First-Price Auctions with Speculative Resale
Part I: Equilibrium and Optimal Revenue

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Abstract

In this paper, we investigate the model of speculative resale in auctions. Resale (or secondary trade) is allowed for each bidder, but in equilibrium we show that there is only speculative resale. We consider a standard first-price auction in the first stage with symmetric independent private values (IPV) among N bidders and one speculator. In the second stage there is resale among the bidders. The winner in the first stage auction uses an optimal mechanism to sell the object to the losing bidders. We establish a supermodularity property without assuming monotonicity or symmetry in bidding. We show the equilibrium bidding function of the regular bidders to be symmetric and increasing. We give a simple computable equilibrium solution and prove it to be unique. The Myerson (1981) revenue formula is extended to our model with speculative resale. Revenue monotonicity holds, and the Bulow-Klemperer (1996) argument favoring participation rather than the choice of optimal reservation price is also valid here with even greater force. The speculator’s active participation enhances the seller revenue, but for optimal revenue, the seller should prevent speculative resale by setting a sufficiently high reservation price. Thus first-price auction provides an implementation of optimal auctions with a unique equilibrium, while allowing speculative resale.

Keywords: Speculative Resale, Revenue Monotonicity, Myerson Revenue formula, Optimal Auctions, Equilibrium Solution

1 Introduction

Whenever there is asymmetry between bidders, a winner of an auction will be interested in resale as the allocation is often inefficient. When we allow resale, many interesting phenomena occur. For example, if you start with private value auctions, but allow resale, then it introduces value dependence between the bidders as the resale revenue is interdependent. This means that the standard benchmark model with private values no longer serves as a good guidance in the analysis of auctions. In auctions with resale, the typical case involves a mixture of private-value and common-value elements. The dependence between valuation among the bidders introduced by resale is probably the most interesting and important type of value dependence in the real world. However it has not received sufficient attention in the literature. Hazlett and Oh (forthcoming) have argued that secondary trade or resale is more important than exact regulation on spectrum rights to avoid the inefficiency from harmful interference. Some of the recent discussions of incentive auctions (Hazlett, Porter and Smith (2012)) have information revelation issues that are also relevant in equilibrium with secondary trade. Resale also is important in explaining many problems of information aggregation or herding behavior when valuation becomes dependent through resale.

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Consideration of resale impact on auctions has its practical importance and relevance in policy making. Many spectrum auctions held across the world prohibit resale after a bidder wins the auction. Such restriction on bidder behavior has poor theoretical basis or grounding. Furthermore such restrictions can be easily circumvented by selling the shares or the whole company owning the license rather than license itself. This has indeed happened. In the 2000 UK 3G auction, resale was prohibited. Orange won the license E, and was quickly sold to France Telecom. The most valuable license A was won by TIW\(^1\). Within three months, Hutchison then sold 35\% of its share in TIW to KPN and NTT DoCoMo, making billions of pounds in the process. Similarly, after the European 3G auctions, Telia, the biggest telecom company in northern Europe, took over Sonera, a smaller and debt-burdened telecom company, and obtained the licenses that Sonera had won in Germany, Italy, Spain, and Norway. In the 2003 UK 3.4G spectrum auction, Pacific Century Cyberworks (PCCW), a Hong-Kong telecom company participated in the auction, lost two licenses to Red Spectrum and Public Hub respectively. After six months it obtained the two licenses back through purchase of the two companies (for more details, see Pagnozzi (2010)). So far we don’t seem to have a clear theoretical guidance as to whether speculators will enhance the auctioneer revenue or over all efficiency.

Most of the recent studies on the optimal design of auctions with resale were based on second-price or English auctions with resale (for instance on the theoretical side: Zheng (2002), Garratt et al. (2009), Mylovanov and Tröger (2009), Lebrun (2012); and on the experimental side: Georganas (2011), Georganas and Kagel (2011), Pagnozzi and Saral (2013), Filiz-Ozbay Lopez-Vargas and Ozbay (2013)). This is understandable because even the actual spectrum auctions were ascending auctions more similar to second-price or English auctions. However in second-price auctions or English auctions with resale, there are typically indeterminacy in equilibria, even infinitely many equilibria. The revenue with resale can be lower or higher than auction revenue without resale depending on which equilibrium prevails (see GT(supp)). Therefore the second-price or English auctions are not the ideal model to evaluate whether resale enhances auction revenue or not. In fact, Lebrun (2012) has argued that optimal auction with resale constructed in his paper and in Zheng (2002), Mylovanov and Tröger (2009) and Garratt et al. (2009) all suffer from the indeterminacy of equilibrium problem aside from the strong assumptions needed for such design. In principle, the lack of determinacy makes it harder to have clear predictions for testing. It is also harder for bidders to have stable equilibrium bidding behavior so that clear patterns of outcomes can be observed and tested. Filiz-Ozbay, Lopez-Vargas and Ozbay (2013) perform experiments showing that there is no significant difference between revenues with or without resale in a multi-unit English auction setting. The first-price or Dutch auctions with resale present itself as an alternative design choice because it has a unique equilibrium.

We will study first-price auctions with resale in this paper. In particular, we will focus on speculative resale involving a speculator who has no value for the object on sale except for the purpose of winning and selling to other losing bidders. Lebrun (2012) also expressed the need to find a unique implementation of optimal auctions with resale. In our model, we will provide a unique implementation of auctions with resale achieving optimal revenue the same as in auctions without resale.

Our model is very general in terms of the specification of the valuation distribution \(F(.)\) over \([0,\beta]\) of the \(N\) regular buyers. In fact, we allow \(F(.)\) to have the possibility of an atom at 0, meaning that a regular buyer may have a positive probability of having 0 value. Take the benchmark model of symmetric private-value auctions for a single object without resale, add a speculator who has no value for the object, and allow the auction winner in the first-stage to sell the object to other bidders in the second stage. That is the framework of our analysis. There are of course many ways of specifying the resale market. We will assume that the winner will sell the object using an optimal mechanism in the second stage\(^2\). We call our model "auctions with speculative resale", because in equilibrium we can show that only the speculator will be a seller in the second stage. By allowing the speculator to maximize revenue using the optimal auctions of Myerson (1981) in the second stage, we are giving the speculator the maximum bargaining power subject to the uncertainty remaining in the resale market. This endogenous price formation in the resale market is probably least controversial, and gives the speculator the highest capacity to be active in the first-stage auction. The auctioneer will also set a reservation price \(\rho\) in the first-stage first-price auction. In specifying this resale market, we do not have the certainty resale game often occurring in the second-price auctions

\(^1\)TIW (Telesystem International Wireless) was a Canadian company based in Montreal and largely owned by Hutchison Whampoa, the Hong Kong conglomerate.

\(^2\)Our resale occurs between bidders in the first-stage auction, sometimes called secondary trade. Bikchandani and Huang (1989), Bose and Deltas (2002) deal with resales to consumers who are not participants in the original auction.
with resale. The share of resale surplus is dependent on the winning bid in the first-stage auction, and the resale outcome is often efficient as resale may fail due to uncertainty.

In between the two stages, there is also a need to specify what information is revealed after the auction and before the resale. We will adopt the natural minimal approach regarding bid revelation: only the winning bid is revealed\(^3\). From this information, the winner of the first-stage auction can only infer about the maximum value of the losers. This information uncertainty means that the object may not be sold, and resale cannot correct all the inefficiency of the first-stage auction. Thus we have an interesting model to study the trade-off of efficiency between the two stages: while resale can correct some inefficiency in the first-stage allocation, the resale itself has its own inefficiency problem. Moreover, resale opportunity may cause more inefficiency in the first-stage outcome, an issue that was widely recognized.

We will first provide a simple equilibrium solution for the model. The equilibrium solution is easily computable as long as \( F(.) \) is analytic (or computable), and equilibrium strategies are given by explicit formulas. The revenue is also easily computable. The equilibrium property is almost as easy to analyze as a private-value model, but has the great advantage of allowing resale considerations and value-dependence in bidding behavior. The revenue equivalence result no longer holds in this model, and there are many issues about auction design that can now be evaluated in the model with speculative resale.

