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Abstract

The paper addresses functional form issues that arise when testing for the existence of homogeneously weakly separable aggregates in producer or consumer theory. Blackorby, Primont and Russell showed that the commonly used flexible functional forms became inflexible when separability restrictions were imposed. The paper proposes functional forms that are: (i) flexible, (ii) parsimonious, and (iii) have the correct curvature, both before and after separability restrictions are imposed. Two functional forms are proposed that are based on the normalized quadratic functional form, and the appropriate tests for the existence of separable aggregates are derived. The second test is a modification of Woodland's separability test.

Key words: Commodity aggregation; Tests of separability; Flexible functional forms
JEL classification: C12; C43

1. Introduction

In empirical applications of producer or consumer theory, the commodities which appear as variables in the production or utility function (e.g., 'blue collar', 'labour', or 'food') are in reality *aggregates* of the underlying microeconomic commodities. How then can we justify empirically estimated production or utility functions using aggregated data as being consistent with microeconomic theory which has the disaggregated micro commodities as the decision variables?

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One method for justifying the use of aggregated data is due to Shephard (1953, pp. 61–71), and it relies on the assumption of *homogeneous weak separability*. Shephard's method can be explained as follows. Suppose that the microeconomic production (or utility) function is F where $u = F(x, y)$ is the output (or utility) produced by the input (or consumption) vectors $x \equiv (x_1, \ldots, x_M)$ and $y \equiv (y_1, \ldots, y_N)$. Let $w \equiv (w_1, \ldots, w_M)$ and $p \equiv (p_1, \ldots, p_N)$ be the corresponding positive price vectors that the producer (or consumer) faces. Then the producer's *cost function* (or the consumer's expenditure function) can be defined as

$$C(u, w, p) \equiv \min_{x,y} \{w^T x + p^T y: F(x, y) \geqslant u\}, \tag{1}$$

where $w^T x \equiv \sum_{m=1}^{M} w_m x_m$ and $p^T y \equiv \sum_{n=1}^{N} p_n y_n$. Shephard (1953, p. 63), who worked in the producer theory context, then postulated that the micro function F had the following representation:

$$F(x, y) = f[g(x), y], \tag{2}$$

where g is a linearly homogeneous function of M variables. If we define the input aggregate as $y_0 \equiv g(x)$, then the cost function that corresponds to g can be written as $c(w)y_0$, where $c(w)$ is the *unit cost function* dual to the micro aggregator function g and c is defined as follows:

$$c(w) \equiv \min_{x} \{w^T x: g(x) = 1\}. \tag{3}$$

The cost function which is dual to the macro function f is C^*, defined as follows:

$$C^*(u, p_0, p) \equiv \min_{y_0, y} \{p_0 y_0 + p^T y: f(y_0, y) \geqslant u\}, \tag{4}$$

where $p_0 > 0$ is the price of a unit of the micro aggregate $y_0 = g(x)$.

Under the assumption that the micro function F has the homogeneous separable representation (2), the micro cost function C defined by (1) has the following representation:

$$\begin{aligned} C(u, w, p) &\equiv \min_{x,y} \{w^T x + p^T y: f[g(x), y] \geqslant u\} \\ &= \min_{x, y_0, y} \{w^T x + p^T y: f[y_0, y] \geqslant u, y_0 = g(x)\} \\ &= \min_{y_0, y} \{c(w) y_0 + p^T y: f[y_0, y\} \geqslant u\} \qquad (5) \\ &= C^*(u, c(w), p). \end{aligned}$$

Thus, under the assumption of homogeneous weak separability,[1] the unit cost function $c(w)$ is equal to the price of the aggregate p_0 and the quantity of the aggregate is equal to $y_0 = g(x)$.[2] The resulting p_0 and y_0 can be treated as if they were genuine microeconomic prices and quantities and the macro cost function C^* can replace the micro cost function C.

The problem that we wish to address can now be stated: given that the microeconomic optimization model (1) is true, how can we devise a statistical test for the validity of the homogeneous weak separability model (2) or (5)?

In order to implement a test, we must choose functional forms for the micro functions F and g (or their dual cost functions C and c) and for the macro function f (or its dual C^*). In choosing functional forms, we will try to satisfy the following three criteria. We want our functional forms to be: (i) *flexible*, (ii) *parsimonious*, and (iii) *curvature correct*. A functional form is flexible if it can provide a second-order approximation[3] to an arbitrary twice continuously differentiable function (which satisfies the appropriate regularity conditions). A functional form is parsimonious[4] if it can provide the second-order approximation using a minimal number of parameters. A functional form is *locally curvature correct* if it satisfies the appropriate convexity or concavity properties[5]

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[1] In the economics literature, the concept of weak separability was independently proposed by Sono (1945) in the consumer context and by Leontief (1947) in the producer context. The term weak is due to Strotz (1959), who also coined the term strong separability to refer to the concept of additive separability. Shephard (1953, p. 63) was the first to realize that the assumption of weak separability by itself [i.e., F has the form (2) but g is not necessarily homogeneous] was not sufficient to imply the existence of a price index $c(w)$ that would be independent of other prices and quantities. Gorman (1959) later found conditions weaker than the linear homogeneity of the micro aggregator function $g(x)$. For expositions and extension of this literature, see Geary and Morishima (1973, pp. 100–103), Blackorby, Primont, and Russell (1977a; 1978, Ch. 5), Blackorby and Schworm (1984), and Blackorby, Schworm, and Fisher (1986).

[2] If we are given the microeconomic price and quantity vectors w^t and x^t for $t = 1, 2, \ldots, T$, then index number techniques can be used to compute the aggregate prices and quantities, p_0^t and y_0^t. For example, if $g(x) \equiv (x^T A x)^{1/2}$ or $c(w) \equiv (w^T B w)^{1/2}$ where A and B are symmetric matrices with one positive eigenvalue, then Diewert (1976, pp. 116, 134) showed that the Fisher (1922) Ideal price and quantity indexes could be used to compute the aggregates as follows:

$$p_0^t \equiv [w^{tT} x^t w^{1T} x^t / w^{tT} x^1 w^{1T} x^1]^{1/2} \quad \text{and} \quad y_0^t \equiv w^{tT} x^t / p_0^t, \qquad t = 1, 2, \ldots, T.$$

[3] Diewert's (1974a, p. 113) original definition of flexibility was called a second-order differential approximation by Lau (1974, p. 184), who distinguished this concept from the more usual concept of a second-order Taylor series approximation. Barnett (1983, pp. 19–20) later showed that the two concepts were equivalent.

[4] The term is due to Fuss, McFadden, and Mundlak (1978, p. 224). As these authors noted, an excessive number of parameters often leads to multicollinearity, instability of parameter estimates, a loss of degrees of freedom and hence a loss in the precision of estimation.

[5] See Diewert, Avriel, and Zang (1981) for definitions and characterizations of the various kinds of convexity and concavity that are used in economics.

at a single point and *globally curvature correct* if it satisfies the correct curvature conditions at all data points in the statistical sample being used. Obviously, we would prefer that our suggested functional forms be globally curvature correct rather than just locally curvature correct.

We have omitted another desirable property for a functional form and that is *monotonicity correctness*; i.e., the cost function should be nondecreasing in its input price arguments. Our reason for omitting this important property is that it will almost always be satisfied in the region spanned by the sample input prices, provided that the derived input demand functions are used in estimating the unknown parameters. This sample local monotonicity property can be extended to global monotonicity by intersecting the isoquants derived from the cost function with the nonnegative orthant. However, if the cost function is to be estimated using only price and cost data, then the imposition of monotonicity becomes a problem, which we do not address in this paper.

There is a relatively large theoretical and empirical literature on testing for weak separability. Berndt and Christensen (1973, 1974) tested for the existence of a capital aggregate and a labour aggregate, respectively, in the context of one-output, three-input translog models of U.S. manufacturing, while Jorgenson and Lau (1975) tested for the existence of separable homogeneous aggregates in the context of a three-good U.S. translog aggregate consumer model. Additional tests using translog functional forms were implemented by Berndt and Wood (1975) and by Denny and Fuss (1977). All of these tests started with the maintained hypothesis that the micro function F (or the dual cost function or indirect utility function) was a flexible translog functional form. However, Blackorby, Primont, and Russell (1977b; 1978, Ch. 8) showed that all of these tests, under the hypothesis of homogeneous weak separability, led to a c or C^* that was necessarily inflexible. In fact, they showed that if we started with any flexible functional form for the micro cost function C that was a transformed quadratic in prices, then under the hypothesis of homogeneous weak separability, the resulting c or C^* (or both) would *necessarily be inflexible.*[6] This rather negative result tended to reduce the flow of papers that tested for homogeneous separability using flexible functional forms.

However, there were two papers which did not suffer from the Blackorby, Primont, and Russell inflexibility results. The first approach was due to Hall (1973) who basically assumed that the micro cost function C had a fourth-order approximation property.[7] With this maintained hypothesis, it proved to be easy to impose homogeneous separability so that the resulting C^* and c functions would be flexible. There are, however, two problems with Hall's approach: (i) in the unrestricted case the micro function is not parsimonious (i.e., a fourth-order approximation requires an enormous number of parameters) and (ii) one cannot impose the correct curvature conditions on these functional forms without destroying their flexibility (i.e., his functional forms were not curvature correct).

[6]Some of Blackorby, Primont, and Russell's results were obtained by Denny and Fuss (1977).

[7]Actually, Hall worked with a joint cost function, which, using our notation, would be defined as $\hat{C}(u, w, y) \equiv \min_x \{w^T x \colon F(x, y) \geqslant u\}$ where y is now interpreted as a vector of other outputs (in addition to the output u). In the homogeneous separable case, $\hat{C}(u, w, y) = h(u, y)c(w)$. Hall used Diewert's (1971) generalized linear functional form for h and his generalized Leontief functional form for the unit cost function c.

The second approach that avoided the Blackorby, Primont, and Russell inflexibility results was due to Woodland (1978). His approach utilized the variable profit function[8] and hence is valid only in the production theory context. His approach satisfied (i) flexibility and (ii) parsimony, but since he used a translog variable profit function,[9] there is no guarantee that the appropriate curvature conditions will be satisfied at even a single data point in empirical applications. Thus, Woodland's approach fails our third criterion for evaluating tests for homogeneous separability: his approach is not in general curvature correct. However, it is relatively easy to overcome this defect of his testing procedure using a different functional form for the micro variable profit function. We pursue his approach in Section 5 below.

A final approach to testing for separability should be mentioned. Blackorby, Schworm, and Fisher (1986) have proposed some tests of homogeneous (and nonhomogeneous) weak separability in the production theory context using the symmetric Generalized Barnett functional form proposed by Diewert and Wales (1987). Unfortunately, when the appropriate curvature conditions are imposed on this functional form, it is not only far from being parsimonious, but it is also no longer fully flexible – although it is quasiflexible to use Diewert and Wales (1987, p. 57) terminology. The Blackorby, Schworm, and Fisher paper is unique, however, in simultaneously considering the aggregation over goods and the aggregation over producers problem.

We shall attempt to avoid the aggregation over producers problem by assuming that the disaggregated technology set exhibits constant returns to scale[10] (or equivalently, that the microeconomic production function F is linearly homogeneous). This is a serious limitation of our analysis, but we feel that the problems involved in modelling nonconstant returns to scale are sufficiently complex that a separate treatment is required.[11] Another limitation of our analysis is that we have not explicitly derived tests for homogeneous separability in the consumer theory context, although our production theory test for separability explained in Section 4 below could be adapted to the consumer context.[12]

[8]See Gorman (1968) and Diewert (1973, pp. 291–294) for detailed definitions and duality theorems.

[9]See Diewert (1974a, pp. 139–140) for this generalization of the original translog functional form due to Christensen, Jorgenson, and Lau (1971).

[10]Actually our models can deal with the diminishing returns to scale case: we need only introduce an artificial fixed input that will absorb the pure profits of the firm.

[11]Spline techniques represent one approach that could be used to model nonconstant returns to scale technologies; see, for example, Diewert and Wales (1992).

[12]A unit cost function would replace the unit profit function and utility u would replace the fixed input. However, the resulting preferences would be homothetic and this is unrealistic. Allowing for general nonhomothetic preferences again leads us to consider spline models; see Diewert and Wales (1993).

We conclude this section by providing an outline of the remainder of the paper.

Section 2 sets up our basic model of producer behaviour. Rather than using production functions and cost functions to characterize the production possibilities set, we use factor requirements functions and profit functions because, in most applications, it is more natural to hold one input fixed (an input which is not adjustable in the short run) and maximize profits subject to this constraint rather than to hold one output fixed and minimize the cost of producing that output.[13] Our basic model of producer behaviour will also allow for multiple outputs and multiple inputs.

In Section 3, we propose functional forms for the separable model that are flexible, parsimonious, and curvature correct. In Section 4, we extend the model of Section 3 to allow for a test for the existence of a homogeneously separable aggregator function. We call this Method I for testing for separability.

In Section 5, we develop an alternative testing procedure (Method II) which is a straightforward adaptation of Woodland's (1978) method. However, we improve upon Woodland's translog functional form by suggesting curvature correct functional forms.

In Section 6, we employ the functional forms introduced in Sections 3 and 4 to illustrate our Method I procedure using two data sets. The first contains data on capital, labour, energy, materials, and output for U.S. manufacturing industries over the period 1947–81. The second contains U.S. national accounts data on consumption, investment, exports, imports, labour, and capital for the period 1948–87. For both data sets we estimate profit functions and carry out various separability tests, both with and without the imposition of curvature. In Section 7, we use these same data sets to illustrate our Method II procedure introduced in Section 5.

