The Economic Journal, 92 (*March* 1982), 73–86
Printed in Great Britain

MULTILATERAL COMPARISONS OF OUTPUT, INPUT, AND PRODUCTIVITY USING SUPERLATIVE INDEX NUMBERS*

Douglas W. Caves, Laurits R. Christensen and W. Erwin Diewert

Early in this century economists began to give serious attention to making comparisons using index number techniques. There was extensive debate as to which index number formulas were the most appropriate for carrying out comparisons.[1] The debate was extensive in no small part due to the lack of agreement as to criteria for preferring one formula over another. In recent decades there has been a resurgence of interest in index numbers, resulting from discoveries that the properties of index numbers can be directly related to the properties of the underlying aggregator functions that they represent. The underlying functions – production functions, utility functions, etc. – are the building blocks of economic theory, and the study of relationships between these functions and index number formulas has been referred to by Samuelson and Swamy (1974) as the economic theory of index numbers.[2]

A key development in the economic theory of index numbers has been the demonstration that numerous index number formulas can be explicitly derived from particular aggregator functions. This development provides a powerful new basis for selecting an index number procedure. Rather than starting the selection process with a number of plausible index number formulas, one can specify an aggregator function with desirable properties and derive the corresponding index number procedure. The resulting index is termed exact for that particular aggregator function. Diewert (1976) makes a strong case for limiting the consideration of aggregator functions to those which are flexible, i.e. those which can provide a second order approximation to an arbitrary aggregator function. He has termed index numbers that are exact for flexible aggregator functions 'superlative'.

There are two superlative index numbers that are of particular interest – the Fisher Ideal index and the Törnqvist-Theil-translog index. Fisher (1922) dubbed the following index Ideal since it best satisfied his several criteria for choosing among index numbers:

$$I_{kl} = \{[\Sigma S_{li}(Z_{ki}/Z_{li})]/[\Sigma S_{ki}(Z_{ki}/Z_{li})]\}^{\frac{1}{2}}, \quad (1)$$

where S_{li} and S_{ki} are value share weights for the two economic entities or time periods being compared, and the Z_i are the corresponding prices or quantities. This index is widely known as the geometric mean of the Laspeyres and Paasche

* This research has been supported in part by the National Science Foundation. Dale Jorgenson provided helpful comments on a previous draft of this paper.

[1] Ruggles (1967) provides an interesting brief survey with extensive references.

[2] See Diewert (1979) for a recent survey of the economic theory of index numbers.

indexes. It has also been long (but not widely) known to be the exact index for the quadratic mean of order two aggregator function.[1] Diewert (1974) showed that this aggregator function is flexible, implying that the Fisher Ideal index is superlative.

The most widely utilised superlative index is the Törnqvist-Theil-translog index,

$$\ln T_{kl} = \Sigma \tfrac{1}{2}(S_{il}+S_{kl}) \ln (Z_{kl}/Z_{il}). \tag{2}$$

It was discussed by Fisher (1922), has been advocated by Törnqvist (1936) and Theil (1965), and has been used extensively by Christensen and Jorgenson (1973) and others. Diewert (1976) established that the translog index is superlative by showing that it is exact for the homogeneous translog aggregator function (Christensen, Jorgenson and Lau, 1971, 1973), a flexible functional form which has been widely used in recent empirical economic research.

The indexes (1) and (2) define binary comparisons. One of the principal issues in the index number literature early in this century was whether use of various indexes gave rise to transitive comparisons. One of Fisher's (1922) criteria for choosing among index numbers was whether they satisfied the following 'circularity' test:[2]

$$I_{kl} = I_{km}/I_{lm}. \tag{3}$$

Fisher found that many indexes that satisfied the circularity test were not attractive on other grounds. On the other hand, his preferred Ideal index did not satisfy circularity. This led Fisher to reject circularity as an important criterion for choosing among index numbers.

In essence Fisher's discomfort with the circularity test was due to the fact that it conflicted with using value shares weights that are specific to the two entities being compared. Dreschler (1973) has used the term 'characteristicity' to indicate the degree to which weights are specific to the comparison at hand. The Fisher Ideal index utilises weights that are perfectly characteristic. Dreschler (1973, p. 17) succinctly summarised Fisher's dilemma: '...characteristicity and circularity are always...in conflict with each other.' The implication is that some degree of characteristicity must be sacrificed to obtain circularity. In this paper we demonstrate that superlative index numbers that maintain circularity and a high degree of characteristicity can be used for making multilateral comparisons.

The problem to which multilateral comparative techniques have most often been applied is that of multicountry output comparisons. The best known studies in this area are those of Gilbert, Kravis, and their associates.[3] These studies have not made use of the economic theory of index numbers. In fact, Kravis and his associates considered and rejected a superlative index as being 'too mechanical'

[1] Diewert (1976, p. 116) attributes proof of this result to Byushgens (1925), Konyus and Byushgens (1926), Frisch (1936), Wald (1939), Afriat (1972) and Pollak (1971). The fact that this result is not widely known is attested to by statements in the literature such as the following by Kravis, Heston and Summers (1978, p. 70): 'The Fisher or "ideal" index – that is, the geometric mean of the partner-weighted and U.S. weighted indexes – is also presented. This index is not easy to justify in theoretical terms....'

