8. Similarity Indexes and Criteria for Spatial Linking

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8.1. INTRODUCTION[1]

One of the most difficult problems in economics from both a theoretical and applied point of view is the problem of making international comparisons of prices and quantities (or volumes) between countries.

Three broad approaches to this problem have been followed in the literature:

- Use the star system where each country in the comparison group is compared to a "star" country using normal bilateral price and quantity indexes;[2]
- Use a symmetric multilateral system where every country's data enters the multilateral formula in a completely symmetric manner;[3] or
- Use spatial or geographic linking of similar countries and eventually find a "tree" that compares all countries using a bilateral index number formula to link each pair of countries.[4] This is the spatial linking method.

If there are incomplete price and quantity data for the countries in the comparison, then it is likely that the third approach is the most promising since the other two approaches require a substantial degree of overlap between prices and quantities of each country. The third method offers the possibility of making comparisons between countries that have the most "similar" price and quantity structures.

However, an essential input into the spatial linking method is a criterion for determining which pair of countries have the most "similar" price or quantity structures. Hill (1995) used the spread between the Paasche and Laspeyres price indexes as an indicator of similarity.[5] Thus let $p^i \equiv [p_1^i,...,p_N^i]$ and $q^i \equiv [q_1^i,...,q_N^i]$ be the price and quantity[6] vectors for country i for i = 1,2. The Laspeyres and Paasche price indexes comparing the prices between the two countries are P_L and P_P defined as follows:

$$P_L(p^1,p^2,q^1,q^2) \equiv p^2\cdot q^1/p^1\cdot q^1 ; \qquad (8.1)$$
$$P_P(p^1,p^2,q^1,q^2) \equiv p^2\cdot q^2/p^1\cdot q^2 \qquad (8.2)$$

where $p^i\cdot q^j \equiv \sum_{n=1}^{N} p_n^i q_n^j$ is the inner product of the vectors p^i and q^j. Hill defined the price structures between the two countries to be more dissimilar the bigger is the spread between P_L and P_P; i.e., the bigger is max $\{P_L/P_P, P_P/P_L\}$. The problem with this measure of dissimilarity in the price structures of the two countries is that we could have $P_L = P_P$ (so

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[2] Kravis (1984; 10) seems to have introduced this terminology.

[3] See Balk (1996) (2001) and Diewert (1999) for surveys of these methods.

[4] Fisher (1922; 271-276) hinted at the possibility of using spatial linking; i.e., of linking countries that are similar in structure. However, the modern literature has grown due to the pioneering efforts of Robert Hill (1995) (1999a) (1999b) (2001).

[5] It turns out that this criterion gives the same results as a criterion that looks at the spread between the Paasche and Laspeyres quantity indexes.

[6] Typically, the quantity vectors will be per capita quantity vectors.

that the Hill measure would register a maximal degree of similarity) but p^1 could be very different than p^2. Thus there is a need for a more systematic study of similarity (or dissimilarity) measures in order to pick the "best" one that could be used as an input into Hill's (1999a) (1999b) (2001) spanning tree algorithm for linking countries.

The present paper will take an axiomatic approach to both relative and absolute indexes of price and quantity dissimilarity.[7] An *absolute index of price dissimilarity* regards p^1 and p^2 as being dissimilar if $p^1 \neq p^2$ whereas a *relative index of price dissimilarity* regards p^1 and p^2 as being dissimilar if $p^1 \neq \lambda p^2$ where $\lambda > 0$ is an arbitrary positive number. Thus the relative index regards the two price vectors as being dissimilar only if *relative prices* differ in the two countries.

The relative index concept seems to be the most useful for judging whether the structure of prices is similar or dissimilar across two countries. However, assuming that the quantity vectors being compared are per capita quantity vectors, then the absolute concept seems to be more appropriate for judging the degree of similarity across countries. If per capita quantity vectors are quite different, then it is quite likely that the rich country is consuming (or producing) a very different bundle of goods and services than the poorer country and hence big disparities in the absolute level of q^1 versus q^2 are likely to indicate that the components of these two vectors are really not very comparable. In any case, it is of some interest to develop the theory for both the absolute and relative concepts.

In addition to being an essential building block for the third approach to making international comparisons listed above, relative indexes of price and quantity similarity or dissimilarity are very useful in deciding how to aggregate up a large number of price and quantity series into a smaller number of aggregates.[8] Finally, absolute indexes of dissimilarity can be useful in deciding when an observation in a large cross sectional data set is an outlier.[9]

In section 8.2 below, we study absolute dissimilarity indexes when the number of commodities being compared is only one. We offer what we think are a fairly fundamental set of axioms or properties that such an absolute dissimilarity index should satisfy and characterize the set of indexes which satisfy these axioms. In section 8.3, we take a somewhat different approach to the determination of a functional form for a dissimilarity index in the case where N = 1. In section 8.4, we add some additional axioms in an attempt to pin down the exact functional form of the absolute index in the case where N = 1.

In section 8.5, we extend the axioms to cover the case where the number of commodities is arbitrary.

Section 8.6 modifies the previous analysis relating to absolute indexes to relative dissimilarity indexes.

Section 8.7 looks at the properties of some of the relative dissimilarity indexes that have been suggested in the literature by using measures of the angle between the vectors x and y.

Sections 8.8 and 8.9 extend the analysis to weighted absolute and relative dissimilarity indexes.

Section 8.10 concludes with a discussion of which of the many functional forms for dissimilarity indexes that are exhibited that we think might be most useful in the spatial linking context.

An Appendix has proofs of the Propositions.

8.2. FUNDAMENTAL AXIOMS FOR ABSOLUTE DISSIMILARITY INDEXES: THE CASE WHERE N EQUALS ONE

We denote an *absolute dissimilarity index* as a function of two variables, $d(x,y)$, where x and y are restricted to be positive scalars. The two variables x and y could be the two prices of the first commodity in the two countries, p_1^1 and p_1^2, or they could be the two per capita quantities of the first commodity in the two countries, q_1^1 and q_1^2. It is obvious that $d(x,y)$ could be considered to be a distance function of the type that occurs in the mathematics literature. However, it turns out that the axioms that we impose on $d(x,y)$ are somewhat unconventional as we shall see.

The 6 fundamental axioms or properties that we think an absolute dissimilarity index should satisfy are the following ones[10]:

A1: *Continuity*: $d(x,y)$ is a continuous function defined for all $x > 0$ and $y > 0$.

[7] Allen and Diewert (1981; 433) made a start on such an axiomatic approach. Sergeev (2001) also pursued an axiomatic approach to similarity indexes.
[8] For applications along these lines, see Allen and Diewert (1981).
[9] See Fox, Hill and Diewert (2004) for examples of this use for a dissimilarity index.
[10] Counterparts to Axioms A2-A6 in the context of relative dissimilarity indexes were proposed by Allen and Diewert (1981; 433). Sergeev (2001; 4) also proposed counterparts to A2, A4 and A6 in the context of similarity indexes (as opposed to dissimilarity indexes).

A2: *Identity*: $d(x,x) = 0$ for all $x > 0$.
A3: *Positivity*: $d(x,y) > 0$ for all $x \neq y$.
A4: *Symmetry*: $d(x,y) = d(y,x)$ for all $x > 0$ and $y > 0$.
A5: *Invariance to Changes in Units of Measurement*: $d(\alpha x,\alpha y) = d(x,y)$ for all $\alpha > 0$, $x > 0$, $y > 0$.
A6: *Monotonicity*: $d(x,y)$ is increasing in y if $y \geq x$.

Some comments on the axioms are in order. The continuity assumption is generally made in order to rule out indexes that behave erratically. The identity assumption is a standard one in the mathematics literature; i.e., the absolute distance between two points x and y is zero if x equals y. A3 tells us that there is a positive amount of dissimilarity between x and y if x and y are different.[11] The symmetry property is very important: it says that the degree of dissimilarity between x and y is independent of the ordering of x and y. A5 is another important property from the viewpoint of economics: since units of measurement for commodities are essentially arbitrary, we would like our dissimilarity measure to be independent of the units of measurement. Finally, A6 says that as y gets bigger than x, the degree of dissimilarity between x and y grows. This is a very sensible property.

It turns out that there is a fairly simple characterization of the class of dissimilarity indexes $d(x,y)$ that satisfy the above axioms; i.e., we have the following Proposition:

Proposition 1: Let $d(x,y)$ be a function of two variables that satisfies the axioms A1-A6. Then $d(x,y)$ has the following representation:

$$d(x,y) = f[\max\{x/y,\ y/x\}] \tag{8.3}$$

where $f(u)$ is a continuous, monotonically increasing function of one variable, defined for $u \geq 1$ with the following additional property:

$$f(1) = 0. \tag{8.4}$$

Conversely, if $f(u)$ has the above properties, then $d(x,y)$ defined by (8.3) has the properties A1-A6.

A proof of this Proposition (and the other Propositions which follow) may be found in the Appendix.

Example 1: **The asymptotically linear dissimilarity index.** Let $f(u) \equiv u + u^{-1} - 2$ for $u \geq 1$. Note that $f'(u) = 1 - u^{-2} > 0$ for $u > 1$, which shows that $f(u)$ is increasing for $u \geq 1$. Note that as u tends to infinity, $f(u)$ approaches the linear function $u - 2$. Hence $f(u)$ is asymptotically linear. Since $f(1) = 0$, we see that $f(u)$ satisfies the required regularity conditions and the associated absolute dissimilarity index is[12]

$$d(x,y) = (x/y) + (y/x) - 2 = [(x/y) - 1] + [(y/x) - 1]\ ; \qquad x > 0\ ;\ y > 0 \tag{8.5}$$

and it satisfies the axioms A1-A6.

Example 2: **The asymptotically quadratic dissimilarity index.** Let $f(u) \equiv [u - 1]^2 + [u^{-1} - 1]^2$ for $u \geq 1$. Note that $f'(u) = 2[u - 1] + 2[u^{-1} - 1](-1)u^{-2} > 0$ for $u > 1$, which shows that $f(u)$ is increasing for $u \geq 1$. Since $f(1) = 0$, we see that $f(u)$ satisfies the required regularity conditions and the associated absolute dissimilarity index is

$$d(x,y) = [(x/y) - 1]^2 + [(y/x) - 1]^2\ ; \qquad x > 0\ ;\ y > 0 \tag{8.6}$$

and it satisfies the axioms A1-A6.

Note that for both of these examples, the resulting d(x,y) is infinitely differentiable.