We will then extend the Myerson revenue formula for the private value model without resale to our model of speculative resale. Let \( F(.) \) defined on \([0, \beta]\) be the use value distribution of a regular buyer (non speculator), and assume there are \( N \) regular buyers. Let the reservation price set by the auctioneer be \( \rho \), and \( r \) is the optimal reservation price according to Myerson (1981). Let \( J(v) \) denote the virtual value according to Myerson (1981). The new formula takes a very simple form. With the speculator competing in the first-stage auction, the winning probability of a regular buyer with value \( v \) is \( t(v) F^{N-1}(v) \), where \( t(v) \) is either 1 or some number between 0 and 1. When a regular buyer has value above \( r \), we have \( t(v) = 1 \), otherwise it is less than one. This is because a regular buyer can buy back from the speculator even if he fails to win in the first-stage, and the buyback occurs with probability 1 when \( v \geq r \), and less than 1 when \( v < r \). We have the following formula for the revenue of an auction with speculative resale

\[
\int_{\rho}^{\beta} t(v) J(v) dF^N(v).
\]

The virtual value, \( J(v) \), is now discounted (keeping the same sign) by \( t(v) \), otherwise the revenue remains the same as before. The discount function \( t(v) \) is also given by an explicit formula related to the bid distribution of the speculator. From the revenue formula, we then provide simple answers to the following questions: (1) Are speculative resale beneficial to the auctioneer revenue? (2) What is the optimal mechanism for revenue when resale is allowed? (3) Do we have revenue monotonicity when there are more buyer competition? The answers to these questions are: (1) Speculative resale increases revenue when the speculator is active in equilibrium, (2) The same reservation price \( r \) is also optimal in this model of speculative resale, and the optimal revenue is the same as well, (3) We have revenue monotonicity. It should be noted that revenue monotonicity is not obvious at all, as the monotonicity of revenue from more participation cannot be taken for granted once we go beyond the benchmark private value symmetric model. It is particularly serious in combinatorial auctions as shown in Ausubel and Milgrom (2006). Even in single object auctions with symmetric buyers, such as common value models or affiliated signal models, the revenue monotonicity may be false (see also Pinkse and Guofu (2005)). Revenue monotonicity is a general result in our model. Hafalir and Krishna (2009) show that resale enhances revenue in a two-bidder model of weak-strong buyers allowing resale, if the value distribution belongs to three families of functions. Our result is quite general in function specifications, but is focused on speculative resale.

We also show that the speculator is inactive when the auctioneer sets the optimal reservation price \( r \). When the speculator is active, the auctioneer revenue will be higher than the auction without the speculator, given the reservation price \( \rho \). However, somewhat paradoxically, in order to achieve the optimal revenue in the auction with speculative resale, it is actually better for the auctioneer to set a sufficiently high reservation price so that the speculator is inactive in equilibrium, and have no impact on the auction revenue.

\(^3\)This is also an issue that causes awkwardness in second-price or English auctions with resale. If the winning bid is announced, as usually is the case in practice, the winner becomes aware of the private information of the highest losing bidder. This information leakage leads to a less interesting resale market for analysis. Lebrun (2010) analyses the problem of bid disclosure.
Determining an optimal reservation price is often difficult and requires plenty of information in practice. Bulow and Klemperer (1996) argued that the revenue is higher with one more bidder without any reservation price than the highest possible revenue from choosing the right reservation price. Thus the auctioneer does better by focusing on expanding participation, rather than choosing the reservation price optimally. We show that in our model of speculative resale, the Bulow-Klemperer result holds with even more persuasion. The difference in revenue is greater when the speculator is active in equilibrium without any reservation price. The question of whether in real world auctions, the reservation price has been optimally chosen has been studied in the literature for a variety of auctions (McAfee and Vincent, 1992; Paarsch, J. (1997); McAfee, Quan, and Vincent, 2002; Athey, Cramton, and Ingraham, 2002; Haile and Tamer, 2003; Tang, 2009). Taken together, the results of these papers found that reserve prices actually observed in real-world auctions are substantially lower than the theoretically optimal ones.

We allow strategies to be non-monotonic, and non-symmetric. An equilibrium strategy is shown to be increasing and symmetric. We say that an equilibrium is symmetric if all regular buyers use the same equilibrium strategy. GT (supp) restrict their analysis to the symmetric equilibrium (quasi symmetric equilibrium in their terminology). It is an open question whether there exists a non symmetric equilibrium in the model. For second-price auctions with resale, it is well-known that there are many non-symmetric equilibria in the model. We show in this paper that the equilibrium must be symmetric for the first-price auction with resale. This is another contribution of the paper. We show that the equilibrium must be unique among all possible non-monotonic, non-symmetric strategies. The uniqueness of equilibrium we obtain in our result is important because it allows us to have a unique implementation of optimal auctions allowing resale.

We show a supermodularity property which has a strong form and a weak form. The strong form is stronger than the single-crossing property discussed in Chapter 4 of Milgrom (2004). The weak form can occur in the model of speculative resale. There is no need for resale in their framework, as the auction is efficient. They allow affiliated signals, and assume that equilibria are symmetric. In our model, the auction with resale need not be efficient. We use the first-price auction with resale which has the symmetry property, and achieves the optimal revenue. The bidding in our model is mutually dependent, but the value dependence is endogenously determined, and the associated common-value model need not have the affiliated signal property. Adding more speculators has no impact on revenue in our model.

In providing a proof of the symmetry property, we establish first a special case of the symmetry result with no speculator. When there is no speculative auction in the auction with resale model, one consequence of our result is that the equilibrium is unique and must be the same as the well-known equilibrium in the symmetric auction without resale. This result, although quite intuitive, and implicitly used by many, has not been shown before, and must be dealt with in our analysis. When you allow resale, it is by no means obvious that the only equilibrium possible is the one without resale. Even though there is no resale in equilibrium, resale may occur out-of-equilibrium. It may also be possible that there is a non-symmetric equilibrium with resale between the bidders as in the case of second-price auctions with resale. We need to rule out the possibility that some bidders may be able to bid lower and buy back later during resale in equilibrium. In fact its proof is almost as involved as the more general case when we have a speculator.

We prove the existence and uniqueness of equilibrium by analyzing the equilibrium properties in active and inactive intervals. Active bid intervals are intervals inside the support of the (cumulative) bid distribution of

\footnote{Bulow and Klemperer (1996) show the result using English auctions without resale. There is no need for resale in their framework, as the auction is efficient. They allow affiliated signals, and assume that equilibria are symmetric. In our model, the auction with resale need not be efficient. We use the first-price auction with resale which has the symmetry property, and achieves the optimal revenue. The bidding in our model is mutually dependent, but the value dependence is endogenously determined, and the associated common-value model need not have the affiliated signal property. Adding more speculators has no impact on revenue in our model.}

\footnote{However, Ostrovsky and Schwarz (2009) argued that the revenue could be increased if the reservation price were raised upward.}

\footnote{The uniqueness of equilibrium is also shown in GT (supp). However, they do this under the assumption that the equilibrium is symmetric and increasing. The uniqueness result we have here is among all possible non-monotonic and non-symmetric strategies.}

\footnote{His result should be easily extendable to include speculators or perhaps even continuous distributions. However, it seems that his property is a weak form rather than a strong form.}
the speculator, and inactive bid intervals are intervals in which the (cumulative) bid distribution is constant. Active value intervals are intervals in which regular buyers bid in active bid intervals, and inactive intervals are intervals in which they bid in inactive intervals. Strategies on active or inactive intervals can be determined relatively easily. More importantly, this is how we obtain a simple solution to the equilibrium strategies, as it can be shown that the equilibrium solution is simple in either active or inactive intervals.

When there are at least two regular buyers, the speculator is often inactive in equilibrium with no reservation price. When a reservation price is set, it is even more likely to lead to an inactive speculator. An interesting question then arises: What are economic factors that make the speculator active in equilibrium? We provide a necessary and sufficient condition for the speculator to be active in equilibrium. The condition basically says that there is a cost function and a revenue function that can be defined from \( F(\cdot) \). The speculator becomes active in equilibrium if and only if the revenue exceeds the cost at some point. The next natural question then arises: If the speculator is active in equilibrium, what are the bid intervals in which she is active? In equilibrium, typically, the speculator is active in disjoint open intervals (an intuitive explanation of this is offered in the beginning of section 4). We provide a simple construction of active intervals. There is a simple algorithm that will determine and compute all the active and inactive intervals.

We focus on the case of finite number of active intervals. The more general case is treated in Part II of the paper which also answers the question of optimal efficient outcome of the model.