[2] This criterion is often referred to as transitivity. In the remainder of the paper we use the terms circularity and transitivity interchangeably to refer to the requirement specified by (3).

[3] See Gilbert and Kravis (1954), Gilbert and associates (1958) and Kravis *et al.* (1975, 1978).

while settling on a non-superlative index that sacrifices much more characteristicity.[1]

Multicountry comparisons of input and productivity are closely related to comparisons of output, but they have received much less attention than output comparisons.[2] In this paper we develop in parallel the methodology of all three types of comparisons. In particular we develop superlative output and input indexes that can be combined to provide a superlative productivity index.

Although multicountry comparisons are a natural application of superlative multilateral indexes, there is wide scope for other applications. Multilateral indexes can be used for time series as well as cross section comparisons and for combinations of time series and cross section data. Multilateral superlative indexes are attractive for firm and industry data in addition to country data.

I. MULTILATERAL OUTPUT AND INPUT COMPARISONS BASED ON THE TRANSLOG TRANSFORMATION FUNCTION

For an economic entity, s, using the vector of inputs $\mathbf{X}^s$ to produce the vector of outputs $\mathbf{Y}^s$, we represent the structure of production by a flexible functional form, the translog transformation function:

$$F(\ln \mathbf{Y}^s, \ln \mathbf{X}^s, s) = 1. \tag{4}$$

We include s as an argument of F to indicate that the structure of production, and hence productivity, is allowed to differ in a non-neutral manner among economic entities. This is achieved by permitting the first order translog parameters to differ across economic entities:

$$\alpha_0^s + \sum_i^I \alpha_i^s \ln Y_i^s + \sum_n^N \beta_n^s \ln X_n^s + \tfrac{1}{2}\sum_i^I \sum_j^I \alpha_{ij} \ln Y_i^s \ln Y_j^s + \tfrac{1}{2}\sum_n^N \sum_m^N \beta_{nm} \ln X_n^s \ln X_m^s$$
$$+ \sum_i^I \sum_n^N \xi_{in} \ln Y_i^s \ln X_n^s = 1, \tag{5}$$

where $\alpha_{ij} = \alpha_{ji}$, and $\beta_{ij} = \beta_{ji}$.[3] We restrict attention to economic entities exhibiting constant return to scale, which is imposed on (5) through the following restrictions:

$$-\sum_i^I \alpha_n^s = \sum_n^N \beta_n^s = 1 \quad \text{for} \quad s = 1, \dots S, \qquad \sum_i^I \alpha_{ij} = 0 \quad \text{for} \quad j = 1, \dots I,$$
$$\sum_n^N \beta_{nm} = 0 \quad \text{for} \quad m = 1, \dots M, \qquad \sum_n^I \xi_{in} = 0 \quad \text{for} \quad n = 1, \dots N,$$
$$\sum_n^N \xi_{in} = 0 \quad \text{for} \quad i = 1, \dots I.$$

[1] Kravis *et al.* (1975, p. 70).

[2] For an exception see Denison (1967).

[3] A functional form can be defined as flexible if it can represent an arbitrary set of revenue shares, cost shares, and elasticities of revenue and cost shares at a specified base point. Our form in equation (5) is flexible in a more general sense because it can represent arbitrary revenue shares and cost shares at multiple base points; each value of s, and hence each entity, constitutes a distinct base point. The form is restricted, however, in the sense that the second order coefficients are independent of s, implying equal elasticities of revenue and cost shares for all entities.

To ease exposition we refer henceforth to the economic entities as countries, without noting further that they could be firms or industries. In this case the comparisons are naturally thought of as cross section comparisons, but they could just as well be time series comparisons or combined cross section, time series comparisons.

Observations from any two countries (values of s) are assumed to fall on the translog transformation function (4). A comparison of output for countries k and l could use country k as a base or country l as a base. Consider using k as the basis of comparison. In this case we define the ratio of output of k to l as the minimum proportional decrease in all elements of $\mathbf{Y}^k$ such that the resulting vector of outputs is producible with the input and productivity levels of l. Since we are using k as the basis of comparison, we denote the factor of proportionality by δ_k, which is the solution to the following equation:

$$F[\ln(\mathbf{Y}^k/\delta_k), \ln \mathbf{X}^l, l] = 1. \tag{6}$$

We can solve for δ_k as a function of all the output relatives, Y_i^k/Y_i^l, by using the quadratic identity (QI) due to Diewert (1976).[1] Using (4) we can rewrite (6) as:

$$F[\ln(\mathbf{Y}^k/\delta_k), \ln \mathbf{X}^l, l] - F(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l) = 0 \tag{7}$$

and applying the QI, we obtain:

$$\sum_i^I \{\tfrac{1}{2} F_i[\ln(\mathbf{Y}^k/\delta_k), \ln \mathbf{X}^l, l] + \tfrac{1}{2} F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l)\}[\ln(Y_i^k/\delta_k) - \ln Y_i^l] = 0, \tag{8}$$

where the F_i are the partial derivatives of F with respect to the logarithms of the individual outputs for either country. The constant returns to scale restrictions, noted below (5), imply:

$$\sum_i^I F_i = -1 \quad \text{and} \quad F_i[\ln(\mathbf{Y}^k/\delta_k), \ln \mathbf{X}^l, l] = F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^l, l). \tag{9}$$