In the following section, we show how a large class of one variable dissimilarity indexes can be defined. Then in section 8.4, we will add some additional axioms in an attempt to narrow down the choice of a particular index to be used in applications.

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[11] It can be shown that A3 is implied by the other axioms.

[12] If $x \geq y$, then max $\{x/y,\ y/x\}$ is x/y and $d(x,y) \equiv f[\max\{x/y,\ y/x\}] = f[x/y] = (x/y) + (y/x) - 2$. If $y \geq x$, then max $\{x/y,\ y/x\}$ is y/x and $d(x,y) \equiv f[\max\{x/y,\ y/x\}] = f[y/x] = (y/x) + (x/y) - 2 = (x/y) + (y/x) - 2$.

8.3. AN ALTERNATIVE APPROACH FOR GENERATING ABSOLUTE DISSIMILARITY INDEXES

Let g and h be continuous monotonically increasing functions of one variable with $g(0) = 0$ and consider the following class of dissimilarity indexes:

$$d_{g,h}(x,y) \equiv g\{|h(y/x) - h(1)|\}. \tag{8.7}$$

Thus we first transform y/x and 1 by the function of one variable h, calculate the difference, $h(y/x) - h(1)$, take the absolute value of this difference and then transform this difference by g.

It is easy to verify that the d defined by (8.7) satisfies all of the axioms A1-A6 with the exception of A4, the symmetry axiom, $d(x,y) = d(y,x)$. However, this defect can be readily overcome. Note that $d_{g,h}(y,x) \equiv g\{|h(x/y) - h(1)|\}$ also satisfies A1-A6 with the exception of A4. Thus, if we take a *symmetric mean*[13] of these two indexes[14], we will obtain a new index which satisfies axiom A4. Hence, let m be a symmetric mean function of two variables and let g and h be continuous monotonically increasing functions of one variable with $g(0) = 0$ and consider the following class of *symmetric monotonic transformation dissimilarity indexes*:

$$d_{g,h,m}(x,y) \equiv m[g\{|h(y/x) - h(1)|\}, g\{|h(x/y) - h(1)|\}. \tag{8.8}$$

Proposition 2: Let g and h be continuous monotonically increasing functions of one variable with $g(0) = 0$ and let $m(a,b)$ be a symmetric mean. Then each member of the class of symmetric monotonic transformation indexes $d_{g,h,m}(x,y)$ defined by (8.8) satisfies the axioms A1-A6.

Let us try and specialize the class of functional forms defined by (8.8). The simplest symmetric mean m of two numbers is the arithmetic mean and so let us set $m(a,b) = (1/2)a + (1/2)b$. It is also convenient to get rid of the absolute value function in (8.8) (so that the resulting dissimilarity index will be differentiable) and this can be done in the most simple fashion by setting $g(u) = u^2$.[15] This leads us to following class of *simple symmetric transformation dissimilarity indexes*, which depends only on the continuous monotonic function h:

$$d_h(x,y) \equiv (1/2)[h(y/x) - h(1)]^2 + (1/2)[h(x/y) - h(1)]^2. \tag{8.9}$$

The two simplest choices for h are $h(u) \equiv u$ and $h(u) \equiv \ln u$.[16] These two choices for h lead to the following concrete dissimilarity indexes:

Example 3: The linear quadratic dissimilarity index:

$$d(x,y) \equiv (1/2)[(y/x) - 1]^2 + (1/2)[(x/y) - 1]^2. \tag{8.10}$$

Note that this example is essentially the same as example 2.

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[13] Diewert (1993; 361) defined a *symmetric mean* of a and b as a function $m(a,b)$ that has the following properties: (1) $m(a,a) = a$ for all $a > 0$ (mean property); (2) $m(a,b) = m(b,a)$ for all $a > 0$, $b > 0$ (symmetry property); (3) m(a,b) is a continuous function for $a > 0$, $b > 0$ (continuity property); (4) $m(a,b)$ is a strictly increasing function in each of its variables (increasingness property).

[14] Our method for converting a measure that is not symmetric into a symmetric method is the counterpart to Irving Fisher's (1922) *rectification* procedure, which is actually due to Walsh (1921). A rectified index number formula satisfies the time reversal test; i.e., it is symmetric in its treatment of time.

[15] There is another good reason for this choice of g. In most applications, we want the slope of $g(u)$ to be zero at $u = 0$ and then increase as u increases. This means the amount of dissimilarity between x and y will be close to zero in a neighborhood of points where x is close to y but the degree of dissimilarity will grow at an increasing rate as x diverges from y. We will formalize these properties as axioms A7 and A8 in the next section. Hence if we want the slope of $g(u)$ to increase at a constant rate as u increases, then $g(u) = u^2$ is the simplest function which will accomplish this task.

[16] Bert Balk in a comment on an earlier version of this paper suggested the following choice for h: $h(u) \equiv u^{1/2}$.

Example 4: The log quadratic dissimilarity index:

$$\begin{aligned} d(x,y) &\equiv (1/2)[\ln(y/x) - \ln(1)]^2 + (1/2)[\ln(x/y) - \ln(1)]^2 \qquad (8.11) \\ &= (1/2)[\ln y - \ln x]^2 + (1/2)[\ln x - \ln y]^2 \\ &= [\ln y - \ln x]^2 \\ &= [\ln(y/x)]^2. \end{aligned}$$

Our conclusion at this point is that even in the one variable case, there are a large number of possible measures of absolute dissimilarity that could be chosen. Hence, in the following section, we add some additional axioms to our list of axioms, A1-A6, in an attempt to narrow down this large number of possible choices.

8.4 ADDITIONAL AXIOMS FOR ONE VARIABLE ABSOLUTE DISSIMILARITY INDEXES

Consider the following axiom:

A7: *Convexity*: $d(x,y)$ is a convex function of y for $y \geq x > 0$.

The meaning of this axiom is that we want the amount of dissimilarity between x and y to grow at a constant or increasing rate as y grows bigger than x. Put another way, we do not want the rate of increase in dissimilarity to *decrease* as y grows bigger than x. Although this property seems to be a reasonable one for many purposes, it must be conceded that this property is not as fundamental as the previous 6 properties.

Proposition 3: The asymptotically linear dissimilarity index defined by (8.5) and the linear quadratic dissimilarity index defined by (8.10) satisfy the convexity axiom A7 but the log quadratic dissimilarity index defined by (8.11) does *not* satisfy A7.[17]

How can we choose between the asymptotically linear dissimilarity index defined by (8.5) and the asymptotically quadratic dissimilarity index defined by (8.6) or (8.10)? Both indexes behave similarly for x close to y but as y diverges from x, the amount of dissimilarity between x and y will grow roughly quadratically in y for the index defined by (8.10) whereas for the index defined by (8.5), the amount of dissimilarity will tend towards a linear in y rate. Hence the choice between the two indexes depends on how fast one wants the amount of dissimilarity between x and y to grow as y grows bigger than x. It should be noted that the index defined by (8.10) will be much more sensitive to outliers in the data so perhaps for this reason, the index defined by (8.5) should be used when there is the possibility of errors in the data.[18]

Another axiom which is also not fundamental but does seem reasonable is the following one:

A8: *Differentiability*: $d(x,y)$ is a once differentiable function of two variables.

The real impact of the axiom A8 is along the ray where $x = y$. If we look at the proof of Proposition 1, we see that if we add A8 to the list of axioms, the effect of the differentiability axiom is to force the derivative of f(u) at $u = 1$ to be 0; i.e., under A8, we must have $f'(1) = 0$. In many applications, this will be a very reasonable restriction on f since it implies that the amount of dissimilarity between x and y will be very small when x is very close to y. All of our examples 1 to 4 above satisfy the differentiability axiom.

We now consider another axiom for $d(x,y)$, which is more difficult to justify, but it does determine the functional form for d:

A9: *Additivity*: $d(x,x + y + z) = d(x,x + y) + d(x,x + z)$ for $x > 0$ $y \geq 0$ and $z \geq 0$.

[17] If $d(x,y)$ is defined by (8.11) so that $d(x,y) = [\ln y - \ln x]^2$, then note that this function is a convex function of $\ln y$ for $\ln y \geq \ln x$. In other words, the failure of convexity of the log quadratic dissimilarity index is not too severe since it is convex in $\ln y$ rather than in y.

[18] If there are possible outliers in the data, then again the index defined by (8.5), the asymptotically linear index, should be preferred to the log quadratic dissimilarity index defined by (8.11).

Proposition 4: Suppose $d(x,y)$ satisfies the axioms A1-A6 and A9. Then d has the following functional form:[19]

$$d(x,y) = \alpha[\max\{x/y,\ y/x\} - 1] \tag{8.12}$$

where $\alpha > 0$.

Let us set $\alpha = 1$ in (8.12) and call the resulting $d(x,y)$, **example 5**, **the linear dissimilarity index**. It can be seen that for large y, the dissimilarity indexes defined by examples 1 and 5 will approach each other. The big difference between the two indexes is along the ray where $x = y$: the linear dissimilarity index will not be differentiable along this ray, whereas the asymptotically linear dissimilarity index will be differentiable everywhere. Also for x close to y, the linear dissimilarity index will be greater than the corresponding asymptotically linear dissimilarity measure.

We conclude this section by indicating a simple way for determining the exact functional form for $d(x,y)$: we need only consider the behavior of $d(1,y)$ for $y \geq 1$. This behavior of the function d determines the underlying generator function $f(u)$ that appeared in Proposition 1. Hence consider the following axioms for d:

A10: $d(1,y) = (y - 1)^{\beta}$; $y \geq 1$, where $\beta > 0$;
A11: $d(1,y) = \ln y$; $y \geq 1$;
A12: $d(1,y) = e^{y}$; $y \geq 1$.

It is straightforward to show that if $d(x,y)$ satisfies A1-A6 and A10, then d is equal to the following function: (**example 6**):

$$d(x,y) \equiv [\max\{x/y,\ y/x\} - 1]^{\beta}\ ; \qquad \beta > 0. \tag{8.13}$$

Of course, if $\beta = 1$, then Example 6 reduces to Example 5.[20]

Similarly, it is straightforward to show that if $d(x,y)$ satisfies A1-A6 and A11, then d is equal to the following function: (**example 7**):[21]

$$d(x,y) \equiv \ln\,[\max\{x/y,\ y/x\}]. \tag{8.14}$$

Finally, if $d(x,y)$ satisfies A1-A6 and A12, then d is equal to the following function: (**example 8**):[22]

$$d(x,y) \equiv e^{\max\{x/y,y/x\}} - e. \tag{8.15}$$

The functional forms for the dissimilarity indexes defined by (8.13)-(8.15) are all relatively simple but they all have a disadvantage: namely, *they are not differentiable along the ray where $x = y$.* Hence, they are probably not suitable for many economic applications.