An example of equilibrium with an active speculator has been given by GT (2006). We offer a different but more transparent example at the end of section 4. The easiest way to find an example with an active speculator is to allow an atom. GT (2006) has such an example with an atom at the end. Our example has an atom\(^8\) at the beginning point 0. This is one reason we allow \( F(0) > 0 \) in our formulation. In our example we show how the revenue function is lifted while the cost function is lowered with an atom at 0. The revenue exceeds the cost everywhere, and the speculator becomes active in equilibrium.

We only allow two stages in the auctions with resale model, and there is only one resale stage in our model. Although this is a restriction, it is rather harmless for the equilibrium strategies. Our symmetry property implies that when the speculator sells in the resale market, she is selling in a symmetric auction which allocates the item efficiently to the buyer with the highest value. Therefore the winner of the resale auction has no more incentive to sell it to other buyers. Therefore, in equilibrium, having only one resale stage is not a limitation. However it could be a limitation when we analyze what could happen out of the equilibrium path in which allocation in the resale stage need not go to the bidder with the highest value. There is also another incentive issue when a seller fails to sell in a stage. The seller may try to sell the same object again. We assume that in this case, the seller keeps the object for himself. The temptation to sell it again at a possibly lower price is an intertemporal issue. This is an issue of commitment rather unrelated to the resale incentive issue. We will assume in this paper there is no commitment problem in the model. Under this assumption, the limitation of one resale opportunity is not a serious one. We should also point out that the optimal reservation price we obtain is the same for all bidders in the first-stage. This is an attractive feature as in equilibrium it could be difficult to identify who is a speculator and who is a regular buyer. The same reservation price applies to all bidders, and an optimal revenue outcome is achieved.

We assume no discounting in our analysis, although the analysis can be easily adapted for this case. Virag (2011) considered auctions with resale with two types of regular buyers. The bid distribution of the two types of buyers are not the same. Ours can be considered a special case of theirs and similar results hold. However, they did not show the symmetry property, and the model of speculative resale here makes it easier to obtain a simple formula of the equilibrium strategy.

In many important auctions, for instance the government sponsored spectrum auctions, revenue is not the main objective. In Part II, we will consider the problem of achieving the most efficient outcome with speculative resale. With this different objective function, we will show that speculator participation may provide the most efficient outcome, an interesting contrast to the optimal revenue case. However other optimal efficient outcome may also arise, depending on the properties of the valuation distribution \( F(\cdot) \). A solution to the efficiency problem with speculative resale is given in Part II.

In section 2, we describe the model of speculative resale. In section 3, we focus on symmetric increasing strategies, and characterize active and inactive intervals and equilibrium properties in such intervals and

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\(^8\) Atoms are neither sufficient nor necessary for the speculator to be active in equilibrium. They just provide simple examples in which we can demonstrate the possibility of active speculation. Examples of active speculation without any atoms are given in Part II of the paper.
show the equilibrium to be unique. In section 4, we give a simple computable solution to the equilibrium under the assumption that they are increasing and symmetric. Section 5 gives the revenue formula using the idea of virtual value in Myerson (1981). It also shows the revenue monotonicity with higher number of regular buyers, and the Bulow-Klemperer result in our model with speculative resale. In section 6, we offer the general version of the supermodularity property of the model and show that the equilibrium must be increasing and symmetric. In section 7, we provide the proofs for the supermodularity property, and the monotonicity and symmetry properties of equilibrium, as well as the lemmata not proved in the main text.

2 A Model of First-price Auction with Speculative Resale

There are \( N \) regular symmetric buyers (referred to with a male pronoun) and one speculator\(^9\) (referred to with a female pronoun) bidding for one object sold by the auctioneer (sometimes called the original seller to distinguish from the seller in the resale market). Regular buyers have use value for the object, but the speculator has no use value for the object and participates in the auction only for resale. A regular buyer has a use value distribution \( F(v) \) over \([0, \beta]\), which is assumed to be \( C^2 \) smooth\(^10\). Let \( f(v) \) denote the density function of \( F(\cdot) \). We allow \( F(0) > 0 \), so that \( F \) may have an atom at 0 as in the case of convex exponential distributions. It is convenient to use the notation \( F(\cdot | v) = \frac{F(\cdot)}{F(v)} \) for the conditional distribution of \( F(\cdot) \) if the use value has an upper bound \( v \). The regular buyers are indexed by \( i = 1, 2, \ldots, N \).

We study the first-price auctions with resale\(^11\). The first-price auction with resale is a two-stage game. The first-stage auction is a first-price auction with a reservation price \( \rho \) for the speculator and regular buyers. A winner of the auction in the first stage may sell to the losers in the second stage when it is profitable. The seller in the resale stage chooses an optimal mechanism. By the revelation principle, we can assume that the resale mechanism is a modified second-price auction with optimally chosen reservation prices. At the end of the first-stage auction and before the beginning of the resale stage, the winning bid (the highest bid) is announced. We will assume no discounting between the first stage and the second stage.

Let \( b_i(v), i = 1, 2, \ldots, N \) be the bidding strategy of regular buyer \( i \). We allow \( b_i(\cdot) \) to be non-monotonic, but exclude the possibility of having an atom in the bid distribution of \( b_i(\cdot) \), except possibly at \( \rho \). This is because atoms above \( \rho \) are not compatible with equilibrium requirements. The speculator uses a mixed strategy of bidding, represented by a cumulative bid distribution function \( H(b) \). We allow \( H(\cdot) \) to be degenerate at 0, or have an atom at or below \( \rho \), but is atomless at any \( b > \rho \). We also assume that \( H(b) \) is weakly increasing. The speculator may become inactive over some interval, and \( H(b) \) is a constant over such an interval. The support of \( H(b) \) may not be connected. Let \( \bar{b}_i, \underline{b}_i, \bar{b}_s, \underline{b}_s \) be the maximum and minimum of the support of the bid distribution of regular buyer \( i \) or the speculator, respectively.

The winner of the first-stage auction, i.e. the seller in the resale auction, will revise his or her belief about the losers’ value distributions in the resale stage. When the speculator or buyer \( i \) wins the first-stage auction with a bid \( b \), the only information known about a losing regular bidder \( j \) is that \( b_j(\cdot) \leq b \). Let \( G_j(\cdot, b) \) be the induced probability distribution of the losing buyer \( j \) conditional on this information after winning, and \( Q_j(b) \) be the probability of buyer \( j \) bidding below \( b \). Let \( w_j(b) \) be the maximum of the support of \( G_j(\cdot, b) \). The function \( w_j(b) \) may not be continuous, but \( Q_j(b) \) is a continuous function. If \( b_j(\cdot) \) is strictly increasing, let \( \phi_j(\cdot) \) be the inverse function of \( b_j(\cdot) \). In this case, the updated belief about buyer \( j \) is simply the conditional distribution \( F(\cdot | \phi_j(b)) \).

\(^9\) An earlier version allows several speculators. The speculators can be combined into one with no change in the outcome of the equilibrium or the behavior of the regular buyers. For simplicity of exposition, we have only one speculator with no loss of generality in our result.

\(^10\) We can allow \( F(\cdot) \) to be piecewise \( C^2 \) smooth. This is a more natural framework for the analysis, as the equilibrium strategy in general is a maximum of two smooth functions. A typical example of a piecewise \( C^2 \) smooth function is the maximum or minimum of two smooth \( C^2 \) functions. Formally a function is called piecewise \( C^2 \) smooth if (i) it is \( C^2 \) except at a countable closed set \( D \), (ii) At a point in \( D \), the left and right derivatives (up to the second order) exist and are continuous from the right and left respectively.

Since the bidding strategies \( b_j(\cdot), j \neq i \), may not be the same for different buyers, the seller in the resale auction faces an asymmetric auction environment. Such revision of beliefs is common knowledge as the winning bid is public information. The game in the resale auction is well-known, and we can summarize the result of the resale auction by a profit function without specifying the second period strategies, so that we can focus on the first-stage bidding behavior anticipating the optimal resale outcome in the second stage.

To define an equilibrium strategy of the auction with resale, let \( \sigma \) denote the strategy profile \( H(\cdot), b_i(\cdot), i = 1, 2, \ldots N \). First consider the objective of the speculator’s bidding behavior. Let \( \pi_s(b, \sigma) \) denote the optimal expected revenue of the speculator during the resale stage. The speculator chooses \( b \) to maximize the overall profit

\[
u_s(b, \sigma) = \pi_s(b, \sigma) - bH(b) \prod_{i=1}^{N} Q_i(b).
\]

Given \( \sigma \), we say that \( b \) is an optimal bid for the speculator if \( b \) is an optimal solution of the above maximization problem. Since the speculator uses a mixed strategy, all bids in the support of the equilibrium bid distribution \( H(b) \) must yield the same payoff to the speculator.