Use of (9) allows us to solve (8) for δ_k:

$$\ln \delta_k = -\sum_i^I [\tfrac{1}{2} F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^l, l) + \tfrac{1}{2} F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l)] \ln(Y_i^k/Y_i^l). \tag{10}$$

Now consider using country l as the basis of comparison. In this case we define the output of k relative to l as the maximum proportional increase in all elements of $\mathbf{Y}^l$ such that the resulting vector of outputs is producible with the input and productivity levels of k. We denote this factor of proportionality by δ_l, which is the solution to the following equation:

$$F(\ln \delta_l \mathbf{Y}^l, \ln \mathbf{X}^k, k) = 1. \tag{11}$$

[1] The QI states the following: Let $\mathbf{z}$ be an n dimensional vector and F be a quadratic function of the elements of $\mathbf{z}$. Denote the vector of first derivatives of F by $\mathbf{f}$. Then we can write,

$$F(\mathbf{z}^1) - F(\mathbf{z}^0) = \tfrac{1}{2}[\mathbf{f}(\mathbf{z}^1) + \mathbf{f}(\mathbf{z}^0)]^{\mathrm{T}} (\mathbf{z}^1 - \mathbf{z}^0),$$

where 1 and 0 represent two observations on the vector $\mathbf{z}$, and T denotes vector transposition.

Following the same procedure as above, we can solve for δ_l as a function of the output relatives Y_i^k/Y_i^l:

$$\ln \delta_l = -\sum_i^I [\tfrac{1}{2} F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^k, k) + \tfrac{1}{2} F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)] \ln(Y_i^k/Y_i^l). \tag{12}$$

We have two alternative definitions of the relative output in k and l. In one case k is treated as the base country and in the other l is treated as the base country. It would be desirable to have a base-country invariant definition of relative output. A natural definition is the geometric mean of δ_k and δ_l, which we denote as δ_{kl}:

$$\ln \delta_{kl} = (\ln \delta_k + \ln \delta_l)/2. \tag{13}$$

Using (10) and (12), equation (13) can be written in the following form:

$$\begin{aligned} \ln \delta_{kl} = & -\sum_i [\tfrac{1}{2} F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k) + \tfrac{1}{2} F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l)] \ln(Y_i^k/Y_i^l) \\ & + \tfrac{1}{4} \sum_i [F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k) - F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^k, k)] \ln(Y_i^k/Y_i^l) \\ & + \tfrac{1}{4} \sum_i [F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l) - F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^l, l)] \ln(Y_i^k/Y_i^l). \end{aligned} \tag{14}$$

The terms in the brackets in the second and third lines of (14) sum to zero, leaving the simplified expression for δ_{kl}:[1]

$$\ln \delta_{kl} = -\sum_i [\tfrac{1}{2} F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k) + \tfrac{1}{2} F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l)] \ln(Y_i^k/Y_i^l). \tag{15}$$

If economic entities maximise revenue conditional on the input levels and output prices, then F_i is equal to the negative of the share of output i in total revenue, $F_i = -P_i Y_i / \sum_j^I P_j Y_j = -R_i$. Thus (15) can be rewritten as:

$$\ln \delta_{kl} = \tfrac{1}{2} \sum_i (R_i^k + R_i^l) \ln(Y_i^k/Y_i^l). \tag{16}$$

Equation (16) provides an index number of the Törnqvist–Theil-translog form.[2] We refer to it as the *translog bilateral output index*. This index has been derived from an unrestricted constant returns to scale translog transformation function; thus,

[1] Note that

$$F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k) = \alpha_i^k + \sum_j^J \alpha_{ij} \ln Y_j^k + \sum_j^N \xi_{ij} \ln X_j^k,$$

and

$$F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^k, k) = \alpha_i^k + \sum_j^J \alpha_{ij} \ln Y_j^l + \sum_j^N \xi_{ij} \ln X_j^k.$$

Thus the second line of (14) is $\sum_i \sum_j^J \alpha_{ij} (\ln Y_j^k - \ln Y_j^l) \ln(Y_i^k/Y_i^l)$. Similarly the third line of (14) is $\sum_i \sum_j^J \alpha_{ij} (\ln Y_j^l - \ln Y_j^k) \ln(Y_i^k/Y_i^l)$. Hence the second and third lines sum to zero.

[2] The derivation of (16) depends upon the equality of second order coefficients, α_{ij}, across entities. If this equality does not hold, it can be shown, from (14) and footnote 1, that $\ln \delta_{kl}$ is equal to the translog bilateral index plus an expression that depends upon the discrepancies in the α_{ij} and the log output ratios. This expression can be used to incorporate estimates of the α_{ij} to correct the translog index or to test the sensitivity of the index to hypothesised differences in the α_{ij}.

the index does not entail separability of inputs and outputs nor neutrality of differences in productivity.