2. Profit functions and homogeneous weak separability

We consider the case of a price-taking, profit-maximizing producer. There are $M + N$ variable inputs and outputs and one fixed input that is not varied in the time period under consideration. The M-dimensional vector $x \equiv (x_1, \ldots, x_M)$ denotes the quantity vector of variable inputs and outputs that may be homogeneously separable and the N-dimensional vector $y \equiv (y_1, \ldots, y_N)$ denotes the quantities of the remaining variable inputs and outputs used or produced. If $x_m > 0$ ($y_n > 0$), then the mth (or nth) good is produced, while if $x_m < 0$ ($y_n < 0$), then the mth (or nth) good is being utilized as an input.

Given the net output vectors x and y, $k = F(x, y)$ is defined to be the minimal amount of the fixed input that is required to produce the net output vectors x and y; F is a *factor requirements function*.[14] We assume that F is a linearly homogeneous, continuous, and convex function over the closed convex set of (x, y) where $F(x, y)$ is finite. This implies that the corresponding production possibilities set $S \equiv \{(x, y - k)\colon F(x, y) \leqslant k,\ k \geqslant 0\}$ is a closed convex cone, so that there are constant returns to scale in production.

[13] However, we do indicate how our model can be adapted to deal with the cost minimization problem.

Let $w \equiv (w_1, \ldots, w_M) \gg 0_M$ and $p \equiv (p_1, \ldots, p_N) \gg 0_N$ be vectors of positive prices that the producer faces for the variable goods. Given that the producer has $k > 0$ units of the fixed input, the producer's *profit function* Π is defined as the optimized objective function for the following profit maximization problem:

$$\Pi(w, p, k) \equiv \max_{x,y} \{w^T x + p^T y\colon F(x, y) \leqslant k\}. \tag{6}$$

The *unit profit function* π for the firm is defined by the firm's profit maximization problem (6) when we set $k = 1$; i.e.,

$$\pi(w, p) \equiv \max_{x,y} \{w^T x + p^T y\colon F(x, y) \leqslant 1\}. \tag{7}$$

Due to our assumption of constant returns to scale, $\Pi(w, p, k) = \pi(w, p)k$, and so the profit function Π can be defined in terms of the unit profit function π. It can be shown[15] that $\pi(w, p)$ will be convex and linearly homogeneous in the components of (w, p). Furthermore, if Π is differentiable at (w, p, k), then by Hotelling's Lemma [see Diewert (1973; p. 294)], the profit-maximizing net output vectors x^* and y^* are equal to the vectors of first-order partial derivatives of Π with respect to the components of w and p, respectively; i.e., $x^* = \Pi_w(w, p, k)$ and $y^* = \Pi_p(w, p, k)$. In terms of the derivatives of the unit profit function π, we will have

$$x^*/k = \nabla_w \pi(w, p) \quad \text{and} \quad y^*/k = \nabla_p \pi(w, p). \tag{8}$$

Thus, given an appropriate functional form for the unit profit function π, the $M + N$ equations in (8) can be used to estimate the unknown parameters in π.

If there exists a homogeneous weakly separable aggregate in x, then the micro factor requirements function $F(x, y)$ will have the following functional structure:

$$F(x, y) = f[g(x), y], \tag{9}$$

where f is a macro factor requirements function and g is a nonnegative, linearly homogeneous micro aggregator function. The unit profit function that

[14] The terminology is due to Zvi Griliches. We define $F(x, y) \equiv +\infty$ if no $k \geqslant 0$ exists which would make (x, y) feasible. We can view $-F(x, y) \equiv t(x, y)$ as a transformation function which satisfies the regularity conditions listed in Diewert (1973, p. 292). For additional material on factor requirements functions in the one input case, see Diewert (1974b).

[15] See Diewert (1973, p. 293) for detailed duality theorems between π and F. Unit profit functions were studied in Diewert and Woodland (1977).

corresponds to $g(x)$ is defined by

$$r(w) = \max_x \{w^T x: g(x) = 1\}. \tag{10}$$

The definition of $r(w)$ will imply that $r(w)$ is a linearly homogeneous and convex function in the components of w. If $r(w) < 0$, then the corresponding aggregate $g(x)$ can be interpreted as an input, while if $r(w) > 0$, then $g(x)$ is an output. Note that in both the input and output case, we have forced $g(x)$ to be positive.

If the homogeneously separable model (9) is true, then we can define the *macro unit profit function P* as follows:

$$P(p_0, p) \equiv \max_{y_0, y} \{p_0 y_0 + p^T y: f(y_0, y) \leqslant 1\}, \tag{11}$$

where the aggregate price p_0 satisfies the following sign conventions: $p_0 < 0$ if $r(w) < 0$ and $p_0 > 0$ if $r(w) > 0$. The aggregate quantity y_0 is always restricted to be nonnegative. Note that P is the macro counterpart to the micro π defined by (7).

Now we are ready to determine the implications of homogeneous weak separability on the micro unit profit function π. If (9) is true, then substitution of (9) into (7) yields the following equalities:

$$\pi(w, p) \equiv \max_{x, y} \{w^T x + p^T y: f[g(x), y] \leqslant 1\}$$

$$= \max_{x, y_0, y} \{w^T x + p^T y: f[y_0, y] \leqslant 1, y_0 = g(x)\}$$

$$= \max_{x, y_0, y} \{w^T (x/y_0) y_0 + p^T y: f[y_0, y] \leqslant 1, 1 = g(x/y_0)\}$$

where we have used the linear homogeneity of g

$$= \max_{y_0, y} \{r(w) y_0 + p^T y: f[y_0, y] \leqslant 1\}$$

using the definition of $r(w)$, (10)

$$= P[r(w), p], \tag{12}$$

where the last equality follows using the definition of P, (11). Thus, under the hypothesis of the homogeneous weak separability of the factor requirements function (9), we find that the micro unit profit function π is also homogeneously weakly separable. Note that the price of the aggregate is $p_0 \equiv r(w)$ and the quantity of the aggregate (in the case where P is differentiable) will be

$$y_0 \equiv \partial P(p_0, p)/\partial p_0 \geqslant 0, \qquad p_0 \equiv r(w). \tag{13}$$

Note that our sign conventions require $P(p_0, p)$ to be nondecreasing in p_0. In addition, we require $P(p_0, p)$ to be linearly homogeneous and convex in (p_0, p) and we require $r(w)$ to be linearly homogeneous and convex in w.

Our task in the next section will be to find a functional form for the micro unit profit function $\pi(w, p)$ that has the structure (12) where the macro unit profit function P and the micro aggregator unit profit function r are: (i) flexible, (ii) parsimonious, and (iii) curvature correct.

3. Normalized quadratic unit profit functions

In this section, we define flexible functional forms for the macro unit profit function P defined by (11) and for the micro aggregator unit profit function r defined by (10).

Suppose that P has the following *normalized quadratic*[16] functional form:

$$P(p_0, p) \equiv a_0 p_0 + a^T p + \tfrac{1}{2}[p_0, p^T] \begin{bmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{bmatrix} \begin{bmatrix} p_0 \\ p \end{bmatrix} \Big/ \alpha^T p, \tag{14}$$

where $[a_0, a^T] \equiv [a_0, a_1, \ldots, a_N]$ is a parameter vector, A_{11}, $A_{12} = A_{21}^T$, and A_{22} are parameter matrices, and $\alpha \equiv [\alpha_1, \ldots, \alpha_N]^T$ is a predetermined vector of nonnegative constants. These vectors and matrices satisfy the following linear restrictions[17] at some predetermined price vector $[p_0^*, p^{*T}] \equiv [p_0^*, p_1^*, \ldots, p_N^*]$ where $p^* \gg 0_N$:

$$\alpha^T p^* = 1, \tag{15}$$

$$A_{11} p_0^* + A_{12} p^* = 0, \tag{16}$$

$$A_{12}^T p_0^* + A_{22} p^* = 0_N. \tag{17}$$

Assuming that $p_0^* \neq 0$, we may use (16) and (17) to solve for the scalar A_{11} and the vector A_{12} in terms of the elements of the N by N matrix A_{22}. Defining $A_{22} \equiv A = [a_{ij}]$, we have

$$A_{12}^T = -Ap^*/p_0^* \quad \text{and} \quad A_{11} = p^{*T} A p^* / [p_0^*]^2. \tag{18}$$

Substituting (18) into (14) yields the following expression for P:

$$P(p_0, p) = a_0 p_0 + a^T p + \tfrac{1}{2}[\alpha^T p]^{-1} p^T A p - [\alpha^T p]^{-1} p^{*T} A p [p_0/p_0^*] + \tfrac{1}{2}[\alpha^T p]^{-1} p^{*T} A p^* [p_0/p_0^*]^2, \qquad A = A^T. \tag{19}$$

Using Theorem 10 *in Diewert and Wales* (1987), *it can be shown that* $P(p_0, p)$ will be globally convex in (p_0, p) if and only if A is a positive semidefinite

[16] See Diewert and Wales (1987, p. 53), where we called this functional form the symmetric generalized McFadden, or Diewert and Wales (1992, p. 707).

[17] Restrictions like (16) and (17) are required in order to permit identification of the unknown parameters in (14).

symmetric matrix. Applying Theorem 11 in Diewert and Wales (1987, p. 54), it can be shown that the P defined by (19) is flexible. There are $1 + N + N(N+1)/2$ a_i's and a_{ij}'s that are not predetermined and hence this functional form is also parsimonious.

In empirical applications, if the estimated $A \equiv [a_{ij}]$ matrix does not turn out to be positive semidefinite, then this property may be imposed without destroying the flexibility of the functional form by setting[18]

$$A = XX^T, \quad X \text{ is lower triangular,} \tag{20}$$

i.e., $X \equiv [x_{ij}]$ is an N by N lower triangular matrix so that $x_{ij} = 0$ if $j > i$. Thus, if we reparameterize the A matrix using (20), the resulting $P(p_0, p)$ defined by (19) will be globally convex in (p_0, p); i.e., P will be globally curvature correct as well as being flexible and parsimonious.

We now turn our attention to the problem of choosing a functional form for $r(w)$. Let $r(w)$ be defined by the following normalized quadratic functional form for $w \gg 0_M$:

$$r(w) \equiv b^T w + \tfrac{1}{2} w^T B w / \beta^T w, \tag{21}$$

where b $[b_1, \ldots, b_M]^T$ is a parameter vector and $B \equiv \{b_{ij}]$ is a symmetric M by M matrix of parameters to be determined empirically. The vector $\beta \equiv [\beta_1, \ldots, \beta_M]^T$ is nonnegative and predetermined with

$$\beta^T w^* = 1, \tag{22}$$

for some predetermined vector of prices $w^* \gg 0_M$. The parameter vector b is assumed to satisfy the following linear restriction:

$$p_0^* = b^T w^*, \tag{23}$$

where p_0^* is the same predetermined price that appeared in (16) and (17). We shall set $p_0^* \equiv 1$ if $w^{*T} x^* > 0$ (in this case, the goods to be aggregated form a net output) or $p_0^* \equiv -1$ if $w^{*T} x^* < 0$ (in this case, the aggregated goods form a net input). In both cases, $x^* \equiv \nabla_w \Pi(w^*, p^*, k^*)$ is the optimal x vector which corresponds to the profit maximization problem defined by (6) when $(w, p, k) = (w^*, p^*, k^*)$. Finally, the symmetric matrix B is assumed to satisfy the following M linear restrictions:

$$Bw^* = 0_M. \tag{24}$$

[18]This method for imposing positive semidefiniteness was originally used by Wiley, Schmidt, and Bramble (1973, p. 318). Drawing on a theorem due to Lau (1978, p. 427), Diewert and Wales (1987, pp. 52–53) show that any positive semidefinite symmetric matrix A can be written as XX^T where X is lower triangular. Explicit formulae, for implementing Lau's method are derived in Diewert and Wales (1988, pp. 336–340).

Again the results of Diewert and Wales can be adapted to show that the r defined by (21) can provide a second-order approximation to an arbitrary unit revenue function $r^*(w)$ at the point $w = w^*$, provided that $r^*(w^*) = 1$ ($p_0^* = 1$ in this case) or $r^*(w^*) = -1$ ($p_0^* = -1$ in this case). Thus, the normalized quadratic micro aggregator unit profit function, $r(w)$ defined by (21)–(24), is flexible. The total number of independent parameters in $r(w)$ is: $M - 1$ b_i's and $M(M-1)/2$ b_{ij}'s, and this is the minimal number that is required in order to be flexible in the present context. Hence, $r(w)$ defined by (21)–(24) is also parsimonious.

In empirical applications, if the estimated B matrix does not turn out to be positive semidefinite, then this property can be imposed without destroying the flexibility of the functional form by setting

$$B = LL^T, \quad L \text{ is lower triangular.} \tag{25}$$

Of course, in order for B to satisfy the restrictions (24), we will require the M by M lower triangular matrix L to satisfy:

$$L^T w^* = 0_M. \tag{26}$$

Thus, the $r(w)$ defined by (21), (22), (23), (25), and (26) will be flexible, parsimonious, and globally curvature correct.

Now set $p_0 = r(w)$ and substitute (21) into (19) to obtain the micro unit profit function $\pi(w, p)$ in this homogeneously separable model. We obtain

$$\begin{aligned}\pi(w,p) &= P[r(w), p] \\ &= a_0[b^T w + \tfrac{1}{2}(\beta^T w)^{-1} w^T B w] + a^T p + \tfrac{1}{2}[\alpha^T p]^{-1} p^T A p \\ &\quad - [p_0^* \alpha^T p]^{-1} p^{*T} A p [b^T w + \tfrac{1}{2}(\beta^T w)^{-1} w^T B w] \\ &\quad + \tfrac{1}{2}[p_0^{*2} \alpha^T p]^{-1} p^{*T} A p^* [b^T w + \tfrac{1}{2}(\beta^T w)^{-1} w^T B w]^2, \end{aligned} \tag{27}$$

where $\alpha \geqslant 0_N$ is predetermined and satisfies (15), $\beta \geqslant 0_M$ is predetermined and satisfies (22), b satisfies (23), A and B are positive semidefinite symmetric matrices and B satisfies (24). The total number of free parameters in (27) is: $1 + N + N(N+1)/2 + M - 1 + M(M-1)/2$ (the independent elements of a_0, a, A, b, and B, respectively) $= M + 2N + N(N-1)/2 + M(M-1)/2$.

