\nabla_{\alpha x^1}^2 f^*] \frac{G c c^T G}{c^T G c} [\nabla_{x^1\alpha} f^*]$$

using (55) and the equality in (38)

$$\leq \nabla_{\alpha\alpha}^2 V(0) \text{ using (57)} \tag{60}$$

and the strict inequality holds in (60) iff

$$c^T G [\nabla_{x^1\alpha}^2 f^*] \equiv [\nabla_{x^1} h_{J+1}^*]^T G [\nabla_{x^1\alpha}^2 f^*] \neq 0. \tag{61}$$

Silberberg ([19], p. 149; [27], p. 165) first obtained the inequality (60) (for an equality constrained problem), except that he assumed that the strict inequality always held in (61). An "economic" interpretation of (59) and (60) might run as follows: the marginal change in the optimal objective function assuming $J + 1$ equality constraints due to a marginal change in a parameter α is generally less than the marginal change in the corresponding optimal objective function when only J equality constraints are imposed on the problem.

Now premultiply both sides of (58) by $\nabla^2_{\alpha x^1} f^* \equiv \nabla^2_{\alpha x^1} f(x^{1*}, 0_{N-S}, 0)$. Using (39), we find that

$$\begin{aligned}[\nabla^2_{\alpha x^1} f^*]\nabla_\alpha x^1_{(J+1)}(0) &= [\nabla^2_{\alpha x^1} f^*]\nabla_\alpha x^1(0) + [\nabla^2_{\alpha x^1} f^*]\frac{Gcc^TG}{c^TGc}[\nabla^2_{x^1\alpha} f^*] \\ &= [\nabla^2_{\alpha x^1} f^*]\nabla_\alpha x^1(0) + \nabla^2_{\alpha\alpha} V_{(J+1)}(0) - \nabla^2_{\alpha\alpha} V(0) \\ &\le [\nabla^2_{\alpha x^1} f^*]\nabla_\alpha x^1(0) \text{ using (60)}\end{aligned} \tag{62}$$

and the strict inequality holds in (62) iff (61) holds.

Now let us further specify that all of the S partial derivatives in $\nabla^2_{\alpha x^1} f^*$ are zero except the ith so that (40) holds; i.e. $\nabla^2_{\alpha x^1} f^* = e_i \partial^2 f^*/\partial\alpha\partial x_i$ where e_i is the ith unit vector. Then using (62) and the appropriate version of (41), we obtain the following inequalities:

$$0 \le \frac{\partial^2 f^*}{\partial\alpha\partial x_i}\frac{\mathrm{d}x}{\mathrm{d}\alpha}(J+1)i(0) \le \frac{\partial^2 f^*}{\partial\alpha\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}\alpha}(0),\ i \in \{1, 2, \ldots, S\}. \tag{63}$$

The first inequality holds strictly iff the columns of $[e_i, C, c]$ are linearly independent while the second inequality in (63) holds strictly iff (61) holds. An "economic" interpretation for the second inequality in (63) is that the magnitude of the response for the second inequality in (63) is that the magnitude of the response of the ith decision variable to a marginal change in a parameter α is generally diminished as an extra equality constraint is added to the list of equality and inequality constraints. Alternatively, paraphrasing Samuelson ([2], p. 38), long run responses are generally bigger in magnitude than short run responses, since in the short run, many decision variables are fixed.

The inequalities (63) were first obtained by Samuelson ([2], 38) (for the case of linear equality constraints and assuming $\partial^2 f^*/\partial\alpha\partial x_i = 1$) and they are known in the economics literature as Samuelson's Le Chatelier principle. Alternative proofs for (63) for the case of general equality constraints and assuming $\partial^2 f^*/\partial\alpha\partial x_i = 1$ were given by Silberberg[19, 27].

5. CONCLUSION

Much of this paper has been concerned with local comparative statics or sensitivity theorems associated with the nonlinear programming problem (1). In particular, the reader will have noticed that many modern microeconomic theorems are intimately associated with the structure of A^{-1} where A was defined by (11).

Economists have also made some contributions to global sensitivity analysis; e.g. see Samuelson ([2], pp. 46–51; pp. 107–111), Bailey [20], Beckmann [21], Kusumoto [1978] and Anderson and Takayama [22].

Finally, it should be mentioned that Silberberg [19] and Hatta [23] have provided alternative approaches to the derivation of local comparative statics theorems.

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