The translog bilateral output index is attractive for making base-country invariant binary output comparisons. But a set of such binary comparisons does not necessarily result in a transitive multilateral comparison. Introduction of any third country, say m, results in three bilateral comparisons that do not in general satisfy the circularity requirement, i.e. $\ln\delta_{kl} \neq \ln\delta_{km} - \ln\delta_{lm}$. It is possible, however, to modify the definition of output comparisons so that transitive results are obtained in the multilateral setting. We define the output of country k relative to the output of all S countries as the geometric mean of the bilateral output comparisons between k and each of the countries:

$$\overline{\ln\delta_k} = \frac{1}{S}\sum_{s}^{S}\ln\delta_{ks}. \tag{17}$$

Substituting the translog bilateral output indexes into (17) yields:

$$\overline{\ln\delta_k} = \tfrac{1}{2}\sum_i[(R_i^k+\overline{R}_i)(\ln Y_i^k-\overline{\ln Y_i})+\overline{R}_i\overline{\ln Y_i}-\overline{R_i\ln Y_i}], \tag{18}$$

where the bar indicates the arithmetic mean over the S countries of the variable or product under the bar. Finally, the *translog multilateral output index*, δ_{kl}^*, is defined as:

$$\ln\delta_{kl}^* = \overline{\ln\delta_k}-\overline{\ln\delta_l} = \tfrac{1}{2}\sum_i(R_i^k+\overline{R}_i)(\ln Y_i^k-\overline{\ln Y_i})-\tfrac{1}{2}\sum_i(R_i^l+\overline{R}_i)(\ln Y_i^l-\overline{\ln Y_i}). \tag{19}$$

Translog multilateral output comparisons are easily seen to be transitive:

$$\ln\delta_{kl}^* = \ln\delta_{km}^* - \ln\delta_{lm}^*. \tag{20}$$

The translog multilateral output index, δ_{kl}^*, reduces to the translog bilateral output index, δ_{kl}, when S is equal to 2.

The translog multilateral output index can be derived in an alternative manner by considering a hypothetical representative country (h) with the output vector $\overline{\ln Y_i}$, the input vector $\overline{\ln X_i}$, and the revenue shares $\overline{R}_i$. A translog bilateral output comparison of country k with this hypothetical country results in the following expression:

$$\ln\delta_{kh} = \tfrac{1}{2}\sum_i(R_i^k+\overline{R}_i)(\ln Y_i^k-\overline{\ln Y_i}), \tag{21}$$

providing an alternative path to (18) $\ln\delta_{kl}^* = \ln\delta_{kh} - \ln\delta_{lh}$.

The foregoing development of output comparisons can be easily adapted to the problem of input comparisons. We denote by ρ_k and ρ_l the proportional changes in input that are analogous to δ_k and δ_l. For example, ρ_k is the minimum proportional increase in all elements of $\mathbf{X}^l$ such that the resulting vector of inputs is sufficient to produce $\mathbf{Y}^k$ with the productivity level prevailing at k; i.e. ρ_k is the solution to the following equation:

$$F(\ln\mathbf{Y}^k, \ln\rho_k\mathbf{X}^l, k) = 1. \tag{22}$$

Proceeding as above we define ρ_{kl} as the geometric mean of ρ_k and ρ_l. If economic entities minimise total cost conditional on output levels and input prices, we obtain the *translog bilateral input index*:

$$\ln\rho_{kl} = \tfrac{1}{2}\sum_n^N(W_n^k+W_n^l)\ln(X_n^k/X_n^l), \tag{23}$$

where the W_n are shares of input n in total cost. Expanding the input comparison to the multilateral setting provides the *translog multilateral input index*, ρ_{kl}^*:

$$\ln\rho_{kl}^* = \overline{\ln\rho_k}-\overline{\ln\rho_l} = \tfrac{1}{2}\sum_i(W_n^k+\overline{W}_n)(\ln X_n^k-\overline{\ln X_n}) - \tfrac{1}{2}\sum_n(W_n^l+\overline{W}_n)(\ln X_n^l-\overline{\ln X_n}). \tag{24}$$

The translog multilateral input index is transitive:

$$\ln\rho_{kl}^* = \ln\rho_{km}^* - \ln\rho_{lm}^*, \tag{25}$$

and it reduces to a translog bilateral input index, $\ln\rho_{kl}$, when S is equal to 2.

Productivity comparisons can be interpreted as comparisons of outputs relative to inputs. Thus, we might expect that the translog output and input indexes could be combined to provide translog productivity indexes. This is verified at the end of the next section.

II. TRANSLOG MULTILATERAL PRODUCTIVITY COMPARISONS

Writers from Solow (1957) to Diewert (1980) have identified changes in productivity with shifts in the production function. This notion of productivity change is unambiguous for infinitesimal shifts in continuous time, but actual productivity comparisons must be based on discrete data points. Productivity comparisons of such points are closely related to output comparisons, as discussed in the previous section. Indeed productivity comparisons can be viewed as output comparisons with input levels controlled to be equal.