Note that the right-hand side of (27) is *not* a normalized quadratic function in the elements of (w, p); i.e., components of the price vector w are raised to the *fourth* power (instead of the *second* power). Thus, our functional form for π in this homogeneously separable case is similar to the Hall (1973) functional form in this respect. It is this aspect of our model that enables us to avoid the Blackorby, Primont, and Russell (1977b) inflexibility results.

Even if $P(p_0, p)$ and $r(w)$ are convex functions in their arguments, it is not immediately evident by looking at (27) that $\pi(w, p) \equiv P[r(w), p]$ is convex in

(w,p). In fact, we do require an additional assumption in order to ensure the convexity of $\pi(w,p)$ as the following proposition shows.

Proposition 1. Define $\pi(w,p) \equiv P[r(w),p]$ *and suppose that the functions* $P(p_0,p)$ *and* $r(w)$ *are both twice continuously differentiable and convex in their arguments. If, in addition,*

$$\partial P(p_0,p)/\partial p_0 \geqslant 0, \tag{28}$$

so that $P(p_0,p)$ *is nondecreasing in* P_0, *then* $\pi(w,p)$ *is convex in* w,p.

Proofs of the Propositions are in the Appendix.

Thus, in empirical applications, even if the estimated A and B matrices which appear in (27) are positive semidefinite, we should check that condition (28) holds for each data point so that our estimated unit profit function $\pi(w,p)$ will satisfy the appropriate curvature conditions; i.e., for each data point (w^t,p^t), check whether

$$y_0^t \equiv \partial P(p_0^t,p^t)/\partial p_0 \geqslant 0 \quad \text{where} \quad p_0^t \equiv r(w^t). \tag{29}$$

Note that y_0^t and p_0^t defined in (29) will be the period t aggregate quantity and price respectively.

The results of this section show that we can satisfactorily model homogeneous weak separability using normalized quadratics for the macro unit profit function $P(p_0,p)$ and for the micro aggregator unit profit function $r(w)$. If we use (20) and (25), the resulting $\pi(w,p) = P(r(w),p)$ is flexible, parsimonious, and globally curvature correct if conditions (29) hold (provided that the null hypothesis of homogeneous separability is valid). In the following section, we define a function involving $N(M-1)$ independent parameters that can be added to the right-hand side of (27) to obtain a fully flexible unit profit function $\pi(w,p)$.

4. Testing for homogeneous separability: Method I

Consider an arbitrary unit profit function $\pi^*(w,p)$ that is twice continuously differentiable at (w^*,p^*). The linear homogeneity of π^* in prices implies the following $1+M+N$ restrictions on the first and second derivatives of π^*:

$$\nabla_w \pi^*(w^*,p^*)w^* + \nabla_p^T \pi^*(w^*,p^*)p^* = \pi^*(w^*,p^*), \tag{30}$$

$$\nabla_{ww}^2 \pi^*(w^*,p^*)w^* + \nabla_{wp}^2 \pi^*(w^*,p^*)p^* = 0_M, \tag{31}$$

$$\nabla_{pw}^2 \pi^*(w^*,p^*)w^* + \nabla_{pp}^2 \pi^*(w^*,p^*)p^* = 0_N. \tag{32}$$

Hence, if a functional form for a unit profit function is flexible, it must have at least $1+M+N+(M+N)(M+N+1)/2-(1+M+N) = M(M+1)/2 + MN + N(N+1)/2$ independent parameters. Thus, the functional form defined by the right-hand side of (27) has $N(M-1)$ too few parameters to be flexible.

Let $P[r(w),p]$ be defined by (27) and the restrictions listed below (27). Define $\pi(w,p)$ as follows:

$$\pi(w),p) \equiv P[r(w),p] + c^Tw + w^TCp/[\gamma^Tw + \delta^Tp], \tag{33}$$

where γ and δ are nonnegative predetermined parameter vectors satisfying

$$\gamma^Tw^* + \delta^Tp^* = 1. \tag{34}$$

The elements of the parameter vector $c \equiv [c_1,\dots,c_M]^T$ satisfy the following restriction:

$$c^Tw^* = 0. \tag{35}$$

The elements of the M by N matrix of parameters $C \equiv [c_{ij}]$ satisfy the following $N+M$ linear restrictions (only $N+M-1$ of them are independent):

$$w^{*T}C = 0_N^T, \tag{37}$$

$$Cp^* = 0_M. \tag{38}$$

We have added $M-1$ independent c_i parameters and $MN-(N+M-1)$ independent c_{ij} parameters, or $N(M-1)$ independent parameters in all to the separable functional form $P[r(w),p]$ discussed in the previous section. Thus, the $\pi(w,p)$ defined by (33)–(38) has just the minimal number of free parameters to be a flexible functional form for a general nonseparable unit profit function $\pi^*(w,p)$.

Proposition 2. $\pi(w,p)$ *defined by (33)–(38) is flexible at* (w^*,p^*).[19]

Thus, the π defined by (33)–(38) is flexible and parsimonious. This general $\pi(w,p)$ collapses into the homogeneously separable functional form $P[r(w),p]$ defined by (27) if the following linear restrictions are true:

$$c = 0_M \quad \text{and} \quad C = 0_{M\times N}. \tag{39}$$

Of course, in view of the maintained restrictions (37) and (38), there are only $M-1+MN-[M+N-1] = N(M-1)$ new restrictions in (39) that are linearly independent.

If the separability restrictions (39) are true, then the correct curvature conditions on $\pi(w,p) = P[r(w),p]$ can be imposed by using the transformations (20),

[19] We require an additional minor assumption; namely, that $w^{*T}\nabla_w\pi^*(w^*,p^*) \neq 0$ where π^* is the arbitrary unit profit function that we are approximating.

(25), and (26) explained in the previous section. However, if the restrictions (39) are not true, then simply setting $A = XX^T$ and $B = LL^T$, where X and L are lower triangular, will *not* ensure that the resulting $\pi(w,p)$ defined by (33)–(38) will be convex in prices, even at the point (w^*,p^*). The problem is that we need to restrict the matrix C in some way to ensure the convexity of $\pi(w,p)$ at (w^*,p^*) (but at the same time, we do not want to destroy the flexibility of π).

Straightforward computations show that the $M+N$ by $M+N$ matrix of second-order derivatives of π evaluated at (w^*,p^*) is

$$\nabla^2\pi(w^*,p^*) = \begin{bmatrix} a_0B + bb^Tp^{*T}Ap^*/p_0^{*2}, & C - bp^{*T}A/P_0^* \\ C^T - Ap^*b^T/p_0^*, & A \end{bmatrix}. \quad (40)$$

Proposition 3. $\nabla^2\pi(w^*,p^*)$ *defined by (40) (where* $Bw^* = 0_M$, $w^{*T}C = 0_N^T$, $Cp^* = 0_M$, *and* $b^Tw^* = p_0^*$), *will be positive semidefinite if and only if the matrix defined by (41) is positive semidefinite:*

$$\begin{bmatrix} a_0B, & C \\ C^T, & A \end{bmatrix}. \quad (41)$$

By Lau's (1978, p. 427) Theorem, if the matrix defined by (41) is positive semidefinite, then we have the following representations:

$$\begin{bmatrix} A, & C^T \\ C, & a_0B \end{bmatrix} = \begin{bmatrix} X, & 0_{N\times M} \\ Y, & Z \end{bmatrix}\begin{bmatrix} X, & 0_{N\times M} \\ Y, & Z \end{bmatrix}^T = \begin{bmatrix} XX^T, & XY^T \\ YX^T, & YY^T + ZZ^T \end{bmatrix}, \quad (42)$$

where X is an N by N lower triangular matrix, Z is an M by M lower triangular matrix, and Y is an unrestricted M by N matrix. Assuming that $a_0 > 0$, we may use Eqs. (42) to impose positive semidefiniteness on $\nabla^2\pi(w^*,p^*)$ without destroying the flexibility of π:

$$A = XX^T, \qquad X \equiv [x_{ij}] \text{ is lower triangular and } N \text{ by } N, \quad (43)$$

$$C = YX^T, \qquad Y \equiv [y_{ij}] \text{ is } M \text{ by } N, \quad (44)$$

$$B = a_0^{-1}[YY^T + ZZ^T], \quad Z \equiv [Z_{ij}] \text{ is lower triangular and } M \text{ by } M. \quad (45)$$

In order for C defined by (44) to satisfy restrictions (37) and (38), we must impose the following $M+N$ restrictions on the elements of Y (only $M+N-1$ of these restrictions are independent):

$$w^{*T}Y = 0_N^T, \quad (46)$$

$$Y(X^Tp^*) = 0_M, \quad (47)$$

where X appears in (43). Finally, for B defined by (45) to satisfy the linear restrictions (24), we must impose the following M linear restrictions on the elements of Z:

$$Z^Tw^* = 0_M. \quad (48)$$

If the transformations and restrictions (43)–(48) are imposed on our model, then the test for homogeneous weak separability becomes

$$c = 0_M \quad \text{and} \quad Y = 0_{M\times N}. \quad (49)$$

In view of the restrictions (35), (46), and (47), there are only $N(M-1)$ linearly independent restrictions in (49).

The above model accomplishes the task set out in the introduction: we have a parsimonious, flexible functional form for a general unit profit function $\pi(w,p)$ and the correct curvature conditions can be imposed at an arbitrary point (w^*,p^*). Unfortunately, we cannot guarantee that the correct curvature conditions will hold globally for the general model. However, if the separability restrictions (49) are true, then the resulting separable functional form, $\pi(w,p) = P(r(w),p)$, will satisfy the appropriate curvature conditions globally [provided that (29) holds].

In the context of cross-sectional data, this would complete our discussion of Model I. However, in the time series context, it is necessary to modify our model to allow for the possibility of technical progress.

Let the scalar variable t be an indicator of technical progress (in our applications, t will be a linear time trend). Recall that $r(w)$ was defined by (21)–(24) and this definition remains unchanged. Our old macro unit profit function P was defined by (27) and the restrictions listed below (27). Define a new macro unit profit function for the separable case P^* in terms of the old P as follows:

$$P^*(p_0,p,t) \equiv P(p_0,p) + [d_0^1p_0 + d^{1T}p]t + [d_0^2p_0 + d^{2T}p]t^2, \quad (50)$$

where d_0^1 and d_0^2 are scalar parameters and $d^1 \equiv [d_1^1,\ldots,d_N^1]^T$ and $d^2 \equiv [d_1^2,\ldots,d_N^2]^T$ are vectors of new parameters. Thus, P^* has $2N+2$ more parameters than P. Recall that our old general nonseparable unit profit function π was defined by (33)–(38). We maintain the restrictions (34)–(38) and define our new general unit profit function π^* as follows:

$$\pi^*(w,p,t) \equiv P^*[r(w),p,t] + c^Tw + [\gamma^Tw + \delta^Tp]^{-1}w^TCp + e^{1T}wt + e^{2T}wt^2, \quad (51)$$

where the new parameter vectors $e^1 \equiv [e_1^1,\ldots,e_M^1]^T$ and $e^2 \equiv [e_1^2,\ldots,e_M^2]^T$ satisfy the following restrictions (in order to permit identification of the parameters):

$$e^{1T}w^* = 0 \quad \text{and} \quad e^{2T}w^* = 0. \quad (52)$$

Thus, π^* has a total of $2(M+N)$ extra independent parameters compared to π. Note that we have simply added linear and quadratic trends in t times terms that are linear in prices.[20]

We can prove a counterpart to Proposition 2 for our new functional form π^*; namely, $\pi^*(w,p,t)$ is flexible at (w^*,p^*,t^*) provided that we scale the technical progress variable so that

$$t^* = 0. \tag{53}$$

In view of (53), the matrix of second-order derivatives of π^* with respect to the components of w and p evaluated at (w^*,p^*,t^*) is still equal to the right-hand side of (40), and hence we can still apply Proposition 3 to our new model and use the substitutions (44)–(48) in order to impose the correct curvature conditions on π^* at (w^*,p^*,t^*). Unfortunately, as was the case with our no technical progress model, there is no guarantee that the correct curvature conditions on π^* will hold at other data points.

The test for homogeneous separability in our new model becomes (39) or (49) and the new restrictions:

$$e^1 = 0_M \quad \text{and} \quad e^2 = 0_M. \tag{54}$$

In view of (52), there are only $2M-2$ independent restrictions in (54). Thus, in our new model, there are $N(M-1)+2M-2$ independent linear restrictions to test.

If the separability restrictions (49) and (54) are true, then $\pi^*(w,p,t) = P^*[r(w),p,t]$ and π^* will be convex in prices at the data point (w^t,p^t,t) provided that

$$y_0^t \equiv \partial P^*(p_0^t,p^t,t)/\partial_0 > 0 \quad \text{where} \quad p_0^t \equiv r(w^t). \tag{55}$$

Note that (55) serves to define the period t aggregate price and quantity, p_0^t and y_0^t, when the separability restrictions (49) and (54) are satisfied.