After a regular buyer \( i \) with use value \( v_i \) submits a bid \( b \), and wins the auction, he updates the belief regarding the other regular buyers, and believes that buyer \( j \) has the value distribution \( G_j(\cdot, b) \). If \( b \geq b_j \), we assume that there is no change in belief regarding buyer \( j \). After winning the first-stage auction, buyer \( i \) may sell the object to buyer \( j \) during the resale stage if \( w_j(b) > v_i \). If buyer \( i \) loses the auction, and the winner is the speculator or some regular buyer \( j \) with \( w_j(b) < v_i \), buyer \( i \) may bid for the object and buy it from the winner during resale. Since the winner will use the winning bid and the bidding strategy of the losing buyer \( i \) in this case to update belief and determine the reservation price, the payoff of buyer \( i \) after losing the auction will depend not only on the strategy profile of the other buyers, but also on his own choice of the bidding strategy. When resale fails to materialize after a winner wins the object in the first stage, the winner keeps the object.

Let

\[\pi_{1i}(v_i, b, \sigma_{-i}) = (v_i - b)H(b) \prod_{j \neq i} Q_j(b)\]

be his payoff in the first-stage auction. Let \( \pi_{wi}(v_i, b, \sigma_{-i}) \) be his expected payoff in the resale market after winning, and \( \pi_{li}(v_i, b, \sigma) \) the corresponding amount after losing. Then the overall payoff from bidding \( b \) is

\[u_i(v_i, b, \sigma) = \pi_{1i}(v_i, b, \sigma_{-i}) + \pi_{wi}(v_i, b, \sigma_{-i}) + \pi_{li}(v_i, b, \sigma).\]

When all regular buyers use the same increasing bidding strategy, more explicit formulas for the payoffs are given below. Buyer \( i \) chooses \( b \) to maximize \( u_i(v_i, b, \sigma) \) in (1). We say that \( b \) is an optimal or equilibrium bid for the regular buyer \( i \) with use value \( v_i \) if it is a solution to the maximization problem. We say that \( \sigma \) is a perfect Bayesian equilibrium of the auction with resale if (i) for the speculator, any bid \( b \) in the support of \( H(b) \) is an optimal bid, (ii) for each regular buyer \( i \), \( b_i(v_i) \) is an optimal bid maximizing (1).

A perfect Bayesian equilibrium of the auction with resale should describe strategies in both stages of the game. For convenience, we shall abuse the language somewhat and refer to \( \sigma \) as a perfect Bayesian equilibrium. This is due to the fact that the resale game is not explicitly spelled out, and only summarized. See section 7.1 for more details about the analysis of the resale auction revenue. We say that the equilibrium has monotonicity and symmetry property if \( b_i(\cdot) = b_j(\cdot) = b(\cdot) \) for any two regular buyers, and \( b(\cdot) \) is strictly increasing. We will show that monotonicity and symmetry must hold for any equilibrium. The equilibrium strategy of a regular buyer will be denoted by one single bidding function \( b(\cdot) \), or one inverse bidding function \( \phi(\cdot) \). We say that the equilibrium is unique if the profile \( \sigma = (b(\cdot), H(\cdot)) \) is uniquely determined. We say that the speculator is inactive in equilibrium if she bids strictly below \( \rho \) with probability 1. We take the convention that when the speculator is inactive in equilibrium, she bids 0 for sure (although bidding below \( \rho \) yields the same outcome). We also take the convention that regular buyers with \( v < \rho \) all bid 0.

2.1 Payoffs Under Symmetry and Monotonicity

When the symmetry property holds, the resale auction by the speculator is a symmetric auction with a single reservation price. We will write down the detailed payoff expression in this case. It is convenient for
the presentation to assume that the virtual value is increasing, so that there is a unique optimal reservation price in the resale stage. Let

\[ J(x, w) = x - \frac{F(w) - F(x)}{f(x)} \]

denote the conditional virtual value at \( x \) when the buyer value upper bound is \( w \). If \( J(x, \beta) \) is strictly increasing in \( x \), then \( J(x, w) \) is also strictly increasing for all \( w \). If the seller has use value \( v_0 \), the optimal reservation price \( r(v_0, w) \) conditioned on the upper bound \( w \) is determined by the solution of the following equation in \( x \)

\[ J(x, w) = v_0. \]

We let \( v_0 = 0 \) when the seller is the speculator. The increasing virtual value property of \( J(x, \beta) \) is made to insure the uniqueness of the optimal reservation price. When the seller is the speculator, we may also use the simpler notation \( r(w) \), or \( r(v) \) when \( w = v \). This optimal reservation price \( r(v) \) is independent of the number of buyers, and satisfies the following equation

\[ r(v)f(r(v)) + F(r(v)) = F(v). \] (2)

For the increasing symmetric strategy \( b(.) = \phi^{-1}(.) \), and \( H(.) \), the payoff in (1) of a regular buyer can now be written more explicitly. Without the speculator and without resale, \( N > 1 \), the payoff function can be written as

\[ u_0(v_i, b, \phi) = F^{N-1}(\phi(b))(v_i - b). \]

When there is a speculator and resale is allowed, the first-stage payoff is given by

\[ \pi_{l1}(v_i, b, \sigma) = H(b)u_0(v_i, b, \phi). \]

Consider a regular buyer \( i \) bidding \( b \) with \( \phi(b) < v_i \), there is zero payoff from resale after winning the first-stage auction (\( \pi_{wi} = 0 \)), but there is payoff after losing. If \( N > 1 \), the winner of the first-stage auction may be a regular buyer. The winner believes that he has the highest value, and is indifferent between no resale or offering it for resale at the price equal to his own value. Hence either there is zero payoff, or the payoff is equal to

\[ \int_b^{\phi(b)} (v_i - \phi(y))H(y)dF^{N-1}(\phi(y)). \] (3)

Let \( \pi_{l10} \) be either 0 or (3). When the speculator wins the first-stage auction with a bid \( y \), the optimal reservation price set by her during resale is \( r(\phi(y)) \). Let \( \bar{y} \) be the unique solution of the equation \( r(\phi(y)) = v_i \). The payoff of buying from the speculator during resale is

\[ \pi_{l1}(v_i, b, \sigma) = \int_b^{\bar{y}} \left( \int_{r(\phi(y))}^{v_i} F^{N-1}(x)dx \right) dH(y) + \pi_{l10}. \] (4)

Hence the over all payoff is given by

\[ u(v_i, b, \sigma) = H(b)u_0(v_i, b, \phi) + \pi_{l1}(v_i, b, \sigma), \quad \text{when } v_i > \phi(b). \] (5)

For the bid \( b \) with \( \phi(b) \geq v_i \), there may be payoff from resale after winning or losing the first-stage auction. If he loses the first-stage auction, and the winner is the speculator, then the resale payoff is given by (4) with \( \pi_{l10} = 0 \). If the winner is a regular buyer, there is zero payoff from resale. If he wins the auction, there is payoff from selling to the losing regular buyers. Let the optimal reservation price in resale be \( r(v_i, \phi(b)) \). The payoff after winning the auction is

\[ \pi_{wi}(v_i, b, \sigma) = H(b) \int_{r(v_i, \phi(b))}^{\phi(b)} (J(x, \phi(b)) - v_i)dF^{N-1}(x). \] (6)

Hence the total payoff in this case is given by

\[ u(v_i, b, \sigma) = H(b)u_0(v_i, b, \phi) + \int_b^{\bar{y}} \left( \int_{r(\phi(y))}^{v_i} F^{N-1}(x)dx \right) dH(y) + \pi_{wi}(v_i, b, \sigma), v_i \leq \phi(b). \] (7)
2.2 Supermodularity

For increasing symmetric strategies, supermodularity can be easily shown. We say that the payoff function has the supermodularity property in the strong form at \((v_i, b)\) if

\[
\frac{\partial^2 u}{\partial b \partial v_i} (v_i, b, \sigma) > 0. \tag{8}
\]

It has the supermodularity in the weak form at \((v_i, b)\) if

\[
\frac{\partial^2 u}{\partial b \partial v_i} (v_i, b, \sigma) > 0. \tag{9}
\]

Consider the case \(\phi(b) > v_i > \rho\). Taking the partial derivatives of (7), we get

\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \frac{\partial}{\partial b} \left[ H(b)F^{N-1}(\phi(b)) \right] > 0.
\]

Hence we have supermodularity in the strong form when \(v_i \leq \phi(b)\).