As in the previous section we assume that observations from any two countries fall on the translog transformation function. A productivity comparison between k and l can be based on either k or l or on some average of them. Consider using k as the basis of comparison. In this case we define the productivity of k relative to l as the minimum proportional decrease (λ_k) in all elements of $\mathbf{Y}^k$ such that the resulting output vector is producible with the input levels of k and the productivity level of l. Thus λ_k is the solution to the following equation:

$$F[\ln(\mathbf{Y}^k/\lambda_k), \ln\mathbf{X}^k, l] = 1, \tag{26}$$

which is equivalent to:

$$F(\ln\mathbf{Y}^l, \ln\mathbf{X}^l, l) - F[\ln(\mathbf{Y}^k/\lambda_k), \ln\mathbf{X}^k, l] = 0. \tag{27}$$

Application of Diewert's QI to (27), along with (9), yields:

$$\ln\lambda_k = -\tfrac{1}{2}\sum_i[F_i(\ln\mathbf{Y}^l,\ln\mathbf{X}^l,l)+F_i(\ln\mathbf{Y}^k,\ln\mathbf{X}^k,l)]\ln(Y_i^k/Y_i^l) - \tfrac{1}{2}\sum_n[F_n(\ln\mathbf{Y}^l,\ln\mathbf{X}^l,l)+F_n(\ln\mathbf{Y}^k,\ln\mathbf{X}^k,l)]\ln(X_n^k/X_n^l), \tag{28}$$

where the $F_i(F_n)$ are the partial derivatives of F with respect to the logarithms of the individual outputs (inputs) for either country.

Now consider using country l as the basis of comparison. In this case we define the productivity of k relative to l as the maximum proportional increase (λ_l) in all elements of $\mathbf{Y}^l$ such that the resulting output vector is producible with the input levels of l and the productivity level of k. Thus λ_l is the solution to the following equation:

$$F(\ln \lambda_l \mathbf{Y}^l, \ln \mathbf{X}^l, k) = 0. \qquad (29)$$

Subtracting $F(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)$ from both sides of (29) and applying the QI yields:

$$\ln \lambda_l = -\tfrac{1}{2}\sum_i [F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, k) + F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)] \ln(Y_i^k/Y_i^l)$$
$$-\tfrac{1}{2}\sum_n [F_n(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, k) + F_n(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)] \ln(X_n^k/X_n^l). \qquad (30)$$

We have two alternative definitions of the productivity of country k relative to the productivity of country l. In one case k is treated as the base country, and in the other l is treated as the base country. A natural base-country invariant definition of the relative productivity of k and l is the geometric mean of λ_k and λ_l, which we denote as λ_{kl}:

$$\ln \lambda_{kl} = (\ln \lambda_k + \ln \lambda_l)/2. \qquad (31)$$

Equation (31) can be written in the following form:

$$\ln \lambda_{kl} = -\tfrac{1}{2}\sum_i [F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l) + F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)] \ln(Y_i^k/Y_i^l)$$
$$-\tfrac{1}{2}\sum_n [F_n(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l) + F_n(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)] \ln(X_n^k/X_n^l)$$
$$-\tfrac{1}{4}[F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, l) - F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l)$$
$$+F_i(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, k) - F_i(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)] \ln(Y_i^k/Y_i^l)$$
$$-\tfrac{1}{4}[F_n(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, l) - F_n(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, l) + F_n(\ln \mathbf{Y}^l, \ln \mathbf{X}^l, k)$$
$$-F_n(\ln \mathbf{Y}^k, \ln \mathbf{X}^k, k)] \ln(X_n^k/X_n^l). \qquad (32)$$

By writing out the derivatives F_i and F_n, it is straightforward to verify that the expression in the third and fourth lines of (32) is equal to zero, as is the expression in the fifth and sixth line of (32).

If economic entities minimise costs conditional on output levels and input prices, and maximise revenues conditional on input levels and output prices, then the F_i are equal to the negative of the revenue shares (R_i), and the F_n are equal to cost shares (W_n). Thus (32) can be rewritten as:

$$\ln \lambda_{kl} = \tfrac{1}{2}\sum_i (R_i^k + R_i^l)(\ln Y_i^k/Y_i^l) - \tfrac{1}{2}\sum_n (W_n^k + W_n^l)\ln(X_n^k/X_n^l), \qquad (33)$$

which is the *translog bilateral productivity index* proposed by Christensen and Jorgenson (1970). Its derivation in this paper is based on an unrestricted constant returns to scale translog transformation function. Thus, it does not entail separability of inputs and outputs, nor neutrality of differences in productivity.

The translog bilateral productivity index is attractive for making base-country invariant binary productivity comparisons. But a set of such binary comparisons

does not satisfy the circularity requirement, i.e. $\ln \lambda_{kl} \neq \lambda_{km} - \lambda_{lm}$. We must extend the definition leading to the bilateral productivity comparison to obtain a transitive multilateral productivity comparison. Proceeding as above, we define the productivity of country k relative to the productivity of all S countries as the geometric mean of the bilateral productivity comparisons between k and each of the S countries.

$$\overline{\ln \lambda_k} = \frac{1}{S}\sum_s^S \ln \lambda_{ks}. \qquad (34)$$

Substituting the translog bilateral productivity indexes into (34) yields:

$$\overline{\ln \lambda_k} = \tfrac{1}{2}\sum_i (R_i^k + \bar{R}_i)(\ln Y_i^k - \overline{\ln Y_i}) - \sum_n (W_n^k + \overline{W}_n)(\ln X_n^k - \overline{\ln X_n})$$
$$+\sum_i (\bar{R}_i \overline{\ln Y_i} - \overline{R_i \ln Y_i}) - \sum_n (\overline{W}_n \overline{\ln X_n} + \overline{W_n \ln X_n}). \qquad (35)$$