In order to estimate the unknown parameters in π^*, we apply Hotelling's Lemma (8) and add errors to the following $M+N$ estimating equations:

$$x^t/k^t = \nabla_w \pi^*(w^t,p^t,t), \qquad y^t/k^t = \nabla_p \pi^*(w^t,p^t,t), \tag{56}$$

where $k^t > 0$ is the amount of the fixed input used in period t, $x^t \equiv [x^t, \dots, x_M^t]^T$ is the period t net output vector for variable goods that are in the homogeneously separable aggregate (the 'separable goods') and $w^t \equiv [w_1^t, \dots, w_M^t]^T$ is the corresponding vector of positive period t prices, $y^t \equiv [y_1^t, \dots, y_N^t]^T$ is the period t net output vector for the variable goods which are not included in the aggregate (the 'other goods') and $p^t \equiv [p_1^t, \dots, p_N^t]$ is the corresponding vector of positive period t prices.

In some applications, it would be desirable to modify our basic model of producer behaviour to allow the fixed input k to be a 'fixed' output. In this case, we initially set $k = -u$, where $u > 0$ is the amount of the 'fixed' output that must be produced. The profit function Π defined by (6) is still valid if k is replaced by $-u$. However, the unit profit function π, previously defined by (7), is now defined as

$$\pi(w,p) \equiv \max_{x,y} \{w^T x + p^T y : F(x,y) \leqslant -1\} = \Pi(w,p,-1), \tag{57}$$

and making use of the linear homogeneity of F, we have $(w,p,-u) = \pi(w,p)u$ where the new unit profit function π defined by (57) is still convex in prices. Thus, all the analysis involving π presented in this section and the previous one is still valid; the only diffeence is that the positive period t output of the fixed good $u^t > 0$ replaces the positive period t fixed input $k^t > 0$ in the estimating equations (56).[21]

One unsatisfactory aspect of Method I for testing for homogeneous separability is that we cannot in general make the (nonseparable) π^* defined by (51) globally curvature correct. Thus, we turn to Method II which will enable us to overcome this defect.

5. Woodland's approach to testing for homogeneous separability: Method II

Let us return to the notation we used in Section 2.

The key to Woodland's (1978) homogeneous weak separability test is to construct the profit function $\Pi(w,p,k)$ defined above by (6) in two stages. In the first stage, the *variable profit function* $V(w,z,k)$ is defined as follows:

$$V(w,z,k) \equiv \max_x \{w^T x : F(x,-z) \leqslant k\}, \tag{58}$$

where F is the factor requirements function which appeared in (6), and we have now defined the net input vector $z \equiv -y$ where $y \equiv [y_1, \dots, y_N]^T$ is the old net output vector of 'other variable goods' which appeared in (6). Thus, the maximization problem in (58) maximizes the net revenue produced by the 'separable goods' in the $x \equiv [x_1, \dots, x_M]^T$ vector given that the amounts $z \equiv [z_1, \dots, z_N]^T$ of the 'other goods' are available as (net) inputs and given that k units of the

[20] Since average rates of technological progress have changed substantially over the last three decades, the assumption of linear trends in t is usually unsatisfactory. In Diewert and Wales (1992), we found that quadratic splines in t worked very well. In the interests of simplicity and readability, we decided not to implement our quadratic spline model in the present paper.

[21] The unit profit function π defined by (57) in this fixed output case is actually equal to minus a unit cost function c; i.e., we have $\pi(w,p) = -c(x,p)$.

fixed input are available.[22] $V(w,z,k)$ will be convex and linearly homogeneous in w and concave and linearly homogeneous in (z,k), provided that the factor requirements function F is linearly homogeneous, an assumption which we made in Section 2 and make here as well.[23]

If $V(w,z,k)$ is differentiable with respect to the components of w, then the solution to the variable profit maximization problem in (58), x^*, will be unique and will equal the vector of first-order partial derivatives of V with respect to the components of w (see Diewert, 1973, p. 294):

$$x^* = \nabla_w V(w,z,k). \tag{59}$$

The second-stage maximization problem is defined in (60), and the resulting optimized objective function is equal to our old profit function defined by (6) (remember, $-z = y$):

$$\max_x \{-p^T z + V(w,z,k)\} = \Pi(p,w,k). \tag{60}$$

The first-order necessary conditions for z^* to solve (60) are equivalent to the following equations:

$$p = \nabla_z V(w, z^*, k). \tag{61}$$

The M equations in (59) and the N equations in (61) can be used as estimation equations in order to determine the unknown parameters which occur in the functional form for the variable profit function V.[24]

Given that z^* satisfies (61), the second-order necessary conditions for z^* to solve (61) are:

$$\nabla^2_{zz} V(w, z^*, k) \text{ is a negative semidefinite } N \text{ by } N \text{ matrix.} \tag{62}$$

We now consider the implications of homogeneous weak separability in the present model; i.e., we assume, as in (9), that $F(x, -z) = f[g(x), -z]$ where f and g are linearly homogeneous functions. As in Section 2, define $r(w)$ by (10).

[22]We have switched from the net output vector y to the net input vector $z \equiv -y$ so that $V(w,z,k)$ will satisfy Conditions III on variable profit functions listed in Diewert (1973, p. 293); his p is our w and his v is our $[z,k]$. If the fixed output k is actually a fixed input, then simply replace k in (58) by $-u$, where $u > 0$ is the quantity of fixed output that must be produced.

[23]For a more precise statement of regularity conditions on V and F (and a duality theorem between the two), see Diewert (1973, pp. 292–294).

[24]Eqs. (59) with $z = z^*$ and (61) are the counterparts to our old estimating equations (8). In our old approach, the price vectors p and w were regarded as exogenous and the quantity vectors x^*/k^* and $y^*/k^* = -z^*/k^*$ were regarded as endogenous. In our new approach, the price vector w and the quantity vector (z^*, k) are regarded as exogenous, while the quantity vector x^* and the price vector p are regarded as endogenous. This endogeneity of p and exogeneity of z^* could be regarded as a limitation of Woodland's method.

If the inequality constraint in (58) is binding, we have

$$\begin{aligned} V(w,z,k) &\equiv \max_x \{w^T x: f[g(x), -z] = k\} \\ &= \max_{x, y_0} \{w^T x: f[y_0, -z] = k,\ y_0 = g(x)\} \\ &= \max_{x, y_0} \{w^T (x/y_0) y_0: f[y_0, -z] = k,\ 1 = g(x/y_0)\} \\ &= \max_{y_0} \{r(w) y_0: f[y_0, -z] = k\} \\ &= \max_{y_0} \{r(w) y_0: y_0 = h(z,k)\} \\ &= r(w) h(z,k), \end{aligned} \tag{63}$$

where $h(z,k)$[25] is the y_0 solution to $f(y_0, -z) = k$. Thus, in the homogeneous separable case (9), the micro variable profit function $V(w,z,k)$ decomposes into the product of two functions, $r(w)$ times $h(z,k)$, where $r(w)$ is the unit profit function dual to the micro homogeneous aggregator function $g(x)$, and $h(z,k)$ is a macro production function. The function $r(w)$ is still defined by (10) and we make the same scaling assumptions on $g(x)$ as we made in Section 2; i.e., $y_0 \equiv g(x)$ is always nonnegative. If the goods to be aggregated form a net output, then $r(w) > 0$, while if the goods to be aggregated form a net input, then $r(w) < 0$. In both cases, $h(z,k)$ must be nonnegative. However, if $r(w) > 0$, then $h(z,k)$ must be nondecreasing in the components of the net input vector (z,k), while if $r(w) < 0$, then h must be nonincreasing in (z,k).

In the separable case when (63) holds, the first-order conditions (61) become:

$$p = r(w) \nabla_z h(z,k), \tag{64}$$

and the second-order necessary conditions (62) become:

$$r(w) \nabla^2_{zz} h(z,k) \text{ is a negative semidefinite } N \text{ by } N \text{ matrix.} \tag{65}$$

Finally, the convexity in w property of $V(w,z,k)$ implies that the following conditions must hold in the separable case:

$$\nabla^2_{ww} r(w) h(z,k) \text{ is a positive semidefinite } M \text{ by } M \text{ matrix.} \tag{66}$$

There remains the problem of choosing functional forms for V in the general case and for r and h in the separable case. We would like these three functions to be flexible, parsimonious, and curvature correct. In addition, we want V to equal the product of r and h in the separable case.

[25]h can be interpreted as a macro production function if the y_0 aggregate is an output or as a macro factor requirements function if y_0 is an input. Note that our model cannot accommodate a situation where the aggregated goods change from being a net output to being a net input or vice versa. The decomposition of V given by (63) is essentially due to Woodland (1977).

Woodland (1978) chose $V(w,z,k)$ to be the translog variable profit function.[26] This functional form satisfies all the criteria listed in the paragraph above except for the curvature correctness property. Since translog models often fail to satisfy the correct curvature conditions when N or M are moderately large, we shall avoid this defect of Woodland's procedure by utilizing normalized quadratic functional forms where the correct curvature conditions can be imposed globally.

We model the case of a separable variable profit function first; recall (63). As in Section 3, let $r(w)$ be defined by (21)–(24). Now define $h(z,k)$ in a manner analogous to the definition of P given by (14):

$$h(z,k) \equiv a_0 k + a^T z + \tfrac{1}{2}[k, z^T]\begin{bmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22}\end{bmatrix}\begin{bmatrix} k \\ z \end{bmatrix} \Big/ [\alpha^T z + \alpha_{N+1} k], \tag{67}$$

where $\alpha_{N+1} \geqslant 0$ and $\alpha \equiv [\alpha_1, \ldots, \alpha_N]^T$ are predetermined with $\alpha_n \geqslant 0$ if $z_n > 0$ (in this case, good n is an output) or $\alpha_n \leqslant 0$ if $z_n < 0$ (in this case, good n is an input), and we also impose the following normalization on α:

$$\alpha^T z^* + \alpha_{N+1} k^* = 1. \tag{68}$$

The parameter matrices A_{ij} satisfy $A_{12} = A_{21}^T$ and $A_{22} = A_{22}^T$. The following linear restrictions are imposed on the elements of the A_{ij} in order to permit identification:

$$A_{11} k^* + A_{12} z^* = 0, \tag{69}$$

$$A_{12}^T k^* + A_{22} z^* = 0_N. \tag{70}$$

Letting $A_{22} = [a_{ij}]$, we use (69) and (70) to eliminate A_{11} and A_{12} and (67) becomes

$$\begin{aligned} h(z,k) = {} & a_0 k + a^T z + \tfrac{1}{2}(\alpha^T z + \alpha_{N+1} k)^{-1} z^T A z - (\alpha^T z + \alpha_{N+1} k)^{-1} z^{*T} \\ & \times A z (k/k^*) + \tfrac{1}{2}(\alpha^T z + \alpha_{N+1} k)^{-1} z^{*T} A z^* (k/k^*)^2. \end{aligned} \tag{71}$$

In order to impose the correct curvature on $r(w)h(z,k)$, where $r(w)$ is defined by (21)–(24) and $h(z,k)$ is defined by (71), we need to consider two cases:

Case (i): $w^{*T} x^* > 0$; i.e., the homogeneous aggregate $y_0^* = g(x^*)$ is an output. In this case, we set $b^T w^* = p_0^* \equiv 1$, the matrix B should be positive semidefinite, and the matrix A should be negative semidefinite. The correct curvature conditions at (w^*, z^*, k^*) can be imposed by using (25) and (26) to define B and by defining A as follows:

$$A = -XX^T, \qquad X \text{ is a lower triangular } N \text{ by } N \text{ matrix}. \tag{72}$$

[26] See Diewert (1974a, p. 139).

If these substitutions for A and B are used, then the correct curvature conditions will be imposed not only at the reference point (w^*, z^*, k^*) but also at all data points (w^t, z^t, k^t) where $r(w^t) \geqslant 0$ and $h(z^t, k^t) \geqslant 0$.

Case (ii): $w^{*t} x^* < 0$; i.e., the aggregate $y_0^* \equiv g(x^*)$ is an input. In this case, we set $b^T w^* = p_0^* \equiv -1$, the matrix B should be positive semidefinite, and the matrix A should also be positive semidefinite. The correct curvature conditions at (w^*, z^*, k^*) can be imposed by using (25) and (26) to define B and by using (20) to define A. If these substitutions for A and B are used, then the correct curvature conditions will be imposed for all data points (w^t, z^t, k^t) where $r(w^t) \leqslant 0$ and $h(z^t, k^t) \geqslant 0$.

Thus, if the substitutions for the matrices A and B indicated in the two cases are used, then our normalized quadratic $r(w)h(z,k)$ will satisfy the correct curvature conditions at the base point (w^*, t^*, k^*) (so it will be locally convex in w and locally concave in z, k) and r and h will also be flexible and parsimonious. In addition, r and h will be globally curvature correct.[27]

We now imbed the separable model in a general flexible model. Let r and h be defined as above, and define V as follows:

$$V(w,z,k) \equiv r(w)h(z,k) + w^T C z + w^T c k, \tag{73}$$

where the M by N matrix $C \equiv [c_{ij}]$ and the vector $c \equiv [c_1, \ldots, c_M]^T$ satisfy the following $N + 1 + M$ linear restrictions:

$$w^{*T}[C, c] = [0_N^T, 0], \tag{74}$$

$$Cz^* + ck^* = 0_M. \tag{75}$$

Only $M + N$ of the restrictions in (74) and (75) are independent. The total number of linearly independent parameters in (73) is $M - 1 + M(M-1)/2 + 1 + N + N(N+1)/2 + MN + M - (M+N) = M(M+1)/2 + MN + N(N+1)/2$ which is just the minimal number of parameters required for V to be flexible.

Proposition 4. V defined by (73)–(75), where r is defined by (21)–(24) and h is defined by (71), is flexible at (w^*, z^*, k^*).