We turn now to N variable measures of absolute dissimilarity.

8.5 AXIOMS FOR ABSOLUTE DISSIMILARITY INDEXES IN THE N VARIABLE CASE

We now let $x \equiv [x_1,...,x_N]$ and $y \equiv [y_1,...,y_N]$ be strictly positive vectors[23] (either price or quantity) that are to be compared in an absolute sense. Let $D(x,y)$ be the absolute dissimilarity index, defined for all strictly positive vectors x and y. The following 6 axioms or properties are fairly direct counterparts to the 6 fundamental axioms that were introduced in section 8.2 above.

[19] The $f(u)$ that corresponds to this functional form is $f(u) \equiv \alpha[u - 1]$ where $\alpha > 0$. The $d(x,y)$ defined by (8.12) also satisfies the convexity axiom A7 but it does not satisfy the differentiability axiom A8.
[20] The $d(x,y)$ defined by (8.13) satisfies the convexity axiom A8 if and only if $\beta \geq 1$.
[21] This $d(x,y)$ does not satisfy A8.
[22] This $d(x,y)$ does satisfy the convexity axiom A8.
[23] Notation: $x >> 0_N$ means that each component of x is positive; $x \geq 0_N$ means that each component of x is nonnegative.

B1: *Continuity*: $D(x,y)$ is a continuous function defined for all $x >> 0_N$ and $y >> 0_N$.
B2: *Identity*: $D(x,x) = 0$ for all $x >> 0_N$.
B3: *Positivity*: $D(x,y) > 0$ for all $x \neq y$.
B4: *Symmetry*: $D(x,y) = D(y,x)$ for all $x >> 0_N$ and $y >> 0_N$.
B5: *Invariance to Changes in Units of Measurement*: $D(\alpha_1 x_1,...,\alpha_N x_N ; \alpha_1 y_1,...,\alpha_N y_N) = D(x_1,...,x_N; y_1,...,y_N) = D(x,y)$ for all $\alpha_n > 0$, $x_n > 0$, $y_n > 0$ for n = 1,...,N.[24]
B6: *Monotonicity*: $D(x,y)$ is increasing in the components of y if $y \geq x$.

The above axioms or properties can be regarded as fundamental. However, they are not sufficient to give a nice characterization Proposition like Proposition 1 in section 2. Hence we need to add additional properties to determine D.

Possible additional properties are the following ones:

B7: *Invariance to the ordering of commodities*: $D(Px,Py) = D(x,y)$ where Px denotes a permutation of the components of the x vector and Py denotes the same permutation of the components of the y vector.
B8: *Additive Separability*: $D(x,y) = \sum_{n=1}^{N} d_n(x_n,y_n)$.

The N functions of two variables, $d_n(x_n,y_n)$, are obviously absolute dissimilarity measures that give us the degree of dissimilarity between the components of the vectors x and y.

Proposition 5: Suppose $D(x,y)$ satisfies B1-B8. Then there exists a continuous, increasing function of one variable, $f(u)$, such that $f(1) = 0$ and $D(x,y)$ has the following representation in terms of f:

$$D(x,y) \equiv \sum_{n=1}^{N} f[\max\{x_n/y_n,\ y_n/x_n\}]. \tag{8.16}$$

Conversely, if $D(x,y)$ is defined by (8.16) where f is a continuous, increasing function of one variable with $f(1) = 0$, then D satisfies B1-B8.

Thus adding the axioms B7 and B8 to the earlier axioms B1-B6 essentially reduces the N dimensional case down to the one dimensional case.

In applications, it is sometimes useful to be able to compare the amount of dissimilarity between two N dimensional vectors x and y to the amount of dissimilarity between two M dimensional vectors u and v. If we decide to use the function of one variable f to generate the dissimilarity index defined by (8.16), then we can achieve comparability across vectors of different dimensionality if we modify (8.16) and define the following family of dissimilarity indexes (which depend on N, the dimensionality of the vectors x and y):

$$D_N(x,y) \equiv \sum_{n=1}^{N} (1/N) f[\max\{x_n/y_n,\ y_n/x_n\}]. \tag{8.17}$$

Recall examples 1 and 2 in section 8.2. We now use the generating functions $f(u)$ for these examples to construct N variable measures of absolute dissimilarity between the positive vectors x and y. Using the generating function $f(u) \equiv [u - 1] + [u^{-1} - 1]$ in (8.16) gives us the following *N dimensional asymptotically linear index of absolute dissimilarity*, which is the N dimensional generalization of Example 1 above, which we now label as **example 9**:

$$D_{AL}(x,y) \equiv (1/N)\sum_{n=1}^{N} [(y_n/x_n) + (x_n/y_n) - 2]. \tag{8.18}$$

Using the generating function $f(u) \equiv [u - 1]^2 + [u^{-1} - 1]^2$ in (8.16) gives us the following *N dimensional asymptotically quadratic index of absolute dissimilarity*, which is the N dimensional generalization of Example 2 above, which we now label as **example 10**:

$$D_{AQ}(x,y) \equiv (1/N)\sum_{n=1}^{N} [(y_n/x_n) - 1]^2 + (1/N)\sum_{n=1}^{N} [(x_n/y_n) - 1]^2. \tag{8.19}$$

The indexes defined by (8.18) and (8.19) are our preferred indexes of absolute dissimilarity.

[24] Note that this axiom implies that D has the homogeneity property $D(\lambda x,\lambda y) = D(x,y)$. To see this, let each $\alpha_n = \lambda$.

It is possible to obtain an axiomatic characterization of a larger class of absolute dissimilarity indexes by dropping the additive separability assumption B8 and replacing it by the following weaker separability assumption:

B9: *Componentwise Symmetry*: $D(x_1,...,x_N;y_1,...,y_N) = D(y_1,x_2,...,x_N;x_1,y_2,...,y_N)$.

What is the meaning of property B9? Suppose that we are comparing the vectors x and y and we have calculated the dissimilarity measure $D(x,y) = D(x_1,...,x_N;y_1,...,y_N)$. Now suppose we interchange the first component of the x vector with the first component of the y vector and we calculate the dissimilarity measure for these new vectors, which will be $D(y_1,x_2,...,x_N;x_1,y_2,...,y_N)$. The axiom B9 says that we get our original dissimilarity measure, $D(x,y)$. Using B7, it can be seen that if we interchange component n of both the x and y vectors and compute the dissimilarity measure for the interchanged vectors, then B9 says that we get the original dissimilarity measure, $D(x,y)$. With the help of this last axiom, we can now derive the following counterpart to Proposition 1:

Proposition 6: Let $D(x,y)$ be a function of 2N variables that satisfies the axioms B1-B7 and B9. Then $D(x,y)$ has the following representation:

$$D(x,y) \equiv f[\max\{x_1/y_1, y_1/x_1\},\max\{x_2/y_2, y_2/x_2\},...,\max\{x_N/y_N, y_N/x_N\}] \quad (8.20)$$

where $f(u)$ is a symmetric continuous, monotonically increasing function of N variables, $u \equiv [u_1,...,u_N]$ defined for $u \geq 1_N$ with the following additional property:

$$f(1_N) = 0. \quad (8.21)$$

Conversely, if $f(u)$ has the above properties, then $D(x,y)$ defined by (8.20) has the properties B1-B7 and B9.

Thus absolute dissimilarity functions satisfying properties B1-B7 and B9 can all be generated (using formula (8.20) above) by a symmetric, continuous, increasing function of N variables f(u) defined for $u \geq 1_N$ which also satisfies (8.21). Examples 11 and 12 below satisfy all of the properties B1-B7 and B9. These examples show that the class of dissimilarity indexes defined by (8.20) in Proposition 6 is indeed larger than the class defined by (8.16) in Proposition 5.

Example 11: Define $f(u) \equiv \{\sum_{n=1}^N (1/N)[u_n - 1]^2\}^{1/2}$ for $u_n \geq 1$ for n = 1,...,N. The resulting $D(x,y)$ is

$$D(x,y) = \{\sum_{n=1}^N (1/N)[\max\{x_n/y_n, y_n/x_n\} - 1]^2\}^{1/2}. \quad (8.22)$$

Example 12 : Define $f(u) \equiv \prod_{n=1}^N [u_n - 1]^{1/N}$ for $u_n \geq 1$ for n = 1,...,N. The resulting $D(x,y)$ is

$$D(x,y) = \prod_{n=1}^N [\max\{x_n/y_n, y_n/x_n\} - 1]^{1/N}. \quad (8.23)$$

We turn now to a discussion of relative dissimilarity indexes in the case of N commodity prices or quantities that must be compared.[25]

8.6 AXIOMS FOR RELATIVE DISSIMILARITY INDEXES IN THE N VARIABLE CASE

In making relative comparisons, we regard x and y as being similar if x is proportional to y or if y is proportional to x; i.e., if $y = \lambda x$ for some scalar $\lambda > 0$. We denote the relative dissimilarity index between two vectors x and y by $\Delta(x,y)$. The earlier axioms B1-B7 for absolute dissimilarity indexes are now replaced by the following axioms:

C1: *Continuity*: $\Delta(x,y)$ is a continuous function defined for all $x >> 0_N$ and $y >> 0_N$.
C2: *Identity*: $\Delta(x,\lambda x) = 0$ for all $x >> 0_N$ and scalars $\lambda > 0$.
C3: *Positivity*: $\Delta(x,y) > 0$ if $y \neq \lambda x$ for any $\lambda > 0$.

[25] The case N = 1 is not relevant in the case of relative dissimilarity indexes (since in this case, any two positive numbers are relatively similar). Hence we assume that N ≥ 2 when discussing relative dissimilarity indexes.

C4: *Symmetry*: $\Delta(x,y) = \Delta(y,x)$ for all $x >> 0_N$ and $y >> 0_N$.
C5: *Invariance to Changes in Units of Measurement*: $\Delta(\alpha_1x_1,...,\alpha_Nx_N ;\alpha_1y_1,...,\alpha_Ny_N) = \Delta(x_1,...,x_N;y_1,...,y_N) = \Delta(x,y)$ for all $\alpha_n > 0$, $x_n > 0$, $y_n > 0$ for n = 1,...,N.
C6: *Invariance to the Ordering of Commodities*: $\Delta(Px,Py) = \Delta(x,y)$ where Px is a permutation or reordering of the components of x and Py is the same permutation of the components of y.
C7: *Proportionality*: $\Delta(x,\lambda y) = \Delta(x,y)$ for all $x >> 0_N$, $y >> 0_N$ and scalars $\lambda > 0$.