Finally, the *translog multilateral productivity index*, $\ln \lambda_{kl}^*$, is defined as:

$$\ln \lambda_{kl}^* = \overline{\ln \lambda_k} - \overline{\ln \lambda_l} = \tfrac{1}{2}\sum_i (R_i^k + \bar{R}_i)(\ln Y_i^k - \overline{\ln Y_i}) - \tfrac{1}{2}\sum_i (R_i^l + \bar{R}_i)(\ln Y_i^l - \overline{\ln Y_i})$$
$$-\tfrac{1}{2}\sum_n (W_n^k + \overline{W}_n)(\ln X_n^k - \overline{\ln X_n}) + \tfrac{1}{2}\sum_n (W_n^l + \overline{W}_n)(\ln X_n^l - \overline{\ln X_n}). \qquad (36)$$

The translog multilateral productivity index is transitive:

$$\ln \lambda_{kl}^* = \ln \lambda_{km}^* - \ln \lambda_{lm}^*, \qquad (37)$$

and it reduces to the bilateral translog productivity index, $\ln \lambda_{kl}$, when S is equal to 2. The alternative approach of comparing each country directly to a hypothetical representative country (h), which has output vector $\overline{\ln Y_i}$, input vector $\overline{\ln X_n}$, revenue shares $\bar{R}_i$, and cost shares $\overline{W}_n$, results in the following expression:

$$\ln \lambda_{kh} = \tfrac{1}{2}\sum_i (R_i^k + \bar{R}_i)(\ln Y_i^k - \overline{\ln Y_i}) - \tfrac{1}{2}\sum_n (W_n^k + \overline{W}_n)(\ln X_n^k - \overline{\ln X_n}). \qquad (38)$$

This approach also leads to the translog multilateral output index,

$$\lambda_{kl}^* = \ln \lambda_{kh} - \ln \lambda_{lh}.$$

At the end of the previous section we suggested that one might expect a relationship between translog output and input indexes and a translog productivity index. It is now clear that the relationship is very simple. The translog bilateral productivity index is the difference between the translog bilateral output and input indexes:

$$\ln \lambda_{kl} = \ln \delta_{kl} - \ln \rho_{kl}. \qquad (39)$$

Similarly, the translog multilateral productivity index is the difference between the translog multilateral output and input indexes:

$$\ln \lambda_{kl}^* = \ln \delta_{kl}^* - \ln \rho_{kl}^*. \qquad (40)$$

III. MULTILATERAL INDEXES IN THE TIME SERIES CONTEXT

As we emphasised at the outset, the multilateral indexes that we have proposed are applicable to cross-section data, time series data, and combinations of cross-section and time series data (panel data). The attractions of multilateral methods are quite clear in the case of cross-section data, but not as clear in the case of time series data. Contrary to cross-section data, where there is generally no natural ordering of the data points, chronological order is considered to be natural for time-differentiated data points. This is the justification for the standard practice of comparing adjacent observations directly, while non-adjacent observations are compared only indirectly, using the intervening observations as intermediaries. This practice, which is called 'chain-linking', results in transitive time series comparisons.

Time series comparisons based on superlative bilateral chain-linked indexes have many desirable properties; their primary unattractive feature is the indirect nature of the comparison of non-adjacent observations. Superlative multilateral indexes compare adjacent and non-adjacent observations directly. Such symmetry of treatment is generally desirable, but in this instance it has the troublesome consequence of destroying the fixity of historical comparisons. As time passes new observations become available that expand the set of comparisons. The chain-linked bilateral index approach leaves the historical comparisons intact, but the multilateral index approach results in new comparisons for the entire time series. The choice between the two approaches appears to rest with the importance attached to the conflicting traits of symmetry of treatment and fixity of historical comparisons.

Although there are no clear grounds for choosing between the multilateral and the chain-linked bilateral approaches for time series data, there are strong grounds for preferring the multilateral approach for comparisons based on panel data. The reason is that the lack of symmetry of treatment with chain-linked bilateral indexes, which might be considered unimportant in the case of a single time series, becomes magnified in the case of several time series, i.e. panel data. The set of time series comparisons could be linked together via any single cross-section, but the results would differ from those obtained by choosing any other cross-section.[1] An equally unattractive alternative would be to construct all the cross-section comparisons and combine them by chain-linking the results through an arbitrarily chosen country; the results would differ from those obtained by choosing any other country. The multilateral approach to panel data comparisons treats all countries and time periods symmetrically and is therefore more desirable than either approach involving chain-linking.[2]

[1] Jorgenson and Nishimizu (1978) provide an example of this approach.

[2] See Caves, Christensen, and Tretheway (1981) for an application of the translog multilateral productivity index to a panel of airline firms.