Thus, the normalized quadratic functional form V defined by (73)–(75) is flexible.

The test for the existence of a homogeneously separable aggregate in the first M goods is

$$C = 0_{M \times N} \quad \text{and} \quad c = 0_M. \tag{76}$$

Of course, in view of the maintained restrictions (74) and (75), there are only $MN + M - (M+N) = N(M-1)$ independent new restrictions in (76), the

[27] However, $V(w,z,k) \equiv r(w)h(z,k)$ will not be globally curvature correct if either r or h changes sign over the sample.

same number of new restrictions that we had in our previous test for homogeneous separability.

If the separability restrictions (76) are true, then the correct curvature conditions on V at (w^*, z^*, k^*) can be imposed by replacing A and B by the appropriate matrices defined above for cases (i) and (ii); see the discussion around (72). Rather surprisingly, these same substitutions for A and B will suffice to impose the correct curvature conditions on the nonseparable V defined by (73)–(75) at (w^*, z^*, k^*). In fact, the correct curvature conditions will generally be imposed *globally* on the nonseparable V: in case (i), all we require is that $r(w^t) \geqslant 0$ and $h(z^t, k^t) \geqslant 0$, and in case (ii), all we require is that $r(w^t) \leqslant 0$ and $h(z^t, k^t) \geqslant 0$ for each data point (w^t, z^t, k^t), where $r(w)$ is still defined by (21)–(24) and $h(z,k)$ is defined by (71). This follows from the fact that even in the nonseparable case, we have $\nabla^2_{ww} V(w,z,k) = \nabla^2_{ww} r(w) h(z,k)$ [recall (66)] and $\nabla^2_{zz} V(w,z,k) = r(w) \times \nabla^2_{zz} h(z,k)$ [recall (65)]. The fact that our adaptation of Woodland's (1978) approach will generally be globally curvature correct is the major advantage of this approach compared to our earlier straightforward approach to testing for homogeneous separability, which could only guarantee the imposition of the correct curvature at the base point.

This completes our discussion of the Woodland approach to testing for homogeneous weak separability in the context of a cross-sectional model. However, in the time series context, we are again obliged to add some additional parameters to our basic model to allow for the possibility of technical progress.

As in the previous section, let the scalar variable t be an indicator of technical progress. Let $t = t^* \equiv 0$ when $(w, z, k) = (w^*, z^*, k^*)$. This normalization of t will ensure that our proof of the flexibility of V in Proposition 4 will still be valid when we modify V to allow for technical progress. Our modified functional form for V is defined as follows:

$$V^*(w,z,k,t) \equiv r(w)h^*(z,k,t) + w^T Cz + w^{1T} ck + (e^{1T} wt + e^{2T} wt^2)(\alpha^T z + \alpha_{N+1} k), \tag{77}$$

where the new parameter vectors $e^1 \equiv [e^1_1, \ldots, e^1_M]^T$ and $e^2 \equiv [e^2_1, \ldots, e^2_M]^T$ satisfy the two restrictions (52), $r(w)$ is still defined by (21)–(24) and h^* is defined in terms of h [defined before by (71)] as follows:

$$h^*(z,k,t) \equiv h(z,k) + [d^1_0 k + d^{1T} z]t + [d^2_0 k + d^{2T} z]t^2, \tag{78}$$

where the parameter scalars d^1_0 and d^2_0 and the parameter vectors $d^1 \equiv [d^1_1, \ldots, d^1_N]^T$ and $d^2 \equiv [d^2_1, \ldots, d^2_N]^T$ have been added to our old h, given by (67). In view of the two restrictions in (52), we have added $2(M+N)$ independent parameters to the unknown parameters which occurred in our original $V(w,z,k)$.

Our new test for the existence of a homogeneous separable aggregate in the first M goods is

$$C = 0_{M \times N}, \quad c = 0_M, \quad e^1 = 0_M, \quad e^2 = 0_M. \tag{79}$$

In view of our maintained restrictions (52), (74), and (75), there are $N(M-1) + 2(M-1)$ independent restrictions in (79).

The correct curvature conditions for our new model can be imposed at the base point (w^*, z^*, k^*, t^*) (for both the separable and nonseparable cases) by using the replacements for A and B in cases (i) and (ii) explained above; see the discussion around (72). The correct curvature conditions will hold for each data point (w^t, z^t, k^t, t) provided that in case (i), we have

$$r(w^t) \geqslant 0 \quad \text{and} \quad h^*(z^t, k^t, t) \geqslant 0, \tag{80}$$

and in case (ii), we have

$$r(w^t) \leqslant 0 \quad \text{and} \quad h^*(z^t, k^t, t) \geqslant 0. \tag{81}$$

The following $M + N$ equations can be used in order to estimate the unknown parameters which occur in the V^* defined by (77) and (78) [recall (59) and (61)]:

$$x^t = \nabla_w V^*(w^t, z^t, k^t, t), \tag{82}$$

$$p^t = \nabla_z V^*(w^t, z^t, k^t, t). \tag{83}$$

To review the notation, we have: $z^t = -y^t$ where $y^t \equiv [y^t_1, \ldots, y^t_N]^T$ is the observed period t net output vector of variable goods that are not to be aggregated and $p^t \equiv [p^t_1, \ldots, p^t_N]^T$ is the corresponding positive period t price vector; $x^t \equiv [x^t_1, \ldots, x^t_M]^T$ is the observed period t net output vector of variable goods that are included in the homogeneous aggregator function $g(x)$ (if it exists) and $w^t \equiv [w^t_1, \ldots, w^t_M]^T$ is the corresponding positive period t price vector, and finally, $k^t > 0$ is the fixed input utilized during period t.

In some of our empirical work, we replaced the positive fixed input k^t by $-u^t$, where $u^t > 0$ is the quantity of some 'fixed' output that must be produced during period t. However, the analysis presented in this section is still valid if k^t is replaced by $-u^t$, including (82) and (83).

We conclude this section by showing how the first- and second-order partial derivatives of the unit profit function $\pi(w,p)$ defined by (7) can be computed from the first- and second-order partial derivatives of $V(w, z^*, 1)$ where V is defined by (58) when $k = 1$ and z^* satisfies the first-order conditions for (60) when $k = 1$, so that we have

$$p = \nabla_z V(w, z^*, 1). \tag{84}$$

Proposition 5. If z^* solves (60) when $k = 1$ and z^* satisfies the resulting first-order necessary conditions (84) and the N by N matrix of second-order derivatives

$E \equiv \nabla^2_{zz} V(w, z^*, 1)$ is negative definite, then the level and first and second derivatives of the unit profit function $\pi(w, p)$ defined by (7) can be expressed in terms of the level and first and second derivatives of the variable profit function $V(w, z^*, 1)$ as follows:

$$\pi(w, p) = V(w, z^*, 1) - p^T z^*, \tag{85}$$

$$\nabla_w \pi(w, p) = \nabla_w V(w, z^*, 1), \tag{86}$$

$$\nabla_p \pi(w, p) = -z^*, \tag{87}$$

$$\nabla^2 \pi(w, p) = \begin{bmatrix} D - FE^{-1}F^T, & FE^{-1} \\ E^{-1}F^T, & -E^{-1} \end{bmatrix}, \tag{88}$$

where $D \equiv \nabla^2_{ww} V(w, z^*, 1)$, $E \equiv \nabla^2_{zz} V(w, z^*, 1)$, and $F \equiv \nabla^2_{wz}(w, z^*, 1)$. Moreover, the matrix on the right-hand side of (88) is a positive semidefinite $M + N$ by $M + N$ matrix.

The formula for $\nabla^2_{ww}\pi(w,p) = D - FE^{-1}F^T = \nabla^2_{ww} V(w, z^*, 1) + F[-E^{-1}]F^T$ that is contained in (88) shows that the substitution matrix for the unit profit function, $\nabla^2_{ww}\pi(w, p)$, will be *more* positive semidefinite (and hence will have *bigger* diagonal elements) than the corresponding substitution matrix for the variable profit function, $\nabla^2_{ww} V(w, z^*, 1)$. This is an example of Samuelson's (1947, p. 38) Le Chatelier Principle.

Formula (88) enables us to compare the price elasticities that correspond to an estimated variable profit function V to the same price elasticities that correspond to an estimated unit profit function π. However, in order to deduce (88), we need $E \equiv \nabla^2_{zz} V(w, z^*, 1)$ to be negative definite. This means that we need the A matrix in our normalized quadratic variable profit function to be negative definite. Unfortunately, in situations where we are forced to replace the A matrix by XX^T or $-XX^T$ in order to obtain the correct curvature conditions, the estimated A matrix will usually be singular and hence we cannot apply (88) in those situations.

We turn now to some empirical illustrations of our separability tests.

6. Method I: Estimation procedure and results

We illustrate the procedures outlined above using two different U.S. data sets. The first contains information for the period 1947–81 on price and output (Y) of U.S. manufacturing industries together with information on prices and quantities for four inputs: capital (K), labour (L), energy (E), and materials (M). These data have been used in studies by Berndt and Wood (1986b, 1987) and are described and published in Berndt and Wood (1986a). The second data set, drawn from the U.S. National Accounts, contains information for the period

1948–87 on prices and quantities for aggregate consumption (C), investment (I), exports (X), imports (M), labour (L), and capital (K), where both C and I include government purchases. These data have been used in two studies by Kohli (1991, 1993); the former contains a listing of the series together with a description of their construction.

For the manufacturing data we estimate nonseparable unit profit functions involving one output (Y), three variable inputs (L, E, M) and one fixed input (K). For the national accounts data, the profit functions involve three outputs (C, I, X), two variable inputs (L, M), and one fixed input (K). For both of these data sets we have chosen K as the fixed factor since it is the one that is least likely to be fully adjustable in the short run. An added advantage of choosing K as fixed is that information on its price is not required in the estimation (calculation of the appropriate rental price of capital often proves to be a difficult task). Of course, our results are dependent on this choice of fixed factor; for comparison purposes, we present results also for the case when output (Y) is fixed, using the manufacturing data only. For the national accounts data, there are three outputs and no obvious reason to choose one as fixed. In addition to the nonseparable (sometimes referred to below as full) models, we estimate various separable models reflecting different possible combinations of inputs and outputs. The appropriateness of these assumptions can then be tested using standard likelihood ratio tests, since by construction the separable models are nested in their nonseparable counterparts. Finally, for both the separable and nonseparable forms, the convexity in prices condition can be imposed at the point of approximation (t^*) if desired. For the separable form, imposition of this condition locally results in global convexity, whereas for the nonseparable form it does not, and consequently, violations of convexity may occur at sample points other than t^* for the nonseparable form.

The nonseparable unit profit function that we estimate is given by (51), and the separable form is obtained from this one by imposing the restrictions (49) and (54). Output supplies and input demands are obtained as usual by differentiating the profit function with respect to prices (input demands are of course negative). The usual adding up condition relating variable profits and these inputs in value terms, both for functions and for data, implies that one equation may be dropped in the estimation. The results will be invariant to the equation dropped under maximum likelihood estimation if the inputs and outputs are expressed in expenditure form. Rather than proceeding along these lines, we follow the standard procedure in the literature and estimate the set of input demands and output supplies in quantity form (after deflating by the quantity of the fixed good). In doing so we are implicitly treating the variable profit function differently from an estimation viewpoint. We consider this issue further when discussing estimation procedures below for Method II. We add a vector of disturbances (e) to each of these demand/supply systems, and assume the e's are independent across observations and that each has a multivariate normal distribution with zero mean and constant covariance matrix. We obtain maximum likelihood estimates making use of an algorithm due to Fletcher (1972).

In the estimation, we pick our reference price vector to be the vector of ones; thus (24) implies that all row sums of B are zero, (23) implies that the b's sum to either plus or minus one, and (35) implies that the c's sum to zero. The data point at which all prices are set to unity is arbitrary and is obtained by scaling the data. We select the midpoint of each sample to be this point, thus we divide the original price and multiply the original quantity of each input or output by its original price in this year. Our scaled data are therefore defined as $\tilde{p}_i^t = p_i^t/p_i^{t^*}$, $w_i^t = \tilde{w}_i^t/w_i^{t^*}$, $\tilde{y}_i^t = y_i^t p_i^{t^*}$, and $\tilde{x}_i^t = x_i^t w_i^{t^*}$, where t^* is the midpoint of the sample. Finally, the variable t is defined as a time trend with unit annual increments and is normalized to zero at t^*.

As is evident from (27), estimation of both the separable and nonseparable models requires us to select values for the vectors α and β. We do this in terms of the scaled data as follows:

$$\alpha_i \equiv |\bar{\tilde{y}}| \Big/ \sum_{j=1}^{N} |\bar{\tilde{y}}_j| = |\bar{y}_i| p_i^{t^*} \Big/ \sum_{j=1}^{N} |\bar{y}_j| p_j^{t^*}, \qquad i = 1, \dots, N, \tag{89}$$

$$\beta_i \equiv |\bar{\tilde{x}}_i| \Big/ \sum_{j=1}^{M} |\bar{\tilde{x}}_j| = |\bar{x}_i| w_i^{t^*} \Big/ \sum_{j=1}^{M} |\bar{x}_j| w_j^{t^*}, \qquad j = 1, \dots, M, \tag{90}$$

where $\bar{x}$, $\bar{\tilde{x}}$, $\bar{y}$, and $\bar{\tilde{y}}$ are the sample means of x, $\tilde{x}$, y, and $\tilde{y}$, respectively. Note that these vectors are nonnegative and satisfy (15) and (22), respectively. Finally, estimation of the nonseparable model (33) requires us to select values for the vectors δ and γ. We set $\gamma \equiv \beta/2$ and $\delta \equiv \alpha/2$, thereby satisfying (34) for the scaled prices, as well as the condition that the vectors be nonnegative.