The last axiom says that the degree of relative dissimilarity between the vectors x and y remains the same if y is multiplied by the arbitrary positive number λ.

The above axioms all seem to be fairly fundamental in the relative dissimilarity index context.[26] We have not developed a counterpart to the absolute monotonicity axiom B6 for relative indexes of dissimilarity because it is not clear what the appropriate relative axiom should be. This is a topic for further research. Also, we do not have any nice characterization theorems for relative dissimilarity indexes that are analogous to the results in Propositions 5 and 6 in the previous section. However, we do have a strategy for adapting the absolute dissimilarity indexes to the relative context.

Our suggested strategy is this. First, find a *scale index* $S(x,y)$ that is essentially a price or quantity index between the vectors x and y and that has the property $S(x,\lambda x) = \lambda$ for all $\lambda > 0$. Second, find a suitable absolute dissimilarity index, $D(x,y)$. Finally, use the scale index S and the absolute dissimilarity index D in order to define the following relative dissimilarity index Δ:

$$\Delta(x,y) \equiv D(S(x,y)x, y). \tag{8.24}$$

Thus in (8.24), we scale up the base vector x by the index number $S(x,y)$ which makes it comparable in an absolute sense to the vector y. We then apply an absolute index of dissimilarity D to the scaled up x vector, $S(x,y)x$, and the vector y. Naturally, in order for the Δ defined by (8.24) to satisfy the axioms C1-C7, it will be necessary for D and S to satisfy certain properties. We will assume that the absolute dissimilarity index D satisfies B1-B5 and B7 in the previous section. We will also impose the following properties on the scale index $S(x,y)$:[27]

D1: *Continuity*: $S(x,y)$ is a continuous function defined for all $x >> 0_N$ and $y >> 0_N$.
D2: *Identity*: $S(x,x) = 1$ for all $x >> 0_N$.
D3: *Positivity*: $S(x,y) > 0$ for all $x >> 0_N$ and $y >> 0_N$.
D4: *Time or Place Reversal*: $S(x,y) = 1/S(y,x)$ for all $x >> 0_N$ and $y >> 0_N$.
D5: *Invariance to Changes in Units of Measurement*: $S(\alpha_1x_1,...,\alpha_Nx_N ;\alpha_1y_1,...,\alpha_Ny_N) = S(x_1,...,x_N;y_1,...,y_N) = S(x,y)$ for all $\alpha_n > 0$, $x_n > 0$, $y_n > 0$ for n = 1,...,N.
D6: *Invariance to the Ordering of Commodities*: $S(Px,Py) = S(x,y)$ where Px is a permutation or reordering of the components of x and Py is the same permutation of the components of y.
D7: *Proportionality*: $S(x,\lambda y) = \lambda S(x,y)$ for all $x >> 0_N$, $y >> 0_N$ and scalars $\lambda > 0$.

Proposition 7: If the scale function $S(x,y)$ satisfies D1-D7 and the absolute dissimilarity index $D(x,y)$ satisfies B1-B5 and B7 listed in the previous section, then the relative dissimilarity index $\Delta(x,y)$ defined by (8.24) satisfies properties C1-C7.

The above Proposition can be used in order to generate a wide class of relative dissimilarity indexes. Obviously, more work needs to be done in order to obtain characterization results that are similar to the Propositions in the previous section.

We conclude this section by giving some examples of how Proposition 7 could be applied in order to define some indexes of relative dissimilarity.

Example 13: Recall the N variable index of dissimilarity $D_{AL}(x,y)$ defined by (8.18) above. It can be verified that this absolute index of dissimilarity satisfies axioms B1-B9. We need to choose a scale index $S(x,y)$ that satisfies the axioms D1-D7. The simplest choice for such an S is:

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[26] Axioms C2-C7 were proposed by Allen and Diewert (1981; 433).

[27] Note that $S(x,y)$ can be interpreted as an elementary quantity index; i.e., an index that compares y to x when price weights are not available. For a discussion of the analogous elementary price index, see Diewert (2004a).

$$S_J(x,y) \equiv \Pi_{n=1}^{N} (y_n/x_n)^{1/N}. \tag{8.25}$$

Thus $S(x,y)$ is the geometric mean of the y_n divided by the geometric mean of the x_n. This functional form (for a price index) is due to Jevons (1865) and it is still used today as a functional form for an elementary price index. It can be verified that S_J satisfies the axioms D1-D7. It should be noted that the following scale indexes do *not* satisfy the time reversal test, D4:

$$S_A(x,y) \equiv \sum_{n=1}^{N} (1/N)(y_n/x_n) ; \tag{8.26}$$
$$S_H(x,y) \equiv [\sum_{n=1}^{N} (1/N)(y_n/x_n)^{-1}]^{-1}. \tag{8.27}$$

Note that S_A is the arithmetic mean[28] of the ratios y_n/x_n and S_H is the harmonic mean of the ratios y_n/x_n.

Inserting S_J defined by (8.25) into formula (8.24) where D is defined by (8.18) leads to the following *asymptotically linear index of relative dissimilarity* (which satisfies C1-C7):

$$\Delta_{AL}(x,y) \equiv D_{AL}(S_J(x,y)x, y) = \sum_{n=1}^{N}(1/N)[(S_J(x,y)x_n/y_n) + (y_n/S_J(x,y)x_n) - 2] . \tag{8.28}$$

Example 14: Recall the N dimensional asymptotically quadratic index of absolute dissimilarity, $D_{AQ}(x,y)$ defined by (8.19) above. It can be verified that this absolute index of dissimilarity satisfies axioms B1-B8. Inserting S_J defined by (8.25) into formula (8.24) where D is defined by (8.19) leads to the following *asymptotically quadratic index of relative dissimilarity* (which also satisfies C1-C7):

$$\begin{aligned} \Delta_{AQ}(x,y) &\equiv D_{AQ}(S_J(x,y)x, y) \\ &= \sum_{n=1}^{N}(1/N)[(S_J(x,y)x_n/y_n) - 1]^2 + \sum_{n=1}^{N}(1/N)[(y_n/S_J(x,y)x_n) - 1]^2. \end{aligned} \tag{8.29}$$

Example 15: Recall the log squared single variable measure of absolute dissimilarity defined by (8.11) above. The additively separable extension of this measure to the N variable case is the following *log squared index of absolute dissimilarity*:

$$D_{LS}(\mathrm{x,y}) \equiv \sum_{n=1}^{N} (1/N)[\ln(y_n/x_n)]^2. \tag{8.30}$$

It can be verified that this absolute index of dissimilarity satisfies axioms B1-B9. Inserting S_J defined by (8.25) into formula (8.24) where D is defined by (8.30) leads to the following *log squared index of relative dissimilarity* (which also satisfies C1-C7):

$$\begin{aligned} \Delta_{LS}(x,y) \equiv D_{LS}(S_J(x,y)x, y) &= \sum_{n=1}^{N} (1/N)[\ln(y_n/S_J(\mathbf{x},\mathbf{y})x_n)]^2 \\ &= (1/N)\sum_{n=1}^{N} [\ln(y_n/x_n) - \ln S_J(x,y)]^2 \\ &= (1/N)\sum_{n=1}^{N} [\ln(y_n/x_n) - \ln\{\Pi_{n=1}^{N} (y_n/x_n)^{1/N}\}]^2. \end{aligned} \tag{8.31}$$

The last line of (8.31) shows that $\Delta_{LQ}(x,y)$ is equal to a constant times the Allen Diewert (1981; 433) measure of nonproportionality between the vectors x and y. Allen and Diewert derived their measure by regressing the N logarithmic ratios, $\ln(y_n/x_n)$, on a constant, obtaining $(1/N)\sum_{n=1}^{N} \ln(y_n/x_n) = \ln\{\Pi_{n=1}^{N} (y_n/x_n)^{1/N}\}$ as the least squares estimator of this constant. They then used the sum of squared residuals from their regression as their measure of nonproportionality, which is N times the last line of (8.31).

Example 16: For this example, we again start off with the absolute dissimilarity index D_{AQ} defined by (8.19) but instead of using the geometric scale function $S_J(x,y)$, we use the harmonic scale function $S_H(x,y)$ defined by (8.27) in the following way:

$$\Delta_A(x,y) \equiv \sum_{n=1}^{N} (1/N)[(S_H(x,y)x_n/y_n) - 1]^2 + \sum_{n=1}^{N} (1/N)[(S_H(y,x)y_n/x_n) - 1]^2 \tag{8.32}$$

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[28] S_A is known in the price index literature as the Carli (1764) index; see Diewert (2004a). Note that the geometric mean of S_A and S_H does satisfy the axioms D1-D7 and hence could be used in place of the Jevons scale index S_J. $S_{AH}(x,y) \equiv [S_A(x,y)S_H(x,y)]^{1/2}$ has been suggested as the functional form for an elementary price index by Carruthers, Sellwood and Ward (1980).

$= \sum_{n=1}^{N} (1/N)[(x_n/y_nS_A(y,x)) - 1]^2 + \sum_{n=1}^{N} (1/N)[(y_n/S_A(x,y)x_n) - 1]^2$

since $S_H(x,y) = [S_A(y,x)]^{-1}$. Note that $S_A(x,y)$ is the arithmetic mean of the N ratios y_n/x_n and $S_A(y,x)$ is the arithmetic mean of the N ratios x_n/y_n. Thus in the first summation on the right hand side of (8.32), we divide each x_n/y_n by the arithmetic mean of these N ratios[29] and in the second summation, we divide each y_n/x_n by the arithmetic mean of these N ratios. Using (8.32), it can be shown that $\Delta_A(x,y) \neq D_{AQ}(S_H(x,y)x,y)$, since the x_n are multiplied by $S_H(x,y)$ in the first summation of terms in (8.32) and by $S_A(x,y)$ in the second summation of terms in (8.32). Nevertheless, the index of relative dissimilarity $\Delta_A(x,y)$ defined by (8.32) *does* satisfy all of the axioms C1-C7. The only axiom which requires a bit of computation to check is the symmetry axiom, namely that $\Delta_A(x,y) = \Delta_A(y,x)$. We have, using definition (8.32):

$$\Delta_A(y,x) \equiv \sum_{n=1}^{N} (1/2N)[(S_H(y,x)y_n/x_n) - 1]^2 + \sum_{n=1}^{N} (1/2N)[(S_H(x,y)x_n/y_n) - 1]^2 \quad (8.33)$$
$$= \sum_{n=1}^{N}(1/2N)[(S_H(x,y)x_n/y_n) - 1]^2 + \sum_{n=1}^{N}(1/2N)[(S_H(y,x)y_n/x_n) - 1]^2 \quad \text{interchanging the two sums}$$
$$\equiv \Delta_A(x,y) \quad \text{using definition (8.32).}$$

We turn now to a rather different approach that has been used to derive measures of relative dissimilarity.