IV. RELATIONSHIPS AMONG METHODS OF INTERNATIONAL COMPARISON

Interest in international comparisons predated recent developments in the economic theory of index numbers. Performing comparisons required that empirical procedures be chosen, and various criteria have been used to make the selection. In some instances the methods chosen turn out to be justified in terms of the analysis developed in this paper.

Dreschler (1973) brought to the attention of English-speaking economists a method of multilateral output comparison (the EKS method) that had been independently proposed by Elteto and Köves (1964) and Szulc (1964). Dreschler (p. 28) described the EKS method as follows:

> '...a method by which circularity can be achieved paying the least possible price for it in respect of characteristicity. The least possible price here means that for the ensemble of the comparisons the deviations of the EKS indices from the characteristic binary (Fisher type) indices is minimised (least squares method).'

Thus, the EKS method attains transitivity while departing as little as possible from the Fisher Ideal bilateral index. It turns out that EKS multilateral comparisons can be obtained from Fisher Ideal bilateral indexes in exactly the same way that translog multilateral indexes are obtained from translog bilateral indexes. It is not known whether the EKS can be derived directly from a flexible transformation function that is non-separable in inputs and outputs and permits non-neutral differences in productivity among countries.[1] Nonetheless, Theorem 6 of Diewert (1978) assures us that the EKS method will provide a second order approximation to multilateral translog comparisons.

As we alluded in the opening section, Kravis, Heston and Summers (1978) rejected the EKS method, as being too mechanical, and adopted instead a procedure that they call the Geary–Khamis method. It does not appear that the Geary–Khamis method has the theoretical justification possessed by the superlative indexes that we have discussed. Thus, we would recommend reconsideration of the methodology used in subsequent comparisons by Kravis and his associates.

In spite of their rejection of the EKS method, Kravis *et al.* (1978, Table 3.4) did complete an EKS output comparison. This allows us to infer whether the different methods can lead to substantially different results. Their Table 3.4 indicates that comparisons among similar countries (developed or developing) did not differ much, but that comparisons among dissimilar countries differed by as much as 20%.[2] Kravis *et al.* have not provided any comparisons using the translog index. They did, however, provide results based on a Walsh index, which – although not a superlative index – is similar to the translog index. The results were quite similar to those of the EKS method, and we conjecture that use of the translog index would also provide results that are close to those of the EKS method.

[1] It is straightforward to derive the EKS index in the separable, neutral case, but we have not succeeded in deriving EKS in the general case.

[2] Per capita output in the United Kingdom in 1970 was estimated to be 6.38 times that in Kenya by the EKS method but only 5.29 by the Geary–Khamis method.

Klock and Theil (1965) have carried out an international comparison of quantities consumed rather than produced. They (KT) argued in favour of making comparisons in logarithmic terms because interest generally focusses on relative differences. This led KT to use bilateral indexes of the translog form. To obtain transitive multilateral comparisons KT argued along lines similar to EKS, which led to a similar least squares criterion. The end result was an index having the form of the translog multilateral indexes derived in Section II. Thus, the KT method of comparing consumption among countries can be viewed as a superlative index based on the translog direct utility function.

V. CONCLUDING REMARKS

We have proposed definitions for bilateral output, input, and productivity comparisons for neoclassical structures of production, and we have shown that a widely used bilateral index, the Törnqvist-Theil-translog, is very attractive for making such comparisons. The indexes can be derived from a flexible multi-product, multifactor representation of the structure of production – the translog transformation function – restricted to constant returns to scale. These functions are flexible, and therefore the resulting indexes are superlative, as discussed by Diewert (1976). The indexes for output and input comparisons have the same form, and their ratio coincides with the appropriate index for productivity comparisons.

We have generalised the definitions for bilateral output, input, and productivity comparisons to the multilateral case. Using these definitions, we have derived the indexes that are implied by the translog transformation function. These indexes provide transitive multilateral comparisons that maintain a high degree of characteristicity.

The superlative multilateral indexes that we have proposed are very attractive for cross section comparisons and for panel data comparisons, but they are not necessarily preferable to chain-linked bilateral indexes for time series comparisons. This follows because chronology provides a natural ordering of time series data that is lacking for cross-section or panel data.

Extensive cross-section and panel data sets, suitable for multilateral comparisons, have become more readily available in recent years. The economic theory of index numbers provides powerful tools that can be used in making such comparisons. Unfortunately, however, these tools have not been widely used in developing empirical methods of comparison. For example, Kravis and his associates, in their large and continuing international comparisons project, have adopted a non-superlative multilateral index procedure that sacrifices a large amount of characteristicity. Their own calculations indicate that some of their results differ substantially from those that would be obtained from the translog multilateral output index proposed in this paper.

University of Wisconsin–Madison
University of British Columbia

Date of receipt of final typescript: June 1981

References

Afriat, S. N. (1972). 'The theory of international comparisons of real income and prices.' In *International Comparisons of Prices and Outputs* (ed D. J. Daly), pp. 13–69. New York: National Bureau of Economic Research.

Byushgens, S. S. (1925). 'Ob odnom klasse giperpoverkhnostey: po povodu "idealnovo indeksa" Irving Fischer' a pokupatelnoi sili deneg.' *Mathematischeskii Sbornik*, vol. 32, pp. 625–31.