In Table 1 we present summary results for the U.S. manufacturing industries, including log-likelihood values for the full models and their nested separable counterparts, together with the calculated chi-square values used in testing the restrictions. These results are obtained with and without imposing the convexity in prices condition at t^*. Recall that for the separable models imposition of convexity at a point is equivalent to global imposition, hence the final column in section (b) of the table consists of zeros by construction. The first three rows in each section contain information used in testing whether any pair of inputs can be aggregated, whereas the next three rows can be interpreted as testing for the existence of a value-added function, with each input in turn aggregated with output. In each of our tests, the number of goods in the separable group is $M = 2$ and the number of remaining goods is $N + 1 = 3$.

Several interesting issues are illustrated in the table. First, when convexity is not imposed, all the separability hypotheses are soundly rejected; indeed, the smallest calculated chi-square value is more than twice as large as the corresponding critical value at the 1% level. On the other hand, when convexity is imposed, the hypothesis that E and M are separable is only marginally rejected

Table 1

Summary statistics for Method I (K fixed), U.S. manufacturing data

	Log-likelihood values		Chi-square value	Convexity violations	
	NS	S		NS	S
(a) Convexity not imposed					
M, L	479.7	464.3	30.8	0	0
E, L	480.9	431.9	98.0	35	35
E, M	489.4	475.9	27.0	35	0
Y, L	479.7	462.3	34.8	0	0
Y, M	486.6	462.7	47.8	0	0
Y, E	487.6	475.3	24.6	35	0
(b) Convexity imposed at t^*					
M, L	479.7	464.3	30.8	0	0
E, L	477.4	431.8	91.2	16	0
E, M	480.8	475.9	9.8	13	0
Y, L	479.7	462.3	34.8	0	0
Y, M	486.6	462.7	47.8	0	0
Y, E	483.0	475.3	15.4	18	0

(1) NS and S refer to the nonseparable and separable models, respectively. (2) The sample contains 35 observations. (3) The separable models contains 14 and the nonseparable model 18 independent parameters; the critical chi-square values with 4 degrees of freedom are 9.5 and 13.3 at the 5% and 1% levels, respectively. (4) The first column of the table indicates the inputs/outputs that form the separable unit. (5) The point t^* is the sample midpoint.

at the 5% level (a chi-square value comparison of 9.8 versus 9.5) and is not rejected at the 1% or even the 4% level. Contributing to this difference is the fact that when convexity is not imposed, the nonseparable model violates convexity at all the data points, whereas when it is imposed (at t^*), convexity is violated at about one-third of the data points. Imposing convexity then reduces the likelihood to a value which does not differ that much from the value for the separable model. Thus, the hypothesis that E and M form a weakly separable group is rejected only marginally if at all when the convexity condition is maintained, but is decisively rejected if it is not maintained. On the other hand all the other hypotheses are rejected at the 5% level even when convexity is imposed, although the Y, E combination is only marginally rejected at the 1% level.

Second, it is interesting to note that when convexity is imposed at the point t^* in the nonseparable models, this always results in convexity being satisfied at more than just that one data point. Indeed, for the three cases in which the nonseparable model violates convexity – rows 2, 3, and 6 in part (a) of the table – imposition of convexity locally results in about one-half of the data points satisfying the condition. Finally, it should be kept in mind that each of the models reported in the table is a minimal flexible form in either the separable or nonseparable class. The fact that likelihood values differ (within a class) depending on which goods are selected for the separable group reflects the biquadratic nature of the profit function when written in terms of the individual goods' prices. For example, from (27) it is clear that prices of goods in the separable group will appear raised to powers as high as four in the profit function, whereas, in the standard normalized quadratic function, the highest power is two.

We have also made use of these data to estimate unit profit functions in which output rather than capital is fixed in the short run, thus giving four variable inputs K, L, E, and M, no variable outputs, and one fixed good Y. We do this in order to provide estimates of a model which is equivalent to the (negative of the) standard constant returns to scale cost function model estimated extensively in the literature. In the computations, K and Y are interchanged and information on the price of K is used, whereas information on the price of Y is not. Of course, modelling behaviour in this way assumes complete capital stock adjustments in the short run, which seems somewhat unrealistic, and that sufficient information is available to permit calculation of a rental cost of capital.

The estimation results are presented in Table 2, which is in the same format as Table 1. As in Table 1, the number of goods in the separable group is $M = 2$ and the number of remaining goods is $N + 1 = 3$. The calculated chi-square values in the first three rows of each section can be compared across tables since the same pairs of inputs are involved. Perhaps the most interesting (and encouraging) result to note is that these values are roughly similar in the two tables, and that they yield the same conclusions concerning separability. In particular, the hypothesis that E and M form a weakly separable group is the only one that is not decisively rejected by the data, and further, this conclusion holds only when the convexity in prices condition is imposed during the estimation. All the other hypotheses concerning separable pairs of inputs are rejected at the 1% level, even when convexity is imposed. In passing, it may be noted that the model in this form (with Y rather than K fixed) appears to violate convexity conditions to a greater extent. Not only do the unconstrained models exhibit more nonconvexities than do those in Table 1, but also the local imposition of convexity leaves more sample points at which the model is nonconvex – a total of 79 for all models in section (b) of Table 2 as compared with 47 in section (b) of Table 1.

In Table 3, we present summary results for the U.S. national accounts data in the same format as Tables 1 and 2. Recall that these data consist of three outputs (C, I, X), two variable inputs (L, M), and one fixed factor (K). The first three rows in sections (a) and (b) contain information useful in testing whether any pair of outputs forms a separable subgroup. Thus, the number of goods in the separable group is $M = 2$ and the number of remaining good is $N + 1 = 4$. Regardless of whether or not convexity is imposed, it is clear that no pairwise combinations

Table 2

Summary statistics for Method I (Y fixed), U.S. manufacturing data

	Log-likelihood values		Chi-square value	Convexity violations	
	NS	S		NS	S
(a) Convexity not imposed					
M, L	567.1	545.5	43.2	35	0
E, L	567.7	521.0	93.4	0	0
E, M	579.5	562.4	34.2	35	35
K, L	568.9	535.8	66.2	35	35
K, M	567.7	552.2	31.0	35	0
K, E	585.6	558.1	55.0	35	35
(b) Convexity imposed at t^*					
M, L	566.9	545.5	42.8	15	0
E, L	567.5	521.0	93.0	17	0
E, M	566.5	562.4	8.2	0	0
E, M	567.1	535.7	62.8	15	0
K,L	567.1	552.2	29.8	17	0
K, E	567.5	557.8	19.4	15	0

(1) NS and S refer to the nonseparable and separable models, respectively. (2) The sample contains 35 observations. (3) The separable model contains 14 and the nonseparable model 18 independent parameters; the critical chi-square values with 4 degrees of freedom are 9.5 and 13.3 at the 5% and 1% levels, respectively. (4) The first column of the table indicates the inputs/outputs that form the separable unit. (5) The point t^* is the sample midpoint.

are acceptable, since all calculated chi-square values are over twice as large as the 1% critical value. Although not recorded in the table, the hypothesis that all three outputs form a single separable group is also decisively rejected. The fourth row in each section addresses the question of whether the two inputs in the model can be combined, and this separability hypothesis is also decisively rejected.

Finally, although not reported in the table, we have considered a model in which I, C, and L form a separable group. This 'trade' model, involving exports, imports, and an aggregate variable domestic good is also clearly rejected. All of these results are, of course, just illustrative; our procedure allows for any combination of inputs/outputs to be considered (provided the value of the aggregate does not change sign over the sample period).

It may be noted that these data appear to violate the convexity conditions to a greater extent than do the manufacturing data discussed above (for the corresponding model with fixed K). When convexity is not imposed, curvature is violated at all sample points for all four combinations using the full model,

Table 3

Summary statistics for Method I (K fixed), U.S. national accounts data

	Log-likelihood values		Chi-square value	Convexity violations	
	NS	S		NS	S
(a) Convexity not imposed					
I, C	610.9	582.2	57.4	40	0
X, C	601.9	542.8	118.2	40	40
X, I	597.7	550.2	95.0	40	40
M, L	596.0	529.2	133.6	40	40
(b) Convexity imposed at t^*					
I, C	598.4	582.2	33.2	2	0
X, C	599.7	538.0	124.8	19	0
X, I	594.1	546.0	96.4	17	0
M, L	593.0	524.8	136.4	35	0

(1) NS and S refer to the nonseparable and separable models, respectively. (2) The sample contains 40 observations. (3) The separable models contains 14 and the nonseparable model 25 parameters; the critical chi-square values with 5 degrees of freedom are 11.5 and 15.1 at the 5% and 1% levels, respectively. (4) The first column of the table indicates the inputs/outputs that form the separable unit. (5) The point t^* is the sample midpoint.

and in three of four cases for the separable model. When convexity is imposed at t^*, this always results in some sample points violating convexity and indeed in one of the cases almost all the sample points violate convexity (row 4).

7. Method II: Estimation procedure and results

Our estimation procedure for Method II differs slightly from that for Method I. For the manufacturing data, the nonseparable variable profit functions now involve four inputs (K, L, E, M) and one output (Y), and for the National Accounts data, three inputs (L, M, K) and three outputs (C, I, X). The nonseparable variable profit function that we estimate is given by (77) and the separable form is obtained by imposing the restrictions (79). Estimation of this model requires us to select values for the vectors $(\alpha^T, \alpha_{N+1}) \equiv (\alpha_1, \ldots, \alpha_N, \alpha_{N+1})$ and $\beta \equiv (\beta_1, \ldots, \beta_M)$, which we define in terms of scaled data. As before, prices of goods in the separable group are scaled to equal one at t^*, but now quantities of goods in the other group are scaled to equal plus one for inputs and minus one for outputs at t^*. Thus, we have $\tilde{w}_i^t = w_i^t / w_i^{t^*}$ and $\tilde{x}_i^t = x_i^t w_i^{t^*}$ for the separable subgroup and $\tilde{z}_i^t = z_i^t / |z_i^{t^*}|$ and $\tilde{p}_i^t = p_i^t |z_i^{t^*}|$ for the other group. The vector β is

again given by (90) while α is defined as

$$\alpha_i \equiv \bar{\bar{p}}_i \left| \sum_{j=1}^{N+1} \bar{\bar{p}}_j \right| = \bar{p}_i^t |z_i^{t^*}| \sum_{j=1}^{N+1} \bar{p}_j^t |z_j^{t^*}|, \qquad i = 1, \ldots, N+1, \tag{91}$$

provided good i is an input, and minus this value if it is an output. Note that in (91), $z_{N+1} \equiv k$. Thus, α satisfies (68) as well as the required sign conventions given below (67).

Output supply and input demand equations for the separable goods are obtained by differentiating the variable profit function with respect to prices. As before, input demands are negative and output supplies positive. Shadow price equations for the goods not in the separable group are obtained by differentiating the variable profit function with respect to quantities, and although these prices are always positive, the corresponding quantities are set negative for outputs and positive for inputs, following the convention outlined above.

The set of goods to be held 'fixed' in the nonseparable variable profit function depends, of course, on the particular separability hypothesis under consideration, and is simply the set of goods that does not form the separable group. For this set of goods, quantities rather than prices are assumed to be exogenous, and these quantities (the z_i's and possibly $z_{N+1} \equiv k$ in the notation above) appear on the right-hand side of the estimating equations. Of course, this represents a different stochastic assumption from that of Method I and a different set of goods must be assumed exogenous for each test that involves different goods in the separable group. In our view, this represents a drawback of the Woodland approach to separability testing. Unfortunately, it is not possible to employ the usual Jacobian correction to adjust the likelihood in a manner that allows for all the prices to be assumed exogenous when testing whether different groups of goods are separable. The reason is that the Jacobian of the transformation from the disturbances to the z's and k requires that the A matrix be nonsingular. However, as mentioned above, imposition of the correct curvature conditions usually results in the estimated A matrix being singular.

As was the case for Method I, an adding-up condition relating variable profits (67) and variable inputs/outputs (82), when expressed in expenditure form, holds both for data and functions. Thus we may drop one equation from the estimation, which we chose to be the variable profit function. Further, in the theoretical analysis above, one of the 'fixed' goods in the nonseparable group was singled out and denoted as k rather than as one of the z's. The reason for this is to enable the two methods to be treated in a similar framework, as is accomplished in (14) and (67). However, careful consideration of Method II reveals that the only practical implication of this distinction from an estimation viewpoint is that k does not appear as a choice variable in the producer's problem and hence there is no corresponding shadow price equation to be estimated. This suggests that if there is one good in the nonseparable group for which information on

Table 4
Summary statistics for Method I (K fixed), U.S. manufacturing data

	Log-likelihood values		Chi-square value	Convexity violations	
	NS	S		NS	S
(a) Curvature not imposed					
M, L	482.4	467.2	30.4	0	0
E, L	483.2	439.9	86.6	0	35
E, M	522.4	507.2	30.4	0	0
Y, L	482.1	460.3	43.6	0	0
Y, M	461.0	446.8	28.4	0	0
Y, E	541.8	538.2	7.2	0	0
(b) Curvature imposed					
M, L	482.4	467.2	30.4	0	0
E, L	483.2	439.4	87.6	0	0
E, M	522.4	507.2	30.4	0	0
Y, L	482.1	460.3	43.6	0	0
Y, M	461.0	446.6	28.4	0	0
Y, E	541.8	538.2	7.2	0	0

(1) K fixed for Method II means that information on the price of K is not used, nor is the shadow price equation for K estimated. (2) See Table 1.

price is unavailable, or possibly of questionable reliability, then it would be a reasonable one to select as k.