8.7 ANGULAR AND LEAST SQUARES MEASURES OF RELATIVE DISSIMILARITY

In this section, we consider a somewhat different approach to obtaining relative dissimilarity measures between two vectors. These methods rely on the Cauchy Schwarz inequality or on the theory of correlation coefficients and they were pioneered by Kravis, Summers and Heston (1982) and Sergeev (2001) (2005).

Let us start with an approach based on the Cauchy Schwarz inequality, which states that for two nonzero vectors x and y, $(x{\cdot}y)^2 \leq (x{\cdot}x)(y{\cdot}y)$, with a strict inequality unless $x = \lambda y$ for some number $\lambda \neq 0$. In economic applications, x and y will be positive vectors in which case, the inequality can be rewritten as follows:

$$0 < (x{\cdot}y)^2/(x{\cdot}x)(y{\cdot}y) \leq 1 \quad (8.34)$$

with the upper bound holding as an equality if and only if $x = \lambda y$. Hence we can define *the Cauchy Schwarz relative dissimilarity index* as

$$\Delta_{CS}(x,y) \equiv 1 - (x{\cdot}y)^2/(x{\cdot}x)(y{\cdot}y). \quad (8.35)$$

What are the properties of Δ_{CS}? It can be verified that Δ_{CS} satisfies all of the axioms C1-C7 except axiom C5, the invariance to changes in the units of measurement property. This is a fatal flaw so we conclude that Δ_{CS} is not suitable as an index of relative dissimilarity.

Let us attempt to overcome this difficulty. Thus define the components of the vectors $r \equiv [r_1,...,r_N]$ and $s \equiv [s_1,...,s_N]$ as follows:

$$r_n \equiv y_n/x_n\ ;\ s_n \equiv x_n/y_n\ ;\ n = 1,...,N. \quad (8.36)$$

Thus the r_n are just the ratios y_n/x_n and the s_n are the reciprocals of these ratios. Now apply the Cauchy Schwarz inequality to the vector r and the vector of 1's, 1_N. We obtain the following counterpart to (8.34):

$$0 < (r{\cdot}1_N)^2/(r{\cdot}r)(1_N{\cdot}1_N) \leq 1 \quad (8.37)$$

with the upper bound holding if and only if $\lambda r = 1_N$ or $x = \lambda y$.[30] Hence we can define a *new Cauchy Schwarz relative dissimilarity index* as

$$\Delta_r(x,y) \equiv 1 - \{(r{\cdot}1_N)^2/(r{\cdot}r)(1_N{\cdot}1_N)\}. \quad (8.38)$$

---

[29] Note that $S_H(x,y)x_n/y_n = (x_n/y_n)/S_A(y,x) = (x_n/y_n)/[\sum_{i=1}^{N} (1/N)(x_i/y_i)]$

[30] Taking into account the positivity of r, the lower bound in (8.37) is 1/N, which can be approached as r tends to e_n, the nth unit vector.

It can be verified that Δ_r satisfies all of the axioms C1-C7 except axiom C4, the symmetry property. This is a fatal flaw so we conclude that Δ_r is also not suitable as an index of relative dissimilarity.

We can also define another *new Cauchy Schwarz relative dissimilarity index* using the vector s instead of r as

$$\Delta_s(x,y) \equiv 1 - (s\cdot 1_N)^2/(s\cdot s)(1_N\cdot 1_N). \qquad (8.39)$$

Of course, Δ_s also satisfies all of the axioms C1-C7 except axiom C4, the symmetry property. This is a fatal flaw so we again conclude that Δ_s is not suitable as an index of relative dissimilarity.[31]

Now let $m(a,b)$ be a symmetric mean of a and b. Use this mean function to define yet *another class of relative dissimilarity indexes* (**example 17**):

$$\Delta_m(x,y) \equiv m[\Delta_r(x,y), \Delta_s(x,y)] \qquad (8.40)$$

where $\Delta_r(x,y)$ and $\Delta_s(x,y)$ are defined by (8.38) and (8.39).

It is straightforward to verify that the class of *symmetric mean relative dissimilarity indexes* defined by (8.38)-(8.40) satisfies all of the axioms C1-C7.[32] If we let $m(a,b)$ be the arithmetic mean, then (8.40) becomes the following *arithmetic mean relative dissimilarity index* (**example 18**):

$$\Delta_a(x,y) \equiv 1 - \{(1/2)(r\cdot 1_N)^2/(r\cdot r)(1_N\cdot 1_N)\} - \{(1/2)(s\cdot 1_N)^2/(s\cdot s)(1_N\cdot 1_N)\}. \qquad (8.41)$$

We conclude that Δ_a defined by (8.41) is a perfectly acceptable index of relative dissimilarity.[33]

There is another way to derive an index of relative dissimilarity that follows the regression approach pioneered by Allen and Diewert (1981; 433) and also utilized by Sergeev (2001) (2005). Let us regress the ratios r_n on a constant and then calculate the resulting sum of squared residuals in order to obtain the following relative dissimilarity index:

$$\begin{aligned}\Delta_{r^*}(x,y) &\equiv r\cdot r - (r\cdot 1_N)^2/(1_N\cdot 1_N) && (8.42)\\ &= \textstyle\sum_{n=1}^N (r_n)^2 - N\, r^{*2} && \text{where } r^* \equiv (1/N)\textstyle\sum_{n=1}^N r_n \text{ is the arithmetic mean of the } r_n\\ &= \textstyle\sum_{n=1}^N (r_n - r^*)^2 \\ &\geq 0.\end{aligned}$$

In a similar manner, we can regress the ratios s_n on a constant and then calculate the resulting sum of squared residuals in order to obtain the following relative dissimilarity index:

$$\begin{aligned}\Delta_{s^*}(x,y) &\equiv s\cdot s - (s\cdot 1_N)^2/(1_N\cdot 1_N) && (8.43)\\ &= \textstyle\sum_{n=1}^N (s_n)^2 - N\, s^{*2} && \text{where } s^* \equiv (1/N)\textstyle\sum_{n=1}^N s_n \text{ is the arithmetic mean of the } s_n\\ &= \textstyle\sum_{n=1}^N (s_n - s^*)^2 \\ &\geq 0.\end{aligned}$$

It is straightforward to verify that the relative dissimilarity indexes $\Delta_{r^*}(x,y)$ and $\Delta_{s^*}(x,y)$ satisfy all of the axioms C1-C7 except the symmetry axiom C4 and the proportionality axiom C7. The failure of the symmetry axiom can be cured if we take a symmetric mean m of $\Delta_{r^*}(x,y)$ and $\Delta_{s^*}(x,y)$. The failure of test C7 can be cured in at least two ways by dividing $\Delta_{r^*}(x,y)$ and $\Delta_{s^*}(x,y)$ by appropriate normalizing factors. Thus for our first method of curing the problem, we divide $\Delta_{r^*}(x,y)$ by the positive number $r\cdot r$ and we divide $\Delta_{s^*}(x,y)$ by the sum of squares $s\cdot s$. This leads to the following *class of relative dissimilarity indexes for each symmetric mean function m:*

$$m[\Delta_{r^*}(x,y)/r\cdot r, \Delta_{s^*}(x,y)/s\cdot s] = \Delta_m(x,y) \equiv m[\Delta_r(x,y), \Delta_s(x,y)] \qquad (8.44)$$

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[31] Note that both $\Delta_r(x,y)$ and $\Delta_s(x,y)$ satisfy the inequalities $0 \leq \Delta(x,y) < 1 - (1/N)$, where we have used the positivity of x and y in deriving these bounds.

[32] In addition, $\Delta_m(x,y)$ will satisfy the inequalities $0 \leq \Delta_m(x,y) \leq 1 - (1/N)$.

[33] The relative dissimilarity indexes defined by (8.46)-(8.49) can be termed *angular measures of dissimilarity* since $x\cdot y/[x\cdot x\ y\cdot y]^{1/2}$ can be interpreted as a measure of the angle between the vectors x and y.

where $\Delta_r(x,y)$ and $\Delta_s(x,y)$ were defined by (8.38) and (8.39). Thus this first method leads back to the class of relative dissimilarity indexes, $\Delta_m(x,y)$, already defined by (8.40).[34] For the second method of curing the defects of (8.42) and (8.43), divide $\Delta_{r^*}(x,y)$ by the positive number r^{*2} and divide $\Delta_{s^*}(x,y)$ by the positive number s^{*2} and take a symmetric mean of these two numbers. This leads to the following *class of relative dissimilarity indexes for each symmetric mean function m* (**example 19**):

$$\begin{aligned} m[\Delta_{r^*}(x,y)/r^{*2}, \Delta_{s^*}(x,y)/s^{*2}] &= m[\{\textstyle\sum_{n=1}^N (r_n-r^*)^2\}/r^{*2}, \sum_{n=1}^N (s_n-s^*)^2/s^{*2}] \qquad \text{using (8.42) and (8.43)} \\ &= m[\textstyle\sum_{n=1}^N (\{r_n/r^*\} - 1)^2, \sum_{n=1}^N (\{s_n/s^*\} - 1)^2]. \end{aligned} \tag{8.45}$$

Let m be the arithmetic mean function and (8.45) becomes (**example 20**):

$$\begin{aligned} (1/2)[\Delta_{r^*}(x,y)/r^{*2}] + (1/2)[\Delta_{s^*}(x,y)/s^{*2}] &= (1/2)\textstyle\sum_{n=1}^N (\{r_n/r^*\} - 1)^2 + (1/2)\sum_{n=1}^N (\{s_n/s^*\} - 1)^2 \\ &= (1/2)\mathrm{N}\, \Delta_A(x,y) \end{aligned} \tag{8.46}$$

where $\Delta_A(x,y)$ is the relative dissimilarity index defined by (8.32) in example 16 above.