Caves, D. W., Christensen, L. R. and Swanson, J. A. (1980). 'Productivity growth, scale economies, and capacity utilization in U.S. railroads, 1955–1974.' Discussion Paper No. 48002, Social Systems Research Institute, University of Wisconsin–Madison, forthcoming in the *American Economic Review*, December 1981.

——, —— and Tretheway, M. W. (1981). 'Total factor productivity of U.S. trunk air carriers, 1972–1977.' In *Productivity Measurement in Regulated Industries* (ed. T. Cowing and R. Stevenson). Academic Press.

Christensen, L. R. and Jorgensen, D. W. (1970). 'U.S. real product and real factor input, 1929–1967.' *Review of Income and Wealth*, series 16, no. 1 (March), pp. 19–50.

—— and —— (1973). 'Measuring economic performance in the private sector.' In *The Measurement of Economic and Social Performance* (ed. M. Moss), pp. 233–338. New York: National Bureau of Economic Research.

——, —— and Lau L. J. (1971). 'Conjugate duality and the transcendental logarithmic production function.' *Econometrica*, vol. 39, pp. 255–6.

——, —— and —— (1973). 'Transcendental logarithmic production frontiers.' *Review of Economics and Statistics*, vol. 55, pp. 28–45.

Denison, E. F. (1967). *Why Growth Rates Differ*. Washington: The Brookings Institution.

Diewert, W. E. (1974). 'Functional forms for revenue and factor requirement functions.' *International Economic Review*, vol. 15, pp. 119–30.

—— (1976). 'Exact and superlative index numbers.' *Journal of Econometrics*, vol. 4, pp. 115–45.

—— (1978). 'Superlative index numbers and consistency in aggregation.' *Econometrica*, vol. 46, pp. 883–900.

—— (1979). 'The economic theory of index numbers: a survey.' Discussion Paper No. 79-09, Department of Economics, University of British Columbia, March.

—— (1980). 'Aggregation problems in the measurement of capital.' In *The Measurement of Capital* (ed. D. Usher). University of Chicago Press.

Dreschler, L. (1973). 'Weighting of index numbers in multilateral international comparisons.' *Review of Income and Wealth*, vol. 19, pp. 17–34.

Eltetö, Ö. and Köves, P. (1964). 'One index computation problem of international comparisons' (in Hungarian). *Statisztikai Szemle*, vol. 7.

Fisher, I. (1922). *The Making of Index Numbers*. Boston: Houghton-Mifflin.

Frisch, R. (1936). 'Annual survey of general economic theory: the problem of index numbers.' *Econometrica*, vol. 4, pp. 1–39.

Gilbert, M. and Associates (1958). *Comparative National Products and Price Levels: A Study of Western Europe and the United States*. Paris: Organisation for European Economic Cooperation.

—— and Kravis, I. B. (1954). *An International Comparison of National Products and the Purchasing Power of Currencies*. Paris: Organisation for European Economic Cooperation.

Jorgenson, D. W. and Nishimizu, M. (1978). 'U.S. and Japanese economic growth, 1952–1974: an international comparison.' ECONOMIC JOURNAL, vol. 88, no. 352 (December), pp. 707–26.

Klock, T. and Theil, H. (1965). 'International comparison of prices and quantities consumed.' *Econometrica*, vol. 33 (July), pp. 535–56.

Konyus, A. A. and Byushgens, S. S. (1926). 'K probleme pokupatelnoi cili deneg.' *Voprosi Konyunkturi*, II: 1, pp. 151–72.

Kravis, I. B., Kennessey, A., Heston, A. and Summers, R. (1975). *A System of International Comparisons of Gross Product and Purchasing Power*. Baltimore: Johns Hopkins University Press.

——, Heston, A. and Summers, R. (1978). *United Nations International Comparison Project: Phase II, International Comparison of Real Product and Purchasing Power*. Baltimore: Johns Hopkins University Press.

Pollak, R. A. (1971). 'The theory of the cost of living index.' Research Discussion Paper No. 11, Office of Prices and Living Conditions, U.S. Bureau of Labor Statistics, Washington, D.C.

Ruggles, R. (1967). 'Price indexes and international price comparisons.' In *Ten Studies in the Tradition of Irving Fisher* (eds W. Fellner *et al.*), pp. 171–205. New York: John Wiley.

Samuelson, P. A. and Swamy, S. (1974). 'Invariant economic index numbers and canonical duality: survey and synthesis.' *American Economic Review*, vol. 64, pp. 566–93.

Solow, R. A. (1957). 'Technical change and the aggregate production function.' *Review of Economics and Statistics*, vol. 39, pp. 312–20.

Szulc, B. (1964). 'Indices for multiregional comparisons' (in Polish). *Przeglad, Statystyczny*, vol. 3.
Theil, H. (1965). 'The information approach to demand analysis.' *Econometrica*, vol. 33, no. 1 (January), pp. 67–87.
Törnqvist, L. (1936). 'The Bank of Finland's consumption price index.' *Bank of Finland Monthly Bulletin*, no. 10, pp. 1–8.
Wald, A. (1939). 'A new formula for the index of the cost of living.' *Econometrica*, vol. 7, pp. 319–31.