For the Berndt–Wood data, we thus initially ignore available price information on capital and estimate a model in which the shadow price equation for capital is deleted. This has the additional advantage in the current study of increasing comparability with Method I in that in both methods information on the price of capital is not used in the estimation. To provide further comparisons with Method I, a second model is then estimated in which information on the price of output is ignored, and the shadow price equation for output is omitted from the estimation.

For the Kohli data there is an additional dependency that arises from the basic National Accounts identity; namely, that in value terms $C + I + G \equiv K + L + M$. The set of six equations consisting of the quantity equations (82), multiplied by their corresponding prices, together with the shadow price equations (83), multiplied by their corresponding quantities, will satisfy this condition, both for functions and data. Thus one of these equations can be dropped in the estimation and the results will be invariant to this choice, provided the equations are expressed in value terms. For this reason, we estimate the input

Table 5
Summary statistics for Method II (Y fixed), U.S. manufacturing data

	Log-likelihood values		Chi-square value	Convexity violations	
	NS	S		NS	S
(a) Curvature not imposed					
M, L	493.5	468.4	50.2	35	0
E, L	489.7	447.2	85.0	0	0
E, M	562.0	517.3	17.4	35	35
K, L	456.9	424.6	64.6	0	0
K, M	497.4	487.1	20.6	35	35
K, E	484.7	476.9	15.6	35	35
(b) Curvature imposed					
M, L	492.2	468.4	47.6	0	0
E, L	489.7	447.2	85.0	0	0
E, M	525.1	516.5	17.2	0	0
Y, L	456.9	424.6	64.6	0	0
K, M	496.9	486.4	21.0	0	0
K, E	471.8	462.1	19.4	0	0

(1) Y fixed for Method II means that information on the price of Y is not used, nor is the shadow price equation for Y estimated. (2) See Table 1.

Table 6
Summary statistics for Method II, U.S. national accounts data

	Log-likelihood values		Chi-square value	Convexity violations	
	NS	S		NS	S
(a) Curvature not imposed					
I, C	652.6	614.3	76.6	40	40
X, C	628.1	577.4	101.4	40	40
X, I	579.0	541.0	76.0	40	40
M, L	700.7	606.2	189.0	40	40
(b) Curvature imposed					
I, C	651.5	613.1	76.8	0	0
X, C	623.7	570.1	107.2	0	0
X, I	578.9	538.7	80.4	0	0
M, L	693.2	601.5	183.4	0	0

(1) As discussed in the text, the estimates are invariant to the choice of equation omitted from the estimation. (2) See Table 3.

demand and output supply equations in expenditure form, and in order to reduce heteroskedasticity arising from the use of expenditures as the dependent variables, we deflate all equations by the product of an exogenous quantity index $(\alpha^T z + \alpha_{N+1} k)$ and an exogenous price index $(\beta^T w)$, where α and β are the vectors defined by (91) and (90), respectively.

We add a vector of disturbances (e) to each of these systems of equations, and assume the e's are independent across observations and that each has a multivariate normal distribution with zero mean and constant covariance matrix. We obtain maximum likelihood estimates as before. In order to provide a unified treatment of Method II, we use this deflated expenditure form of the equations in estimations involving the Berndt–Wood data as well as the Kohli data.

In Tables 4–6 we present the separability test results for Method II corresponding to those in Tables 1–3 for Method I. One of the main differences between the sets of tables appears in the column recording the number of curvature violations in the nonseparable model when curvature is imposed. In Tables 4–6 there are no violations since correct curvature is imposed globally, while in Tables 1–3 there are a substantial number of violations since it can only be imposed at a point. This illustrates one of the major attractions of Method II.

With regard to separability possibilities, the tables yield consistent results for the Kohli data. In both Tables 3 and 6 all four hypotheses are clearly rejected, whether or not separability is imposed. For the Berndt–Wood data, the results are inconclusive. For the models in which no use is made of information on the price of capital (Tables 1 and 4), all hypotheses are clearly rejected except for the E, Y and E, M subgroups. Separability of E and Y as a group is not rejected at the 5% level with Method II, and is marginally rejected at the 1% level with Method I. Separability of E and M is only marginally rejected at the 5% level with Method I, but is soundly rejected with Method II. For models in which the price of output is not used (Tables 2 and 5), all hypotheses are rejected at the 1% level using Method II, although with curvature imposed the chi-square associated with the E, M subgroup is the lowest. For Method I, however, separability of the E, M subgroup is not rejected at the 5% level. In general then, there appears to be weak evidence for combining either energy and materials, or energy and output.

Several additional points are of interest. First, in many cases when the imposition of curvature results in a reduction in the log-likelihood value, this reduction is very small, particularly for Method II. For example, in Table 4 the log-likelihood values for the curvature constrained and unconstrained models differ only slightly for the separable model, when testing the separability of E and L as a group. This occurs also in Table 5 for the nonseparable models when testing the L, M and K, M subgroups, and in Table 6 in several instances. For Method I, the log-likelihood changes are generally somewhat larger.

Second, it should be kept in mind that a major difference between the sets of tables is that log-likelihood values for nonseparable models are comparable within a table for the first three tables, but not for the next three. Thus, the differences in the log-likelihoods in the first column of Tables 1, 2, or 3, in either part (a) or (b), are minor in comparison with the differences in Tables 4, 5, or 6. This follows because with Method II, the variable profit function approach, whether a price or quantity is the dependent variable depends on the separability hypothesis under consideration. For goods in the separable group, prices are assumed exogenous and quantities are dependent variables, while for goods in the other group the opposite holds. On the other hand, prices are always assumed exogenous and quantities are the dependent variables for Method I (except, of course, for the single fixed factor that is used in conjunction with the unit profit function to define total profits). Hence, the set of dependent variables is always the same for Method I (within a table), and since the nonseparable forms are flexible, the log-likelihood values should be reasonably close to each other. It is encouraging to note that this is indeed the case. Of course, for the separable models the log-likelihood values in the first three tables reflect the extent to which the data are consistent with the particular separability assumption under consideration. Thus, within each of these tables, one would generally expect higher log-likelihood values to be associated with lower chi-square values, which they are.

Since the focus of this paper is on separability testing, we present only a brief account of our findings on the usual summary statistics generally reported for (variable) profit function models–price elasticities, R squared values, and technical change effects. Although the price elasticity estimates vary according to the model estimated, they all lie within the general range of estimates in the literature. Generally, input demands are very inelastic, and own price elasticity estimates obtained from the variable profit function models are, of course, lower in absolute value than those obtained from the profit function models, for reasons discussed at the end of Section 5. For both models, R-squared values are high considering that the dependent variables are input–capital or input–output ratios with Method I and expenditures deflated by the product of a price and quantity index with Method II. The lowest R-squared value is 0.75, and most are above 0.90. Finally, both the linear and quadratic time trend variables are highly significant in all models, as measured by their effect on log-likelihood values. For both samples the derivative of the profit function (or variable profit function) with respect to time is positive, and this derivative as a fraction of output falls monotonically over the sample period. This suggests the existence of technical progress in the postwar U.S. economy, but at a rate that is decreasing over time.

8. Conclusion

The main purpose of this research is to propose and implement empirically tests for homogeneous weak separability using flexible, parsimonious, and curvature correct functional forms. We suggest two methods, both of which are based on normalized quadratic forms. The first is a straight forward implementation of the definition of homogeneous weak separability in terms of a standard profit or cost function. The second is an extension of Woodland's (1978) work on separability, which is in terms of a *variable* profit or cost function. We employ both of these methods to test various separability hypotheses with two different postwar U.S. data sets.

Our empirical results were rather discouraging from the viewpoint of applying microeconomic theory using aggregated data: conditional on our maintained hypotheses, virtually all forms of homogeneous weak separability were rejected.

In our separability tests, both the macro aggregator function f and the micro aggregator function g were assumed to be linearly homogeneous; see Eq. (2). It would be useful to extend our tests to cover the case where f and g are general functions, as in Blackorby, Schworm, and Fisher (1986). It would also be useful to extend our Method I to cover the consumer context.[28] However, given the complexity of the present paper, these extensions must be left to the future.

Appendix

Proofs of propositions

Proposition 1. We need only show that the $M + N$ by $M + N$ matrix of second-order partial derivatives $\nabla^2\pi(w,p)$ is positive semidefinite. The convexity of $P(p_0, p)$ implies that the $1 + N$ by $1 + N$ matrix of second-order derivatives $\nabla^2 P(p_0, p)$ is positive semidefinite and the convexity of $r(w)$ implies that the M by M matrix of second-order derivatives $\nabla^2 r(w)$ is also positive semidefinite.

Letting $p_0 = r(w)$, straightforward differentiation of $\pi(w,p) \equiv P[r(w), p]$ yields the following expression for $\nabla^2\pi(w,p)$:

$$\begin{bmatrix} \nabla^2_{p_0 p_0} P(p_0,p)\nabla_w r(w)\nabla_w^T r(w) + \nabla_{p_0} P(p_0,p)\nabla^2_{ww} r(w), & \nabla_w r(w)\nabla^2_{p_0 p} P(p_0,p) \\ \nabla^2_{p p_0} P(p_0,p_0)\nabla_w^T r(w), & \nabla^2_{pp} P(p_0,p) \end{bmatrix}. \quad \text{(A.1)}$$

Let u be an arbitrary M-dimensional column vector and v be an arbitrary N-dimensional column vector. Using (A.1),

$$\begin{aligned} &[u^T, v^T]\,\nabla^2\pi(w,p)[u^T, v^T]^T \\ &= [u^T\nabla_w r(w), v^T]\,\nabla^2 P(p_0,p)[u^T\nabla_w r(w), v^T]^T + \nabla_{p_0} P(p_0,p) v^T \nabla^2_{ww} r(w) v \geqslant 0, \end{aligned} \quad \text{(A.2)}$$

[28] This is straightforward; however, Method II does not have a straightforward extension to the consumer context.

since the first term on the right-hand side of (A.2) is nonnegative by the positive semidefiniteness of $\nabla^2 P(p_0, p)$ and the second term is nonnegative because $\nabla_{p_0} P(p_0, p) = \delta P(p_0, p)/\delta p_0 \geqslant 0$ by assumption and $v^T\nabla^2 r(w) v \geqslant 0$ by the positive semidefiniteness of $\nabla^2 r(w)$.

Proposition 2. Let $\pi^*(w,p)$ be an arbitrary unit profit function which is twice continuously differentiable at (w^*, p^*). Hence, the derivatives of π^* (and the derivatives of π as well) satisfy the restrictions (30)–(32). We need to equate the first- and second-derivatives of π^* to those of π at (w^*, p^*). This means we have to find a_0, vectors a, b, and c, and matrices A, B, and C that satisfy (A.3)–(A.7) below as well as the restrictions (23), (24), (35)–(38).

$$\begin{aligned} \nabla_w\pi(w^*,p^*) &= a_0 b + c &&= \nabla_w\pi^*(w^*,p^*) \equiv b^* \; (= x^*/k^*), && \text{(A.3)} \\ \nabla_p\pi(w^*,p^*) &= a &&= \nabla_p\pi^*(w^*,p^*) \equiv a^* \; (= y^*/k^*), && \text{(A.4)} \\ \nabla^2_{pp}\pi(w^*,p^*) &= A &&= \nabla^2_{pp}\pi^*(w^*,p^*) \equiv A^*, && \text{(A.5)} \\ \nabla^2_{wp}\pi(w^*,p^*) &= C - bp^{*T}A/p_0^* &&= \nabla^2_{wp}\pi^*(w^*,p^*) \equiv C^*, && \text{(A.6)} \\ \nabla^2_{ww}\pi(w^*,p^*) &= a_0 B + bb^T p^{*T}Ap^*/p_0^{*2} &&= \nabla^2_{ww}\pi^*(w^*,p^*) \equiv B^*. && \text{(A.7)} \end{aligned}$$

In calculating the derivatives of π evaluated at (w^*, p^*), we have used the restrictions (15), (22), and (34). We also assume $w^{*T}b^* \neq 0$.

Obviously, we can set $a = a^*$ and $A = A^*$ and then (A.4) and (A.5) will be satisfied. We define a_0 by

$$a_0 \equiv |w^{*T}b^*| \neq 0. \quad \text{(A.8)}$$

To determine the remaining parameters of π, we have to consider two cases.

Case (*i*): $w^{*T}B^*w^* > 0$. In this case, define b by

$$b \equiv [w^{*T}B^*w^*]^{-1}B^*w^*b^{*T}w^*/|b^{*T}w^*|. \quad \text{(A.9)}$$

Therefore, $b^Tw^* = b^{*T}w^*/|b^{*T}w^*| = p_0^*$ where $p_0^* = 1$ if $b^{*T}w^* > 0$ and $p_0^* = -1$ if $b^{*T}w^* < 0$. Thus b satisfies (23). Having defined a_0 and b, now define c by $c \equiv b^* - a_0 b$, and thus (A.3) is satisfied. Using (A.8) and (A.9), it is straightforward to verify that $c^Tw^* = 0$, and so (35) is satisfied. Since b and A have already been defined, define $C \equiv C^* + bp^{*T}A/p_0^*$, and (A.6) is satisfied.

Note that (31) and (32) can be rewritten as follows:

$$B^*w^* + C^*p^* = 0_M, \quad \text{(A.10)}$$

$$C^{*T}w^* + A^*p^* = 0_N. \quad \text{(A.11)}$$

Premultiply (A.10) by w^{*T} and premultiply (A.11) by p^{*T}. We get

$$w^{*T}C^*p^* = -w^{*T}B^*w^* = -p^{*T}A^*p^*, \quad \text{(A.12)}$$

$= w$

$= 0_N^T$

and so C satisfies the restrictions (37).