Thus, in this section, we have related both the Cauchy Schwarz and least squares regression type relative dissimilarity indexes to the relative dissimilarity indexes that were studied in Section 6 above.

Sergeev (2005) did not actually propose any of the above angular measures of *dissimilarity*. In fact, Sergueev preferred to work with *similarity* measures and his preferred (unweighted) *measure of relative similarity* was (**example 21**):

$$\begin{aligned} S_S(x,y) &\equiv [r\cdot 1_N\, s\cdot 1_N/r\cdot r\; s\cdot s]^{1/2} \\ &= [r\cdot 1_N/(r\cdot r\; 1_N\cdot 1_N)^{1/2}]^{1/2}\, [s\cdot 1_N/(s\cdot s\; 1_N\cdot 1_N)^{1/2}]^{1/2}\, [\mathrm{N}/(r\cdot r)]^{1/4}\, [\mathrm{N}/(s\cdot s)]^{1/4}. \end{aligned} \tag{8.47}$$

Using the positivity of r and s and the Cauchy Schwarz inequality, it can be seen that the first two terms on the right hand side of (8.47) are between 1/N and 1. Since the s_n are the reciprocals of the r_n, it can be verified by solving a minimization problem that:

$$(r\cdot r)^{1/2}\,(s\cdot s)^{1/2} \geq \mathrm{N} \tag{8.48}$$

which shows that the last two terms on the right hand side of (8.47) are bounded from above by 1 (and the positivity of r and s implies that the last two terms are also bounded from below by 0). Hence Sergeev's similarity measure is bounded from above by 1 (maximum similarity of the two vectors being compared) and bounded from below by 0 (minimum similarity).[35] Hence $1 - S_S(x,y)$ is a dissimilarity index.[36] It is straightforward to show that it satisfies the axioms C1-C7.

We turn now to weighted absolute and relative dissimilarity indexes.

8.8 WEIGHTED ABSOLUTE DISSIMILARITY INDEXES

The analysis up to this point has implicitly assumed (using the axioms B7 or C6) that the amount of dissimilarity between each component of the x and y vectors is equally important and hence gets an equal weight in the overall index of dissimilarity. In many applications, this assumption is not justified, which suggests that the individual component measures of dissimilarity should be weighted according to the *economic importance* of that commodity. However,

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[34] This shows that our earlier class of angular relative dissimilarity indexes $\Delta_m(x,y)$ is in fact closely related to the radial or circular type dissimilarity indexes defined by (8.40) and (8.41).

[35] Sergeev (2005) argues that the fact that his measure is bounded from both above and below by a finite number is an advantage of his relative similarity measure over the relative dissimilarity measures that follow along the Allen and Diewert (1981) analysis where there is a finite lower bound but an infinite upper bound. However, if m is an Allen-Diewert type measure of dissimilarity that takes on values between zero and plus infinity, then $m/(1+m)$ is a transformation of the original measure that takes on values between 0 and 1. A more important consideration is to obtain dissimilarity measures that are comparable across situations where N differs. Hence, in the present version of this paper, we have tried to suggest measures that are comparable across varying N.

[36] Note the relationship of $1 - S_S(x,y)$ to the dissimilarity indexes defined by (8.38) and (8.39).

there are several ways that this economic importance could be measured. If we are constructing an index of price dissimilarity, then it might be natural to weight by either the *quantities transacted* in the two situations or by the *expenditures* pertaining to that component. However, if the prices of a large country are being compared to those of a small country, then using either of these two methods of weighting will perhaps give too much weight to the large country. Hence, we will follow the example of Theil (1967; 136-137) and weight the importance of commodities by their *expenditure shares* in the two countries.[37] Thus define the *expenditure share* of commodity n in country i as

$$s_n{}^i \equiv p_n{}^i q_n{}^i / p^i \cdot q^i \; ; \qquad i = 1,2 \; ; \; n = 1,...,N. \qquad (8.49)$$

Let $m(a,b)$ be a symmetric mean of the positive numbers a and b and let $f(u)$ be an increasing continuous function of one variable, defined for $u \geq 1$ with the property that $f(1) = 0$. Then we can use the functions m and f in order to define the following *weighted absolute indexes of price and quantity dissimilarity*, D_P and D_Q:

$$D_P(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N m(s_n{}^1,s_n{}^2) f[\max\{p_n{}^1/p_n{}^2, p_n{}^2/p_n{}^1\}] \; ; \qquad (8.50)$$
$$D_Q(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N m(s_n{}^1,s_n{}^2) f[\max\{q_n{}^1/q_n{}^2, q_n{}^2/q_n{}^1\}]. \qquad (8.51)$$

It can be seen that we have just used the characterization of $D(x,y)$ in the unweighted case given by Proposition 5 and weighted the commodities according to their economic importance, which is reflected in the weights $m(s_n{}^1,s_n{}^2)$.[38]

It will be necessary to make concrete choices for the mean function m and the generator function f in empirical examples. As in the earlier sections, on the grounds of simplicity, we choose the arithmetic mean so that

$$m(a,b) = (1/2)a + (1/2)b. \qquad (8.52)$$

Our two preferred choices for f were made at the end of section 8.2 and in examples 9 and 10 in section 8.5. With the first preferred choice, (8.50) and (8.51) become the *weighted asymptotically linear index of absolute dissimilarity*: (**example 22**):

$$D_{PAL}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n{}^1 + s_n{}^2)\,[(p_n{}^1/p_n{}^2) + (p_n{}^2/p_n{}^1) - 2] \; ; \qquad (8.53)$$
$$D_{QAL}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n{}^1 + s_n{}^2)\,[[(q_n{}^1/q_n{}^2) + (q_n{}^2/q_n{}^1) - 2]. \qquad (8.54)$$

With the second preferred choice, (8.50) and (8.51) become the *weighted asymptotically quadratic index of absolute dissimilarity*: (**example 23**):

$$D_{PAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n{}^1 + s_n{}^2)\,[\{(p_n{}^1/p_n{}^2) - 1\}^2 + \{(p_n{}^2/p_n{}^1) - 1\}^2] \; ; \qquad (8.55)$$
$$D_{QAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n{}^1 + s_n{}^2)\,[\{(q_n{}^1/q_n{}^2) - 1\}^2 + \{(q_n{}^2/q_n{}^1) - 1\}^2]. \qquad (8.56)$$

We can follow Theil (1967; 138) and give the following statistical interpretation of the right hand side of (8.50) when m is defined by (8.52). Define *the absolute dissimilarity of the nth price ratio* between the two countries, r_n, by:

$$r_n \equiv f[\max\{p_n{}^1/p_n{}^2, p_n{}^2/p_n{}^1\}] \quad \text{for } n = 1,\ldots,N. \qquad (8.57)$$

Now define the discrete random variable, R say, as the random variable which can take on the values r_n with probabilities $\rho_n \equiv (1/2)[s_n{}^0 + s_n{}^1]$ for n = 1,…,N. Note that since each set of expenditure shares, $s_n{}^0$ and $s_n{}^1$, sums to one, the probabilities ρ_n will also sum to one. It can be seen that the expected value of the discrete random variable R is:

$$E[R] \equiv \sum_{n=1}^N \rho_n\, r_n = \sum_{n=1}^N (1/2)(s_n{}^0 + s_n{}^1)\, f[\max\{p_n{}^1/p_n{}^2, p_n{}^2/p_n{}^1\}] = D_P(p^1,p^2,q^1,q^2). \qquad (8.58)$$

using (8.50) and (8.57). Thus $D_P(p^1,p^2,q^1,q^2)$ can be interpreted as *the expected value of the absolute dissimilarities of the price ratios between the two countries*, where the N discrete price dissimilarities are weighted according to Theil's probability weights, $\rho_n \equiv (1/2)[s_n{}^0 + s_n{}^1]$ for n = 1,…,N.

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[37] Recent papers that also pursue this weighted approach are Heston, Summers and Aten (2001) and Sergeev (2005).

[38] In (8.17), we used the normalizing factor (1/N) in place of our present normalizing factor, $m(s_n{}^1,s_n{}^2)$. It can be seen that the dissimilarity measures defined by (8.50) and (8.51) are comparable for differing N.

A similar interpretation can be given to $D_Q(p^1,p^2,q^1,q^2)$ defined by (8.51) when m is defined by (8.52). Thus $D_Q(p^1,p^2,q^1,q^2)$ can be interpreted as *the expected value of the absolute dissimilarities of the quantity ratios between the two countries*, where the N discrete absolute quantity dissimilarities, $f[\max\{q_n^1/q_n^2, q_n^2/q_n^1\}]$, are weighted according to Theil's probability weights, $\rho_n \equiv (1/2)[s_n^0 + s_n^1]$ for n = 1,…,N.

We note that it is not clear how to define weighted (for economic importance) absolute dissimilarity indexes in the nonseparable case. This is a topic for further research.

8.9 WEIGHTED RELATIVE DISSIMILARITY INDEXES

Let $P(p^1,p^2,q^1,q^2)$ and $Q(p^1,p^2,q^1,q^2)$ be the "best" bilateral price and quantity indexes that one could choose.[39] We want the index number formulae P and Q to satisfy counterparts to the axioms D1-D7 listed above.[40] Adapting the strategy outlined in section 8.6 above, we again use the functions m and f in order to define the following *weighted relative indexes of price and quantity dissimilarity*, Δ_P and Δ_Q:

$$\Delta_P(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N m(s_n^1,s_n^2) f[\max\{P(p^1,p^2,q^1,q^2)p_n^1/p_n^2, p_n^2/P(p^1,p^2,q^1,q^2)p_n^1\}] ; \quad (8.59)$$
$$\Delta_Q(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N m(s_n^1,s_n^2) f[\max\{Q(p^1,p^2,q^1,q^2)q_n^1/q_n^2, q_n^2/Q(p^1,p^2,q^1,q^2)q_n^1\}]. \quad (8.60)$$

As in the previous section, we specialize m to be the arithmetic mean. With this choice, (8.59) and (8.60) become the following *weighted relative indexes of price and quantity dissimilarity*:

$$\Delta_P(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2) f[\max\{P(p^1,p^2,q^1,q^2)p_n^1/p_n^2, p_n^2/P(p^1,p^2,q^1,q^2)p_n^1\}] ; \quad (8.61)$$
$$\Delta_Q(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2) f[\max\{Q(p^1,p^2,q^1,q^2)q_n^1/q_n^2, q_n^2/Q(p^1,p^2,q^1,q^2)q_n^1\}] \quad (8.62)$$

where $f(u)$ is an increasing continuous function of one variable, defined for $u \geq 1$ with the property that $f(1) = 0$.