Consider the other case \(v > \phi(b) > \rho\). When \(\pi_{i0} = 0\), from the expression in (5), we have

\[
\frac{\partial}{\partial v_i} \pi_{i0} (v_i, b, \sigma) = \int_b^\theta F^{N-1}(v_i) dH(y).
\]

We also have

\[
\frac{\partial}{\partial v_i} H(b)u_0 (v_i, b, \phi) = H(b)F^{N-1}(\phi(b))\tag{10}
\]

Hence

\[
\frac{\partial}{\partial v_i} u(v_i, b, \sigma) = H(b) \frac{d}{db} F^{N-1}(\phi(b)) > 0.
\]

When \(\pi_{i0}\) is given by (3), we have

\[
\frac{\partial}{\partial v_i} \pi_{i0} (v_i, b, \sigma) = \int_b^\theta F^{N-1}(v_i) dH(y) + \int_{b(v_i)} F(v_i) d\Phi^{N-1}(\phi(y)),
\]

the right-hand side above is simply the probability of winning the object during resale (either from the speculator or a regular buyer). While the right-hand side of (10) is the probability of winning the object in the first-stage auction. We have

\[
\frac{\partial}{\partial b} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \frac{\partial}{\partial b} \left[ \frac{\partial}{\partial v_i} H(b)u_0 (v_i, b, \phi) + \frac{\partial}{\partial v_i} \pi_{i0} (v_i, b, \sigma) \right] = 0. \tag{11}
\]

Thus we only get the weak form in this case.

The intuition in the equation (11) is that the sum of the two terms inside the bracket above does not depend on \(b\). When the regular buyer bids higher, the probability of winning in the first-stage auction becomes larger, but is cancelled by the lower probability of winning the object during resale. As long as \(b\) is inside the region in which there is zero payoff from selling the object after winning, and there is resale (for sure) from buying after losing the auction in the first-stage, then the two probabilities must compensate each other and the sum stays the same. This intuition remains the same when we consider asymmetric and non-monotone strategies, and will be generalized later.

The above supermodularity property is sufficient for the first-order condition \(\frac{\partial}{\partial b} u(v_i, b, \sigma)\big|_{b = b(v_i)} = 0\) to yield the optimality property for the strategy \(b(v_i)\). Take \(b' > b(v_i)\), \(\frac{\partial}{\partial b} u(v_i, b, \sigma)\big|_{b = b(v_i)} = 0\), (8) implies \(\frac{\partial}{\partial b} u(v_i, b, \sigma)\big|_{b = b'} < 0\). For \(b' < b(v_i)\), we have \(\frac{\partial}{\partial b} u(v_i, b, \sigma)\big|_{b = b'} > 0\). Hence \(b(v_i)\) is an optimal bid. We will see later that the regular buyer’s strategy is also uniquely determined. It is also clear that the supermodularity property above is also sufficient to insure that the equilibrium strategy \(b(v)\) of a regular buyer must be strictly increasing.

The supermodularity property, when generalized to asymmetric and non-monotone strategies, will allow us to conclude that an equilibrium strategy must be increasing and the first-order condition will be a sufficient condition for an optimal strategy. We will first use the supermodularity property for the symmetric increasing strategies established here to find a solution of the symmetric equilibrium.
3 Equilibrium Properties

Once the equilibrium has been shown to be symmetric and increasing, GT (supp) have proved that the equilibrium exists and is unique. They use the uniqueness result of the solution of a system of differential equations to prove uniqueness. We shall adopt a different mode of analysis that is more transparent and eventually leads to an explicit simple solution of the equilibrium strategy.

The main ideas for our analysis will be implemented in several steps. We give an informal description before presenting the formal analysis. First we give the unique symmetric equilibrium solution of the model with resale but with no speculator. Secondly, we introduce some useful functions which are important for our analysis and provide some preliminary lemmata. Thirdly we prove the first-order conditions of equilibrium. Fourthly, we give a simple characterization of active and inactive intervals based on revenue and cost function comparisons. In the fifth step, we show that active and inactive intervals do not depend on which equilibrium we take, and are uniquely determined. In the sixth step, we show that the cost function dominates the revenue function in inactive intervals, while the reverse occurs in active intervals. In the final step, done in section 4, we define the equilibrium strategy to be the maximum of the cost and revenue functions. The bid distribution of the speculator is also easily determined from the first-order conditions. This construction gives us an equilibrium strategy profile. The uniqueness of the equilibrium holds because it holds on each active and inactive intervals.

3.1 Equilibrium with No Speculator

If there is a speculator, \( N > 1 \); and resale is not allowed, the equilibrium strategy is known to be strictly increasing, symmetric and unique. Let \( \phi(.) \) be the inverse of the equilibrium bidding strategy. The first-order condition of the equilibrium satisfies

\[
\frac{\partial}{\partial b} u_0(v, b, \phi)|_{b=b(v)} = 0,
\]

\[
\frac{d}{db} \ln F(\phi(b)) = \frac{1}{(N-1)(\phi(b) - b)}.
\]  (12)

We can rewrite the first-order condition (12) in the variable \( v = \phi(b) \) as follows

\[
b'(v) = (N - 1)(v - b(v)) \frac{f(v)}{F(v)}.
\]  (13)

Let \( b_{\rho}(.) \), \( \phi_\rho(.) = b^{-1}_{\rho}(.) \) denote the equilibrium bidding strategy and its inverse when the reservation price is \( \rho \). We have the formula

\[
b_{\rho}(v) = v - \int_{\rho}^{v} F^{N-1}(x|v)dx = \rho F^{N-1}(\rho|v) + \int_{\rho}^{v} xdF^{N-1}(x|v), \text{ for } v \geq \rho.
\]  (14)

Note that for \( N = 1 \), we have \( b_{\rho}(v) = \rho \) for \( v \geq \rho \).

When resale is allowed, but there is no speculator (or \( H(0) = 1 \)), we can write the payoff of a regular buyer with value \( v \) bidding \( b, \phi(b) < v \), as

\[
u(v, b, \phi) = u_0(v, b, \phi) + \int_{b}^{\phi(b)} (v - \phi(y))dF^{N-1}(\phi(y)),
\]

and we have the first-order condition

\[
\frac{\partial}{\partial b} u(v, b, \phi) = \frac{\partial}{\partial b} u_0(v, b, \phi) = 0.
\]  (15)
When \( \phi(b) \geq v \), we have
\[
    u(v, b, \phi) = u_0(v, b, \phi) + \int_{r(v, \phi(b))}^{\phi(b)} (J(x, \phi(b)) - v) dF^{N-1}(x),
\]
and we have the same first-order condition (15). Therefore if we allow resale with no speculator, the first-order condition of an equilibrium is the same as that of a model with no speculator and no resale. Combine this with the supermodularity property, and we know that equilibrium must be unique in a model with a speculator but allowing resale, and is the same one well-known in the literature. Once we establish the increasing symmetry property of the equilibrium, the uniqueness follows from the well-known result of the auction without resale model. Hence we immediately have the following result.

**Theorem 1** If there is no speculator, the equilibrium with resale is unique and is the same as the one without resale given by (14).

### 3.2 Equilibrium Conditions of the Speculator

Let \( \phi(.) \) be the inverse equilibrium strategy of the regular buyers. After winning the auction bidding \( b \), the speculator sells to \( N \) regular buyers with the value distribution \( F(,|\phi(b)) \) in the resale market. Let \( v = \phi(b) \), and \( B(v) \) be the expected total revenue during the resale. The function \( B(v) \) can be defined for any \( v \), and according to the Myerson (1981) revenue formula, it is given by
\[
    B(v) = \int_{r(v)}^{v} J(x, v) dF^{N}(x|v) = \frac{N}{F^{N}(v)} \int_{r(v)}^{v} [xf(x) + F(x) - F(v)F^{N-1}(x)] dx.
\]
(16)

We shall call this the revenue function of the speculator. When \( N = 1 \), we have \( B(v) = r(v)(1 - \frac{F(r(v))}{F(v)}) \). The function \( B(.) \) is not defined at \( v = 0 \), but we can let \( B(0) = 0 \), which will make \( B(v) \) a continuous function over \([0, \beta]\).