Since $a_0 \neq 0$ and b and A have been defined, we may define B as

$$B \equiv a_0^{-1}\{B^* - bb^T p^{*T} A p^*/p_0^{*2}\}, \tag{A.13}$$

and B satisfies (A.7). Note that since $p_0^* = \pm 1$, $p_0^{*2} = 1$. Thus,

$$\begin{aligned} Bw^* &= a_0^{-1}\{B^*w^* - bb^T w^* p^{*T} A^* p^*\} && \text{using } A = A^* \text{ and (A.13)} \\ &= a_0^{-1}\{B^*w^* - B^*w^* p^{*T} A^* p^*/w^{*T} B^* w^*\} && \text{using (A.9)} \\ &= a_0^{-1}\{B^*w^* - B^*w^*\} && \text{using (A.12)} \\ &= 0_M, \end{aligned}$$

and so B satisfies (24). Thus, all of (A.3)–(A.7) are satisfied (along with the appropriate restrictions) for case (i).

Case (ii): $w^{*T} B^* w^* = 0$. Since $B^* \equiv \nabla^2_{ww}\pi^*(w^*, p^*)$ is a positive semidefinite matrix, $w^{*T}B^*w^* = 0$ implies that w^* must be an eigenvector of B^* that corresponds to a zero eigenvalue. Thus, we actually have

$$B^*w^* = 0_M. \tag{A.14}$$

Recall that (A.10) and (A.11) hold in general. Thus (A.10) and (A.14) imply that

$$C^*p^* = 0_M. \tag{A.15}$$

Premultiply (A.11) by p^{*T} and use (A.15) to obtain

$$p^{*T}A^*p^* = 0. \tag{A.16}$$

Since $A^* \equiv \nabla^2_{pp}\pi^*(w^*, p^*)$ is positive semidefinite, (A.16) implies

$$A^*p^* = 0_N. \tag{A.17}$$

Now use (A.17) and (A.11) to deduce that

$$C^{*T}w^* = 0_N. \tag{A.18}$$

Define $c \equiv 0_M$, and thus c satisfies (35). Define $b \equiv b^*/a_0$, and (A.3) is satisfied. Since $a_0 = |b^{*T}w^*|$, we have $b^T w^* = b^{*T}w^*/|b^{*T}w^*| \equiv p_0^*$, and (23) is satisfied. Finally, define B and C by

$$C \equiv C^*, \qquad B \equiv a_0^{-1}B^*. \tag{A.19}$$

Using (A.17), we see that (A.6) and (A.7) are satisfied. Thus, our a_0, a, b, c, A, B, and C for case (ii) satisfy (A.3)–(A.7). Using (A.19) and (A.14), (24) is satisfied. Using (A.19) and (A.18), (37) is satisfied. Finally, (A.19) and (A.15) imply the restrictions in (38).

Proposition 3.

$$\begin{bmatrix} I_M, & bp^{*T}/p_0^* \\ 0_{N\times M}, & I_N \end{bmatrix} \nabla^2\pi(w^*, p^*) \begin{bmatrix} I_M, & 0_{M\times N} \\ p^*b^T/p_0^*, & I_N \end{bmatrix} = \begin{bmatrix} a_0 B, & C \\ C^T, & A \end{bmatrix},$$

where $\nabla^2\pi(w^*, p^*)$ is defined by (40).

Proposition 4. Using (22), (24), (68), (74), and (75), we can calculate the derivatives of V as indicated in (A.20)–(A.26) below and equate these derivatives to the corresponding derivatives of V^*:

$$\begin{aligned} \nabla_w V(w^*, z^*, k^*) &= b(a_0k^* + a^Tz^*] = \nabla_w V^*(w^*, z^*, k^*) \equiv x^*, && \text{(A.20)} \\ \nabla_z V(w^*, z^*, k^*) &= b^T w^* a = \nabla_z V^*(w^*, z^*, k^*) \equiv p^*, && \text{(A.21)} \\ \nabla_k V(w^*, z^*, k^*) &= b^T w^* a_0 = \nabla_k V^*(w^*, z^*, k^*) \equiv p_{N+1}^*, && \text{(A.22)} \\ \nabla^2_{ww} V(w^*, z^*, k^*) &= B[a_0k^* + a^Tz^*] = \nabla^2_{ww} V^*(w^*, z^*, k^*) \equiv B^*, && \text{(A.23)} \\ \nabla^2_{wz} V(w^*, z^*, k^*) &= C + ba^T = \nabla^2_{wz} V^*(w^*, z^*, k^*) \equiv C^*, && \text{(A.24)} \\ \nabla^2_{wk} V(w^*, z^*, k^*) &= c + ba_0 = \nabla^2_{wk} V^*(w^*, z^*, k^*) \equiv c^*, && \text{(A.25)} \\ \nabla^2_{zz} V(w^*, z^*, k^*) &= b^T w^* A = \nabla^2_{zz} V^*(w^*, z^*, k^*) \equiv A^*. && \text{(A.26)} \end{aligned}$$

Using Euler's theorem on homogeneous functions (see Diewert 1973, p. 308), the linear homogeneity of $V^*(w, z, k)$ in w and in (z, k) implies

$$\begin{aligned} V^*(w^*, z^*, k^*) &= w^{*T}\nabla_w V^*(w^*, z^*, k^*) \\ &= z^{*T}\nabla_z V^*(w^*, z^*, k^*) + k^*\nabla_k V^*(w^*, z^*, k^*), \end{aligned} \tag{A.27}$$

and the linear homogeneity of $V^*(w,z,k)$ in (z,k) implies

$$\nabla^2_{zz} V^*(w^*,z^*,k^*)z^* + \nabla^2_{zk} V^*(w^*,z^*,k^*)k^* = 0_N, \tag{A.28}$$

$$\nabla^2_{kz} V^*(w^*,z^*,k^*)z^* + \nabla^2_{kk} V^*(w^*,z^*,k^*)k^* = 0. \tag{A.29}$$

Since $V(w,z,k)$ has the same homogeneity properties as V^*, equations analogous to (A.27)–(A.29) holds with V replacing V^*. Thus, in order to equate the level and all first- and second-order partial derivatives of V and V^* at (w^*,z^*,k^*), we need only find a_0, a, b, c, A, B, and C which satisfy (A.20)–(A.26). In addition, we require b to satisfy (23), B to satisfy (24), and $[C,c]$ to satisfy (74) and (75).

We assume $w^{*T}x^* \neq 0$ and define b as follows:

$$b \equiv x^*/|w^{*T}x^*| \tag{A.30}$$

$$\therefore$$

$$\begin{aligned} w^{*T}b &= w^{*T}x^*/|w^{*T}x^*| \\ &= 1 \quad \text{if} \quad w^{*T}x^* > 0 \\ &= -1 \quad \text{if} \quad w^{*T}x^* < 0 \\ &\equiv p_0^*. \end{aligned} \tag{A.31}$$

Thus, b satisfies (23) as required. Define a_0 and a by

$$a_0 \equiv p^*_{N+1}/w^{*T}b = p^*_{N+1}/p^*_0, \tag{A.32}$$

$$a \equiv p^*/w^{*T}b \quad = p^*/p^*_0. \tag{A.33}$$

Thus, (A.21) and (A.22) are satisfied. Now calculate:

$$\begin{aligned} b[a_0k^* + a^Tz^*] &= b[p^*_{N+1}k^* + p^{*T}z^*]/p^*_0 && \text{using (A.32) and (A.33)} \\ &= b[w^{*T}x^*]/p^*_0 && \text{using (A.27) and (A.20)–(A.22)} \\ &= x^*w^{*T}x^*/|w^{*T}x^*|p^*_0 && \text{using (A.30)} \\ &= x^* && \text{using (A.31),} \end{aligned}$$

and thus (A.20) is satisfied. The above equalities show that

$$a_0k^* + a^Tz^* = w^{*T}x^*/p^*_0 = |w^{*T}x^*| > 0. \tag{A.34}$$

Define B as

$$B \equiv [|w^{*T}x^*|]^{-1}B^* = [a_0k^* + a^Tz^*]^{-1}B^* \quad \text{using (A.34),} \tag{A.35}$$

and thus (A.23) is satisfied. Note that B must be positive semidefinite since it is a positive multiple of $B^* \equiv \nabla^2_{ww}V^*(w^*,z^*,k^*)$. Also, the linear homogeneity of $V^*(w,z,k)$ in w implies

$$B^*w^* = \nabla^2_{ww}V^*(w^*,z^*,k^*)w^* = 0_M. \tag{A.36}$$

Thus, (A.35) and (A.36) imply that $Bw^* = 0_M$ which is (24).

Define A as follows:

$$A \equiv (p^*_0)^{-1}A^*. \tag{A.37}$$

Using (A.31), (A.37) implies that (A.26) is satisfied. Note that $A^* \equiv \nabla^2_{zz}V^*(w^*,z^*,k^*)$ is negative semidefinite by (62) applied to V^*. Thus, A is also negative semidefinite if $p^*_0 = 1$ (so $w^{*T}x^* > 0$) or A is positive semidefinite if $p^*_0 = -1$ (so $w^{*T}x^* < 0$ in this case).

Define C as follows:

$$\begin{aligned} C &\equiv C^* - x^*p^{*T}/w^{*T}x^* && \text{(A.38)} \\ &= C^* - x^*a^Tp^*_0/w^{*T}x^* && \text{using (A.33)} \\ &= C^* - ba^T|w^{*T}x^*|p^*_0/w^{*T}x^* && \text{using (A.30)} \\ &= C^* - ba^Tp^*_0/p^*_0 && \text{using (A.31)} \\ &= C^* - ba^T, \end{aligned}$$

and so (A.24) is satisfied. The linear homogeneity of $V^*(w,z,k)$ in w implies

$$\begin{aligned} w^{*T}\nabla^2_{wz}V^*(w^*,z^*,k^*) &= \nabla^T_z V(w^*,z^*,k^*) \quad \text{or} && \text{(A.39)} \\ w^{*T}C^* &= p^{*T} && \text{using (A.21) and (A.24).} \end{aligned}$$

Premultiply both sides of (A.38) by w^{*T}. Using (A.39), we get $w^{*T}C = 0^T_N$ and thus the first N restrictions in (74) are satisfied by our C.

Finally, define the vector c by

$$\begin{aligned} c &\equiv c^* - x^*p^*_{N+1}/w^{*T}x^* && \text{(A.40)} \\ &= c^* - x^*a_0p^*_0/w^{*T}x^* && \text{using (A.32)} \\ &= c^* - x^*a_0/|w^{*T}x^*| && \text{using (A.31)} \\ &= c^* - ba_0 && \text{using (A.30),} \end{aligned}$$

and thus (A.25) is satisfied. The linear homogeneity of $V^*(w,z,k)$ in w implies

$$\begin{aligned} w^{*T}\nabla^2_{wk}V^*(w^*,z^*,k^*) &= \nabla_k V^*(w^*,z^*,k^*) \quad \text{or} && \text{(A.41)} \\ w^{*T}c^* &= p^*_{N+1} && \text{using (A.22) and (A.26).} \end{aligned}$$

Premultiply both sides of (A.40) by w^{*T} and using (A.41) yields $w^{*T}c = 0$ which is the last restriction in (74). The linear homogeneity of $V^*(w,z,k)$ in (z,k)

implies that

$$\nabla^2_{wz} V^*(w^*,z^*,k^*)z^* + \nabla^2_{wk} V^*(w^*,z^*,k^*)k^* = \nabla_w V^*(w^*,z^*,k^*) \quad \text{or} \tag{A.42}$$

$$C^*z^* + c^*k^* = x^* \qquad \text{using (A.20), (A.24), (A.25).}$$

Finally, use definitions (A.38) and (A.40) and calculate

$$\begin{aligned} Cz^* + ck^* &= C^*z^* + c^*k^* - (w^{*T}x^*)^{-1}x^*[p^{*T}z^* + p^*_{N+1}k^*] \\ &= x^* - (w^{*T}x^*)^{-1}x^*[w^{*T}x^*] \quad \text{using (42) and (27)} \\ &= 0_N, \end{aligned}$$

which is (75).

Proposition 5. Using (7), (58), and (60), it can be shown that

$$\pi(w,p) = \max_z \{ -p^T z + V(w,z,k) \}. \tag{A.43}$$

By assumption, z^* satisfies the second-order sufficient conditions for the unconstrained maximization problem in (A.43). Thus, (85) follows, and by Samuelson's (1947, p. 34) Envelope Theorem, (86) and (87) follow.

Since $E \equiv \nabla^2_{zz} V(w,z^*,1)$ is negative definite by assumption, its inverse exists. The Implicit Function Theorem applied to (84) with $z^* = z(w,p)$ yields

$$\nabla_p z(w,p) = E^{-1} \quad \text{and} \quad \nabla_w z(w,p) = -E^{-1}F^T. \tag{A.44}$$

Now replace z^* in (86) and (87) by $z(w,p)$ and differentiate the resulting equations with respect to the component of w and p. Using (A.44), we obtain (88). The positive semidefiniteness of (88) follows from the positive semidefiniteness of D and the positive definiteness of $-E^{-1}$; i.e., we have

$$\begin{bmatrix} I_M, & F \\ 0_{N\times M}, & I_N \end{bmatrix} \begin{bmatrix} D - FE^{-1}F^T, & FE^{-1} \\ E^{-1}F^T, & -E^{-1} \end{bmatrix} \begin{bmatrix} I_M, & F \\ 0_{N\times M}, & I_N \end{bmatrix} = \begin{bmatrix} D, & 0_{M\times N} \\ 0_{N\times M}, & -E^{-1} \end{bmatrix}.$$

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