Example 24: Consider the following special case where we choose $f(u) \equiv [\ln u]^2$. The resulting *weighted log quadratic index of relative price dissimilarity* using the bilateral index number formula P is:

$$\Delta_{PLQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\,[\ln(p_n^2/P(p^1,p^2,q^1,q^2)p_n^1)]^2 . \quad (8.63)$$

The above formula is a generalization of the Allen Diewert (1981) unweighted formula (8.31) above. The Törnqvist Theil (1967) bilateral index number formula $P_T(p^1,p^2,q^1,q^2)$ seems to be the appropriate generalization of the unweighted Jevons formula to use in (8.63) for $P(p^1,p^2,q^1,q^2)$ but any superlative price index formula $P(p^1,p^2,q^1,q^2)$ could be used in (8.63).[41]

Example 25: Consider the following special case of (8.59) where $f(u) \equiv [u + u^{-1} - 2]$ for $u \geq 1$. The resulting *weighted asymptotically linear index of relative price dissimilarity* using the bilateral index number formula P is:

$$\Delta_{PAL}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\{(p_n^2/P(p^1,p^2,q^1,q^2)p_n^1) + P(p^1,p^2,q^1,q^2)(p_n^1/p_n^2) - 2\}. \quad (8.64)$$

The above formula is the weighted generalization of the unweighted relative formula (8.28) above.

Example 26: Consider the following special case of (8.59) where $f(u) \equiv (1/2)[u - 1]^2 + (1/2)[u^{-1} - 1]^2$ for $u \geq 1$. The resulting *weighted asymptotically quadratic index of relative price dissimilarity* using the bilateral index number formula P is:

$$\Delta_{PAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^N (1/2)(s_n^1 + s_n^2)\{[(p_n^2/P(p^1,p^2,q^1,q^2)p_n^1) - 1]^2 + [(P(p^1,p^2,q^1,q^2)(p_n^1/p_n^2) - 1]^2\}. \quad (8.65)$$

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[39] Diewert (1992) argued that the Fisher (1922) price and quantity indexes are "best" from the axiomatic point of view but Balk (1995), Von Auer (2001) and Diewert (2004b) argue for some other choices as well.

[40] The Fisher ideal indexes satisfy these properties.

[41] See Diewert (1976) for examples of superlative indexes. Our preferred superlative index is the Fisher (1922) ideal index.

The above formula is the weighted generalization of the unweighted relative formula (8.29) above.

Our preferred index of relative dissimilarity when there is the possibility of outliers in the data is the weighted asymptotically linear index of relative price dissimilarity defined by (8.64). If the data are regarded as being quite reliable, then the weighted log quadratic and the weighted asymptotically quadratic indexes of relative price dissimilarity defined by (8.63) and (8.65) could be used. Of course, analogous indexes can be defined for quantities rather than prices.

8.10 CONCLUSION

Our tentative conclusion is that linking between countries should be based on the sum of a *weighted absolute dissimilarity index of quantities* and a *weighted relative dissimilarity index of prices*.

We have exhibited many different functional forms for these two dissimilarity indexes but until more theoretical and empirical research becomes available, we recommend the use of the *asymptotically linear* or *asymptotically quadratic* functional forms. Both of these functional forms are differentiable when the price vectors being compared are proportional and when the quantity vectors being compared are equal but the asymptotically quadratic functional form penalizes large deviations between the two vectors much more heavily than does the asymptotically linear functional form.[42] Thus we are specifically recommending either the *weighted asymptotically linear index of relative price dissimilarity* $\Delta_{PAL}(p^1,p^2,q^1,q^2)$ defined by (8.64) where the price index $P(p^1,p^2,q^1,q^2)$ is the Fisher (1922) ideal formula or the *weighted asymptotically quadratic index of relative price dissimilarity* $\Delta_{PAQ}(p^1,p^2,q^1,q^2)$ defined by (8.65) as our preferred measures of relative price dissimilarity.[43] Similarly, we are specifically recommending either the *weighted asymptotically linear index of absolute quantity dissimilarity* $D_{QAL}(p^1,p^2,q^1,q^2)$ defined by (8.54) or the *weighted asymptotically quadratic index of quantity dissimilarity* $D_{QAQ}(p^1,p^2,q^1,q^2)$ defined by (8.56) as our preferred measures of absolute quantity dissimilarity. These indexes satisfy all of the important axioms that we have discussed.

APPENDIX: PROOFS OF PROPOSITIONS

Proof of Proposition 1: Using A5 with $\alpha = x^{-1}$, we have:

$$d(x,y) = d(1,y/x). \qquad \text{(A1)}$$

Now use A5 with $\alpha = y^{-1}$ and we find:

$$\begin{aligned} d(x,y) &= d(x/y,1) && \text{(A2)} \\ &= d(1,x/y) && \text{using A4} \\ &= d(1,y/x) && \text{using A1 and A2.} \end{aligned}$$

For $u \geq 1$, define the continuous function of one variable, $f(u)$ as

$$f(u) \equiv d(1,u)\ ; \qquad u \geq 1. \qquad \text{(A3)}$$

Using A1 and definition (A3), we have

$$f(1) = d(1,1) = 0. \qquad \text{(A4)}$$

Using A6, we deduce that $f(u)$ is an increasing function of u for $u \geq 1$. Now if $x \geq y$, then from (A2) and definition (A3), we deduce that $d(x,y) = f(x/y)$. If however, $y \geq x$, then from (A2) and (A3), we deduce that $d(x,y) = f(y/x)$. These two results can be combined into the following result:

[42] Researchers who prefer the sum of absolute deviations as a measure of dispersion will probably be comfortable with the asymptotically linear functional form whereas researchers who prefer the variance as a measure of dispersion will probably be more comfortable using the asymptotically quadratic functional form.

[43] An alternative to the use of (8.65) is the use of (8.63) with the use of the Törnqvist Theil (1967) bilateral index number formula $P_T(p^1,p^2,q^1,q^2)$ as the price index.

$$d(x,y) = f[\max\{x/y,\ y/x\}] \tag{A5}$$

which completes the first part of the Proposition. Going the other way, if $f(u)$ is an increasing, continuous function for $u \geq 1$ with $f(1) = 0$, then if we define $d(x,y)$ using (A5), it is easy to verify that $d(x,y)$ satisfies the axioms A1-A6.

Proof of Proposition 2: Straightforward computations except for axiom A6, which we now verify. Let $y'' > y' \geq x > 0$. Then

$$\begin{aligned} d_{g,h,m}(x,y'') &\equiv m[g\{|h(y''/x) - h(1)|\}, g\{|h(x/y'') - h(1)|\}] && \text{(A6)}\\ &= m[g\{h(y''/x) - h(1)\}, g\{h(1) - h(x/y'')\}] \\ &\qquad \text{using } y'' > x \text{ and the monotonicity of } h \\ &> m[g\{h(y'/x) - h(1)\}, g\{h(1) - h(x/y'')\}] \\ &\qquad \text{using } y'' > y',\ x > 0 \text{ and the monotonicity of } h, g \text{ and } m \\ &> m[g\{h(y'/x) - h(1)\}, g\{h(1) - h(x/y')\}] \\ &\qquad \text{using } y'' > y',\ x > 0 \text{ and the monotonicity of } h, g \text{ and } m \\ &= m[g\{|h(y'/x) - h(1)|\}, g\{|h(x/y') - h(1)|\}] \\ &\qquad \text{using } y' > x \text{ and the monotonicity of } h \\ &\equiv d_{g,h,m}(x,y'). \end{aligned}$$

Q.E.D.

Proof of Proposition 3: Let $y \geq x > 0$ and define $f(y) \equiv d(x,y)$. For the $d(x,y)$ defined by (8.5), we find that $f''(y) = 2x/y^3 > 0$ and so the asymptotically linear dissimilarity index defined by (8.5) is convex in y.

For the $d(x,y)$ defined by (8.11), we find that $f''(y) = 2(x/y)^2[1 - x^{-1}\ln(y/x)]$ which is negative for y large enough and hence the log quadratic dissimilarity index defined by (8.11) does not satisfy A7.

For the $d(x,y)$ defined by (8.10), we find that:

$$f''(y) = x^{-2} + 3x^2y^{-4} - 2xy^{-3} \equiv g(y). \tag{A7}$$

Let us attempt to minimize $g(y)$ defined in (A7) over $y \geq x$. We have:

$$g'(y) = -\ 12x^2y^{-5} + 6xy^{-4} = 0. \tag{A8}$$

The positive roots of (A8) are $y^* = 2x$ and $y^{**} = +\infty$. We find that $g(y)$ attains a strict local minimum at $y = 2x$ and this turns out to be the global minimum of $g(y)$ for $y \geq x$. Thus we have for $y \geq x$:

$$f''(y) \geq f''(2x) = x^{-2} + 3x^2\,(2x)^{-4} - 2x\,(2x)^{-3} > 0 \tag{A9}$$

and hence the linear quadratic dissimilarity index defined by (8.10) satisfies A7. Q.E.D.

Proof of Proposition 4: If $d(x,y)$ satisfies A1-A6, then by Proposition 1, $d(x,y) = f[\max\{x/y,\ y/x\}]$ where $f(u)$ is continuous, increasing for $u \geq 1$ with $f(1) = 0$. Substitute this representation for $d(x,y)$ into A9 and letting $x > 0$, $y \geq 0$ and $z \geq 0$, we find that f satisfies the following functional equation:

$$f(1 + (y/x) + (z/x)) = f(1 + (y/x)) + f(1 + (z/x))\ ; \qquad x > 0,\ y \geq 0,\ z \geq 0. \tag{A10}$$

Define the variables u and v as follows:

$$u \equiv y/x\ ;\quad v \equiv z/x\ . \tag{A11}$$

Substituting A11 into A10, we find that f satisfies the following functional equation:

$$f(1 + u + v) = \mathrm{f}(1 + u) + \mathrm{f}(1 + v)\ ; \qquad u \geq 0,\ v \geq 0. \tag{A12}$$

Define the function g as follows:

$g(u) \equiv f(1+u)$. (A13)

Using A13, A12 can be rewritten as follows:

$g(u+v) = g(u) + g(v)$; $u \geq 0, v \geq 0$. (A14)

But A14 is Cauchy's first functional equation or a special case of Pexider's (1903) first functional equation[44] and has the following solution:

$g(x) = \alpha x$; $x \geq 0$ (A15)

where α is a constant. Using A13 and A15,

$f(u) = \alpha(u-1)$; $u \geq 1$. (A16)

Equation A16 implies that d is equal to the right hand side of (8.12). However, in order that $f(u)$ be increasing for $u \geq 1$, we require that $\alpha > 0$, which completes the proof. Q.E.D.