Another function that is closely related to the revenue function will now be introduced. For \( z \leq w \), let
\[
    B^t(z, w) = z - \int_{r(w)}^{z} F^{N-1}(x|w) dx.
\]
The function \( B^t(z, w) \) is the revenue contribution of a regular buyer with value \( z \) to the speculator in the resale market when the upper bound of the value is \( w \). When \( N = 1 \), \( B^t(v, w) = r(w) \). We use a simpler notation when \( v = z = w \):
\[
    B^t(v) = v - \int_{r(v)}^{v} F^{N-1}(x|v) dx.
\]
For comparison, note that the revenue function can be written, using integration by parts, as
\[
    B(v) = v - \int_{r(v)}^{v} F^{N-1}(x|v) dJ(x, v).
\]
The function \( B^t(v) \) is the amount of payment to the speculator by the "top" type when the speculator sells to \( N \) symmetric buyers with the use value distribution \( F(,|v) \). We have the following useful alternative formula for \( B(v) \):
\[
    B(v) = \int_{r(v)}^{v} B^t(x, v) dF^{N}(x|v). \tag{17}
\]
Since \( B^t(x, v) \) is strictly increasing in \( x \), we have \( B^t(x, v) < B^t(v) \) when \( x < v \). Hence we have
\[
    B(v) < B^t(v) \int_{r(v)}^{v} dF^{N}(x|v) < B^t(v).
\]
For convenience, we state this as a lemma.
Lemma 1  We have 

\[ B'(v) > B(v) \text{ for all } v \in (0, \beta]. \]

The following gives an important derivative formula for \( B(v) \).

Lemma 2  The derivative of the function \( B(v) \) in (16) is given by

\[ B'(v) = \frac{Nf(v)}{F(v)} \left[ B'(v) - B(v) \right] > 0, \text{ for } v \in (0, \beta]. \]  

(18)

The speculator profit, given the regular buyer’s strategy \( \phi(.) \), can be written as

\[ u_s(b, \phi) = F^N(\phi(b))(B(\phi(b)) - b). \]

Equilibrium profit for the speculator bidding \( b \) in the support of the \( H(.) \) must be a constant independent of \( b \). It is known that the speculator makes zero profit in equilibrium. This has been proved in Lemma 7 of GT(supp). It is useful to provide the reasons here as it is a central part of our analysis. Let \( b_s \geq \rho > 0 \) be the minimum bid of the speculator. The minimum bid of a regular buyer must be \( b_s \) as well. Let a regular buyer with value \( v \) bid \( b_s \) in equilibrium. Since non-negative profit condition implies \( B(v) \geq b_s \), we must have \( v > b_s \). If \( H(b_s) = 0 \), the regular buyer with value \( v \) gets 0 payoff bidding \( b_s \). The deviation to a slightly higher bid yields a positive profit, hence we have a contradiction. If \( H(b_s) > 0 \), again a regular buyer can increase the probability of winning by bidding slightly higher, and increase the profit. Hence we have another contradiction. This implies that we must have \( b_s < \rho \), or \( \rho = 0 \). In either case, the speculator profit is zero. Another useful intuition we want to mention here is that in equilibrium the speculator bids less aggressively than a regular buyer (see Theorem 5). This means she bids below \( \rho \) with positive probability, and the profit is zero when she does so. For convenience, we state it as a lemma here.

Lemma 3  In any equilibrium, the speculator makes zero profit.

Lemma 2 implies that the function \( B(.) \) is strictly increasing. Let the inverse function of \( B(.) \) be denoted by \( \eta(.) \). From the zero profit condition, we know that if \( b \) is in the support of the \( H(.) \), we must have \( B(\phi(b)) - b = 0 \), or \( \phi(b) = B^{-1}(b) = \eta(b) \). In other words, \( \eta(.) \) is the inverse equilibrium bidding strategy of the regular buyers. Thus the inverse equilibrium bidding strategy of a regular buyer over an interval in the support of \( H(.) \) is uniquely determined. The speculator may become inactive, so that the support of \( H(.) \) is the set \{0\}. We need to determine whether the speculator is active or not in equilibrium. The revenue function \( B(.) \) plays essential roles in understanding whether the speculator is active or not in equilibrium and, if active, in determining the support of the bid distribution (or intervals in which they are active).

The equilibrium condition for the speculator is that \( \pi_s(b, \phi) = 0 \) for \( b \) in the support of \( H(.) \), and \( \pi_s(b, \phi) \leq 0 \) outside the support of \( H(.) \). An important part of our analysis is to understand the difference in equilibrium conditions in different intervals, depending on whether the speculator is active or not in the intervals. We shall give a formal definition of this idea.

Let \( S_H \) be the support of \( H(.) \). We say that the speculator is inactive in equilibrium if \( H(0) = 1 \), or \( S_H = \{0\} \). She bids 0 for sure in equilibrium in this case, and has no impact on the outcome of the auction. We say that she is fully active in equilibrium if the support of \( H(.) \) is the same as that of the equilibrium bid distribution of a regular buyer. In equilibrium, the speculator typically bids only in certain intervals with positive probability. To define an interval in which the speculator is active, we say that a bid interval \([a_1, a_2]\) or \((a_1, a_2]\) is an active interval if it is subset of the support of \( H(.) \). A maximal active interval is a closed active interval which is not a proper subset of another closed active interval. A maximal active interval is simply a maximal connected component of \( S_H \). When \([a_1, a_2]\) is a maximal active interval, we say that \( a_1 \) is the beginning of an active interval, and \( a_2 \) is the end of an active interval. An interval \([a_1, a_2] \) is inactive if it contains no active intervals. This is the same as saying \( H(.) \) is constant over the interval. Since most of our functions are defined in the space of use values, It is convenient to call an interval \([z_1, z_2]\)
of use values an active interval if \([B(z_1), B(z_1)]\) is an active interval of bids. Note that the definition of an active bid interval depends only on the support of \(H(.)\). If we can show that the support of any two equilibria \(H_1(.)\), \(H_2(.)\) are the same, then we have the same collection of active intervals. Since we know \(B(.)\) must be the equilibrium bidding strategy over an active interval \([z_1, z_2]\), an active value interval is simply the value-types of a regular buyer who bids inside an active bid interval in equilibrium. A maximal value interval can be similarly defined, so are the beginning and end of an active value interval, and the inactive intervals. The concept of active and inactive intervals plays an important part of our analysis.

If two continuous functions \(g_1(.)\), \(g_2(.)\) defined over \((c, d)\) are not the same over any open interval in \((c, d)\), then \(g_1(.) = g_2(.)\) can only occur on a countable closed subset.

### 3.3 Equilibrium Conditions in Inactive Intervals

The equilibrium condition for a regular buyer in inactive intervals is very similar to the equilibrium condition with no speculator. It can be easily seen that in inactive intervals, (12) is also the first-order condition of equilibrium. Therefore if the boundary condition is determined, then (12) uniquely determines the bidding behavior of the regular buyers in inactive intervals. When we solve (13) with the initial condition \(b(z) = b_0\) at \(z > \rho\), the solution is given by

\[
b(v) = b_0 F^{N-1}(z|v) + \int_z^v xdF^{N-1}(x|v), \quad v \in [z, \beta],
\]

We often take \(z\) to be either \(\rho\), the beginning or the end of an active interval, and \(b_0 = B(z)\). In this case, we use the notation \(B^*_z(.)\) to denote the solution (19). When \(b_0 = B(z) = B(z)\), we have

\[
B^*_z(.) = B(z) F^{N-1}(z|v) + \int_z^v xdF^{N-1}(x|v).
\]  

(19)

In a symmetric increasing equilibrium only the speculator will sell the object after winning it in the first-stage auction. In determining whether the speculator is active or not, the two functions \(b_\rho(.)\), \(B^*_z(.)\) can be regarded as "cost functions" for the speculator because it is the cost needed to win the object to be able to sell to \(N\) regular buyers with value distribution \(F(.,|v)\). We also call \(B^*_z(.)\) an endogenous cost function as it depends on the equilibrium bid \(B(z)\) at \(z\), while \(b_\rho(.)\) does not. We can regard \(b_\rho(.)\) as a special case of \(B^*_z(.)\) when \(z = \rho = b_0\). Both functions are strictly increasing (when \(N > 1\)) and continuous. From (13), we immediately have the following.

**Lemma 4** When \(N > 1\), the function \(B^*_z(.)\) has the following properties: (i) \(B^*_z(z) = B(z)\), (ii) \(B^*_z(.)\) is strictly increasing in \([z, \beta]\), and (iii) The derivative is given by

\[
B^*_z(v) = \frac{(N - 1) f(v)}{F(v)} (v - B^*_z(v)).
\]

Another function closely related to \(B^t(v)\) we will need often in the theory is now introduced. Let

\[
B^c(v) = v - N \int_{r(v)}^v F^{N-1}(x|v) dx.
\]

Note that, by definition, we have

\[
v - B^c(v) = N(v - B^t(v)).
\]  

(20)

When \(N = 1\), \(B^c(v) = B^t(v) = r(v)\).

The following is a key lemma on the derivative of the profit function of the speculator \(B(.) - B^*_z(v)\).
Lemma 5 We have
\[
B'(v) - B^*_z(v) = \frac{N f(v)}{F(v)} \left[ \frac{1}{N} B^c(v) + \frac{N - 1}{N} B^*_z(v) - B(v) \right], \quad v > 0,
\] (21)

The following gives the crossing properties of the three functions \(B^c(.), B(.), B^*_z(.)\).

Lemma 6 If any two of the functions \(B(.), B^*_z(.), B^c(.)\) are the same over some interval \([v_1, v_2]\), then all three functions are equal to each other over \([v_1, v_2]\).

A countable closed subset of an interval \((c, d)\) is the intersection of \((c, d)\) with a countable closed set in the set of real numbers. The following says that at the point where \(B^c(.)\) crosses \(B(.)\), we also have the crossing of \(B^*_z(.)\) and \(B(.)\) in the other direction.

Lemma 7 Let \(N > 1\), and \(B^*_z(.)\) be defined in \((z, v)\) or \((v', z)\) with the boundary condition \(B^*_z(z) = B(z)\), or \(B^*_z(.) = b_0(.)\) at \(z = 0\). Then the following three statements hold (with the understanding that the inequalities hold in \((z, v)\) or \((v', z)\) except a countable closed subset):

(a) \(B^c(.) > B(.)\) if and only if \(B(.) > B^*_z(.)\),
(b) \(B^c(.) < B(.)\) if and only if \(B(.) < B^*_z(.)\),
(c) \(B^c(.) = B(.)\) if and only if \(B(.) = B^*_z(.)\).

3.4 Equilibrium Conditions in Active Intervals

One useful function in simplifying the first-order condition of a regular buyer in an active interval will now be introduced. Let
\[
\mathcal{L}(v) = \frac{B^c(v) - B(v)}{B'(v) - B(v)}, \quad L(b) = \mathcal{L}(\eta(b)) = \mathcal{L}(v).
\] (22)

When \(N = 1\), we have \(B^c(v) = B^t(v)\), hence \(\mathcal{L}(v) = 1\). In general \(\mathcal{L}(v) \leq 1\), and may take negative values.

The following gives the derivative of a regular buyer’s payoff with respect to \(b\).

Lemma 8 We have the following derivative
\[
\frac{\partial u(v, b, \sigma)}{\partial b} |_{b=b(v)} = H'(b)F^{N-1}(\phi(b))[B'(\phi(b)) - b] + H(b)\frac{\partial u_0(v, b, \phi)}{\partial b} |_{b=b(v)} \text{ for } b > \rho.
\] (23)

Proof. For the case \(\phi(b) < v\), the term \(\pi_{t|0}\) is either 0 or given by (3), with \(vi\) now denoted by \(v\). Taking the derivative with respect to \(b\), we have

\[
\frac{\partial}{\partial b} \pi_{t|0} = \frac{\partial}{\partial b} \int_b^{b(v)} (v - \phi(y))H(y)dF^{N-1}(\phi(y)) |_{b=b(v)} = 0.
\]

Taking the derivative of (5) with respect to \(b\), we have
\[
\frac{\partial u(v, b, \sigma)}{\partial b} |_{b=b(v)} = H'(b)F^{N-1}(\phi(b))[v - b - \int_{r(\phi(b))}^v F^{N-1}(x|\phi(b))dx] + H(b)\frac{\partial u_0(v, b, \phi)}{\partial b} |_{b=b(v)},
\] (24)
which is the same as (23). The formula (23) holds in the other case \( \phi(b) \geq v \) as well if

\[
\frac{\partial u_{u}(v, b, \phi)}{\partial b}|_{b=b(v)} = 0.
\]

Since \( J(\phi(b), \phi(b)) = v, J(r(v, \phi(b)), \phi(b)) = v, r(\phi(b), \phi(b)) = \phi(b) \), we have

\[
\frac{\partial u_{u}(v, b, \phi)}{\partial b}|_{b=b(v)} = H'(b) \int_{r(v, \phi(b))}^{\phi(b)} (J(x, \phi(b)) - v) dF_{N-1}(x)|_{b=b(v)} = 0.
\]

Hence the derivative in the case \( \phi(b) \geq v \) leads to the same formula (23). \( \square \)

In active intervals, we must have \( \phi(b) = \eta(b) \). We have the following expression for the second term in the first-order condition.

**Lemma 9** We have

\[
\frac{\partial u_{b}(v, b, \eta)}{\partial b}|_{b=B(v)} = -\frac{1}{N} F_{N-1}(v) L(v).
\]

**Proof.** First change the variable from \( b \) to \( z = \eta(b) \), and then use the formula for \( B'(v) \) in Lemma 2 twice. We get

\[
\frac{\partial u_{b}(v, b, \eta)}{\partial b}|_{b=B(v)} = \frac{1}{B'(v)} \frac{\partial}{\partial z} \left[ F_{N-1}(z)(v - B(z)) \right]|_{z=v}
\]

\[
= \frac{F_{N-1}(v)}{B'(v)} \left[ \frac{(N-1)f(v)}{F(v)} (v - B(v) - B'(v)) \right]
\]

\[
= \frac{F_{N-1}(v)}{B'(v)F(v)} \left[ \frac{(N-1)f(v)}{F(v)} (v - B(v) - \frac{Nf(v)}{F(v)} (B'(v) - B(v)) \right]
\]

\[
= \frac{f(v)F_{N-1}(v)}{B'(v)F(v)} \left[ N(v - B'(v)) - (v - B(v)) \right]
\]

\[
= -\frac{F_{N-1}(v) B'(v) - B(v)}{N B'(v) - B(v)},
\]

and the lemma is proved by the definition (22). \( \square \)

The following summarizes the first-order conditions of a regular buyer in active and inactive intervals.

**Lemma 10** Let \( H(., .), b(.) \) be an equilibrium strategy profile, \( \phi(.) = b^{-1}(.) \). In inactive intervals, we have

\[
\frac{\partial u_{b}(v, b, \phi)}{\partial b}|_{b=b(v)} = 0, b > \rho.
\]

In active intervals, we have \( \phi(b) = \eta(b) \), and

\[
\frac{H'(b)}{H(b)} = L(b) \frac{d}{db} \ln F(\eta(b)), b > \rho.
\]

**Proof.** In inactive intervals, \( H'(.) = 0 < H(.) \) implies (25). Let the speculator be active in any open interval \( I \). By Lemma 3, we must have \( \phi(b) = \eta(b) \) in the interval. Setting the derivative (23) equal to zero in equilibrium, and apply Lemma 9, we have

\[
H'(b) F_{N-1}(\eta(b)) [B'(\eta(b)) - b] - H(b) \frac{1}{N} F_{N-1}(v) L(v) = 0.
\]
Since \( b = B(v) \), from Lemma 2, we have
\[
\frac{H'(b)}{H(b)} = \frac{\mathcal{L}(v)}{N(B'(v) - B(v))} = \frac{L(b)}{N(B'(\eta(b) - b))} = L(b) \frac{d}{db} \ln F(\eta(b)).
\] (27)

Since \( H'(b) > 0 \) except a countable closed subset in an active interval, (26) implies \( \mathcal{L}(v) > 0 \) except a countable closed subset in an active interval. When \( N > 1 \). We may have \( \mathcal{L}(v) < 0 \), and (26) implies a restriction on where the speculator can be active. The following is an immediate implication of Lemma 10.

**Lemma 11** In an active open interval \( I \) above \( \rho \) of the speculator, we must have \( L(b) > 0 \), except a countable closed subset, or equivalently \( B'(v) > B(v) \) in the active interval \( \eta(I) \), except a countable closed subset.