Proof of Proposition 5: Using B2 and B8, we have

$D(1_N,1_N) = \sum_{n=1}^{N} d_n(1,1) = 0.$ (A17)

Thus

$$\begin{aligned} D(x,y) &= D(x,y) - D(1_N,1_N) && \text{using (A17)} \quad \text{(A18)} \\ &= \sum_{n=1}^{N} d_n(x_n,y_n) - \sum_{n=1}^{N} d_n(1,1) && \text{using B8} \\ &= \sum_{n=1}^{N} d_n^*(x_n,y_n) \end{aligned}$$

where the $d_n^*(x_n,y_n)$ are defined as:

$d_n^*(x_n,y_n) \equiv d_n(x_n,y_n) - d_n(1,1)$; n = 1,2,...,N. (A19)

It is easy to check that the d_n^* functions satisfy the following restrictions:

$d_n^*(1,1) = 0$; n = 1,2,...,N. (A20)

Using (A18), we have:

$$\begin{aligned} D(x_1,1_{N-1},y_1,1_{N-1}) &= d_1^*(x_1,y_1) + \sum_{n=2}^{N} d_n^*(1,1) && \text{(A21)} \\ &= d_1^*(x_1,y_1) && \text{using (A20).} \end{aligned}$$

Properties B1-B6 on D imply that $d_1^*(x_1,y_1)$ will satisfy properties A1-A6 listed in section 8.2 above. Hence, we may apply Proposition 1 and conclude that $d_1^*(x_1,y_1)$ has the following representation:

$d_1^*(x_1,y_1) = f[\max\{x_1/y_1,\ y_1/x_1\}]$ (A22)

for some continuous, increasing function of one variable f(u) defined for $u \geq 1$ with $f(1) = 0$:
Using B7, we deduce that

$$\begin{aligned} d_n^*(x_n,y_n) &= d_1^*(x_n,y_n) && \text{(A23)} \\ &= f[\max\{x_n/y_n,\ y_n/x_n\}] \ ; && \text{for n = 2,...,N using (A22)} \end{aligned}$$

[44] See Eichhorn (1978; 49) for a more accessible reference.

and this establishes (8.16). The second half of the Proposition is straightforward. Q.E.D.

Proof of Proposition 6: Using B5 with $\alpha_n \equiv (x_n)^{-1}$ for n = 1,...,N, we deduce that

$$D(x,y) = D(1_N, y_1/x_1, \ldots, y_N/x_N). \tag{A24}$$

For $u \geq 1_N$, define the continuous function of N variables $f(u)$ as follows:

$$f(u) \equiv D(1_N, u_1, \ldots, u_N). \tag{A25}$$

Using B7, we deduce that $f(u)$ is symmetric in u. Using B6, we deduce that $f(u)$ is increasing in the components of u for $u \geq 1_N$. Using B2, we deduce that

$$f(1_N) = 0. \tag{A26}$$

Using (A24), (A25) and B9, it is straightforward to verify that D and f satisfy equation (8.20) above. Note that we required only properties B1, B2, B5, B6, B7 and B9 to establish the first half of Proposition 3.[45] The converse part of the Proposition is also straightforward. Q.E.D.

Proof of Proposition 7: Properties C1 and C5 are obvious. Now check property C2:

$$\begin{aligned} \Delta(x,\lambda x) &\equiv D(S(x,\lambda x)x, \lambda x) && \text{using definition (8.24)} && \text{(A27)} \\ &= D(\lambda S(x,x)x, \lambda x) && \text{using D7} \\ &= D(\lambda x, \lambda x) && \text{using D2} \\ &= 0 && \text{using B2.} \end{aligned}$$

Check property C3: given x and y, suppose that $y \neq \lambda x$ for any $\lambda > 0$. Using definition (8.24), we have:

$$\begin{aligned} \Delta(x,y) &\equiv D(S(x,y)x, y) && && \text{(A28)} \\ &= D(\mu x, y) && \text{where } \mu = S(x,y) > 0 \text{ using D3} \\ &> 0 && \text{using B3 since } y \neq \mu x. \end{aligned}$$

Check property C4:

$$\begin{aligned} \Delta(x,y) &\equiv D(S(x,y)x, y) && \text{using definition (8.24)} && \text{(A29)} \\ &= D(1_N, y_1/x_1S(x,y), \ldots, y_N/x_NS(x,y)) && \text{using B5} \\ &= D(1_N, S(y,x)y_1/x_1, \ldots, S(y,x)y_N/x_N) && \text{using D4} \\ &= D(x, S(y,x)y) && \text{using B5 again} \\ &= D(S(y,x)y, x) && \text{using B4} \\ &\equiv \Delta(y,x) && \text{using definition (8.24) again.} \end{aligned}$$

Property C6 follows from Properties B7 and D6.

Finally, check Property C7. Let $x >> 0_N$, $y >> 0_N$ and $\lambda > 0$. Then by definition (8.24),

$$\begin{aligned} \Delta(x,\lambda y) &\equiv D(S(x,\lambda y)x, \lambda y) && && \text{(A30)} \\ &= D(\lambda S(x,y)x, \lambda y) && \text{using D7} \\ &= D(S(x,y)x, y) && \text{using B5 with all } \alpha_n = \lambda \\ &= \Delta(x,y) && \text{using definition (8.24).} \end{aligned}$$

Q.E.D.

References

[45] Property B3 is implied by properties B2 and B6 and property B4 is implied by Property B7 and B9.

Allen, R.C. and W.E. Diewert (1981), “Direct versus Implicit Superlative Index Number Formulae”, *The Review of Economics and Statistics* 63, 430-435.

Balk, B.M. (1995), “Axiomatic Price Index Theory: A Survey”, *International Statistical Review* 63:1, 69-93.

Balk, B.M. (1996), “A Comparison of Ten Methods of Multilateral International Price and Volume Comparisons”, *Journal of Official Statistics* 12, 199-222.

Balk, B.M. (2001), “Aggregation Methods in International Comparisons: What Have we Learned?”, Erasmus Research Institute of Management, Erasmus University Rotterdam, June.

Carli, Gian-Rinaldo, (1804), “Del valore e della proporzione de' metalli monetati”, pp. 297-366 in *Scrittori classici italiani di economia politica*, Volume 13, Milano: G.G. Destefanis (originally published in 1764).

Carruthers, A.G., D.J. Sellwood and P.W. Ward (1980), “Recent Developments in the Retail Prices Index”, *The Statistician* 29, 1-32.

Diewert, W.E. (1976), “Exact and Superlative Index Numbers”, *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1992), “Fisher Ideal Output, Input and Productivity Indexes Revisited”, *Journal of Productivity Analysis* 3, 211-248; reprinted as pp. 317-353 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.

Diewert, W.E. (1993), “Symmetric Means and Choice under Uncertainty”', pp. 355-433 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1999), “Axiomatic and Economic Approaches to International Comparisons”, pp. 13-87 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth Volume 61, NBER, Chicago: The University of Chicago Press.

Diewert, W.E. (2002), “Similarity and Dissimilarity Indexes: An Axiomatic Approach”, Discussion Paper 02-10, Department of Economics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z1.

Diewert, W.E. (2004a), “Elementary Indices”, Chapter 20 in *Producer Price Index Manual: Theory and Practice*, P. Armknecht (ed.), Washington D.C.: International Monetary Fund.

Diewert, W.E. (2004b), “A New Axiomatic Approach to Index Number Theory”, Discussion Paper 04-05, Department of Economics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z1

Eichhorn, W. (1978), *Functional Equations in Economics*, Reading, MA: Addison-Wesley Publishing Company.

Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton Mifflin.

Fox, K.J., R.J. Hill and W.E. Diewert (2004), “Identifying Outliers in Multi-Output Models”, *Journal of Productivity Analysis* 22, 73-94.

Heston, A., R. Summers and B. Aten (2001), “Some Issues in Using Chaining Methods for International Real Product and Purchasing Power Comparisons”, Paper presented at the Joint World Bank and OECD Seminar on Purchasing Power Parities, January 30-February 2, Washington D.C.

Hill, R.J. (1995), *Purchasing Power Methods of Making International Comparisons*, Ph. D. dissertation, Vancouver: The University of British Columbia.

Hill, R.J. (1999a), "Comparing Price Levels across Countries Using Minimum Spanning Trees", *The Review of Economics and Statistics* 81, 135-142.

Hill, R.J. (1999b), "International Comparisons using Spanning Trees", pp. 109-120 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth Volume 61, NBER, Chicago: The University of Chicago Press.

Hill, R.J. (2001), "Measuring Inflation and Growth Using Spanning Trees", *International Economic Review* 42, 167-185.

Jevons, W.S., (1865), "The Variation of Prices and the Value of the Currency since 1782", *Journal of the Statistical Society of London* 28, 294-320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119-150.

Kravis, I.B. (1984), "Comparative Studies of National Incomes and Prices", *Journal of Economic Literature* 22, 1-39.

Kravis, I.B., A. Heston and R. Summers (1982), *World Product and Income: International Comparison of Real Gross Product*, Baltimore: John Hopkins University Press.

Pexider, J.V. (1903), "Notiz über Funktionaltheoreme", *Monatshefte Mathematische Physik* 14, 293-301.

Sergeev, S. (2001), "Measures of the Similarity of the Country's Price Structures and their Practical Application", Conference on the European Comparison Program, U. N. Statistical Commission. Economic Commission for Europe, Geneva, November 12-14, 2001.

Sergeev, S. (2005), "Aggregation Methods on the Basis of Structural International Prices", Chapter 11 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Theil, H. (1967), *Economics and Information Theory*, Amsterdam: North-Holland Publishing.

von Auer, L. (2001), "An Axiomatic Checkup for Price Indices", working Paper No. 1/2001, Faculty of Economics and Management, Otto von Guericke University Magdeburg, Postfach 4120, 39016 Magdeburg, Germany.

Walsh, C.M. (1921), "Discussion of the Best Form of Index Number", *Journal of the American Statistical Association* 17, 537-544.