### 3.5 Characterization of active and inactive intervals

It can occur quite often that the speculator is not active in equilibrium. When the speculator is not active in equilibrium, she has no influence on the outcome of the auction, and regular bidders bid as if there is no speculator. We now give a condition that tells us precisely when the speculator is inactive in equilibrium. Assume that initially there is no speculator, and according to the standard first-price symmetric auction, bidders are bidding \( b_\rho(.) \) in equilibrium. If the revenue \( B(.) \) exceeds the cost \( b_\rho(.) \) at some \( v = \phi(b) \), then there is profit to be made, and the speculator will probably enter and bid actively. Otherwise, there will be no entry, and we should expect the speculator to be inactive in this case. Thus, intuitively, for any possible entry of the speculator in the bidding, there should be a bid \( b > \rho \) at which the speculator can make a profit. This intuition can be formally verified in the following theorem.

Let \( \pi_\rho(b) = B(\phi_\rho(b)) - b, b \in [\rho, b_\rho(\beta)] \) be the profit function for the speculator when the regular bidders use the bidding function \( b_\rho(.) \). When \( N = 1 \), the function \( \pi_\rho(b) \) is defined only for \( b = \rho \).

**Theorem 2** The speculator is inactive in equilibrium if and only if \( \pi_\rho(b) \leq 0 \) for all \( b \in [\rho, b_\rho(\beta)] \). In an equilibrium in which the speculator is inactive, the equilibrium is the same as if there is no speculator and we get the same equilibrium in Theorem 1.

**Proof.** From the assumption, \( \pi_\rho(b) = B(\phi_\rho(b)) - b \leq 0 \), we have \( B(v) - b_\rho(v) \leq 0 \) for all \( v \). Let \( \phi(.) \) be the inverse equilibrium bidding strategy of the regular bidders. If the speculator is active in equilibrium, by Lemma 3, there exists an active interval \( (b_1, b_2) \) such that \( \eta(b) = \phi(b) \) on the interval. We claim that on the interval \( [\eta(b_1), \eta(b_2)] = [x_1, x_2] \), we have \( B(.) = b_\rho(.) \). Otherwise, there exists an open interval \( I \) in \( (b_1, b_2) \) such that \( \pi_\rho(b) = B(\phi_\rho(b)) - b > 0 \) contradicting Lemma 3. However, Lemma 6 implies that \( B'(\cdot) = B'(\cdot) \) over \( [x_1, x_2] \) as well, hence we have \( B'(\cdot) = B'(\cdot) \) over \( [x_1, x_2] \) and this contradicts Lemma 11. The contradiction implies that the speculator cannot be active in equilibrium. We obtain the same equilibrium \( b_\rho(.) \) in the model without resale as shown in Theorem 1. If \( \pi_\rho(b) > 0 \) for some \( b \in [\rho, b_\rho(\beta)] \), and the speculator is inactive in an equilibrium, then regular bidders will bid \( b_\rho(.) \) in equilibrium, but in this case the speculator can make a strictly positive profit by bidding \( b \), contradicting the zero profit condition. Hence the theorem is proved. ■

Theorem 2 tells us precisely when the speculator is inactive in equilibrium. It can be rephrased as follows: The bid interval \([\rho, b(\beta)]\) is an inactive interval (equivalently, \([\rho, \beta]\) is an inactive value interval) if and only if \( B(.) \leq b_\rho(.) \) in \([\rho, \beta]\). We now generalize this result to all inactive intervals and give a simple characterization of inactive intervals by endogenous cost functions. Given any equilibrium strategy \( b(.) \) and interval \([z, z']\), we can define the endogenous cost function \( B^*_2(.) \) with the initial condition \( B^*_2(z) = b(z) \). Let \( \phi^*_2(.) \) be the inverse of \( B^*_2(.) \). Define \( \pi^*(b) = B(\phi^*_2(b)) - b \) for \( b \geq b(z) \). If the speculator is inactive in the interval, we know that the first-order condition (12) must be satisfied, and both \( B^*_2(.) \) and \( b(.) \) have the same initial condition, therefore we have \( B^*_2(.) = b(.) \) in \([z, z']\). The speculator makes maximum profit 0 in equilibrium, and we must have \( \pi^*(b) \leq 0 \). This condition also is sufficient for being an inactive interval as stated in the following. The proof is the same as that of Theorem 2.
Lemma 12 The interval \([z, z']\) is an inactive interval if and only if \(\pi^*(b) \leq 0\) for all \(b \in [B(z), B^*_z(z')]\), or equivalently, \(B(.) \leq B^*_z(.)\) in \([z, z']\).

Note that a corollary of the characterization of inactive equilibrium is that when the speculator is inactive in one equilibrium, she must be inactive in another equilibrium. Similarly, if an interval is inactive in one equilibrium, it must be inactive in another equilibrium, if there is another one.

Lemma 13 If an interval is inactive in one equilibrium, it must be inactive in another equilibrium. Similarly, if an interval is active in one equilibrium, it must be active in another equilibrium.

Proof. It is sufficient to prove this for a maximal inactive interval. Let \([z, z']\) be a maximal inactive interval in the equilibrium \(H(\cdot), b(\cdot)\). Either we have \(z = \rho\), or \(b(z) = B(z)\). If \(z = \rho\), then the non-positive profit condition implies \(B(.) \leq b(.)\) in \([z, v]\), and Lemma 7 implies that \(B^c(.) \leq B(.)\) in \([z, v]\). If \(b(z) = B(z)\), define \(B^*_z(.)\) with the initial condition \(B^*_z(z) = B(z)\), then Lemma 7 shows that we must have \(B^c(.) \leq B(.)\) in \([z, v]\) as well. Assume that there is another equilibrium \(\bar{H}(\cdot), \bar{b}(\cdot)\) such that \([z, z']\) is not an inactive interval. There is a maximal active interval \((z_0, z'')\) of \(\bar{H}(\cdot)\) that overlaps with \([z, z']\) in some open interval \((x, x')\). By continuity, the speculator profit must be 0 at \(z_0\), hence \(B(z_0) = \bar{b}(z_0)\). Define \(\bar{B}^*_z(.)\) with the initial condition \(\bar{B}^*_z(z_0) = B(z_0)\). By Lemma 11, we have \(\bar{B}^c(.) > B(.)\). This is a contradiction. The contradiction proves that \([z, v]\) must be inactive in another equilibrium. Since the active intervals are complementary to inactive intervals, we also conclude that the active intervals are uniquely determined regardless of the \(H(\cdot)\) used to define it. ■

Example 1 Suppose there are two regular buyers and one speculator and \(\rho = 0\). If \(F(v) = v\) is the uniform distribution, we have \(b_0(v) = \frac{1}{2} v, B(v) = \frac{v}{2} \leq b_0(v)\). Hence Theorem 2 implies that the speculator is inactive in equilibrium when \(F(v)\) is a uniform distribution. This is true for any number of regular buyers. If \(F(v) = v^2\), we have \(b_0(v) = \frac{v^2}{3}, r(v) = \frac{v}{\sqrt{3}}, B(v) = \frac{2}{v^2} \int_{v/\sqrt{3}}^{v} (3x^2 - v^2)x^2 dx = 0.58465v < b_0(v)\).

Hence the speculator is inactive in equilibrium, when \(F(v) = v^2\), and this is also true for any number of regular buyers.\(^{12}\)

From Lemma 11 and Lemma 7, we immediately have the following necessary condition of an active interval.

Lemma 14 If \((v_1, v_2), v_1 \geq \rho\), is an active interval, and \(b(.)\) is an equilibrium strategy, then for any \(B^*_z(.)\) defined at \(z < v_1\) with the initial condition \(B^*_z(z) = b(z)\), we must have \(B^c(.) > B(.) > B^*_z(.)\) over \((v_1, v_2)\) except a countable closed subset.

\(^{12}\)In fact the same conclusion holds for any power function, concave function, or many often-used value distributions. This is treated more extensively in Part II of the paper.
Remark 1 The conditions in Lemma 14 are in fact also sufficient for the interval \((v_1, v_2)\) to be an active interval. In Part II, we will show how an equilibrium can be constructed such that it is an active interval in the equilibrium. In fact, it is known (as shown in GT supp) that an equilibrium exists, and in any equilibrium, Lemma 10 implies that \(H'(\cdot) > 0\) in the interval when \(B'(\cdot) > B(\cdot)\). By definition, this means that the interval must be an active interval in the equilibrium or any equilibrium.

The following says that the end of an active interval occurs at the place where \(B'(\cdot)\) crosses \(B(\cdot)\) from above.

Lemma 15 If \([z_1, z_2]\) is a maximal active interval, then (i) \(B'(z_2) = B(z_2), \) if \(z_2 < \beta;\) (ii) \(B'(z_1) = B(z_1)\) if \(z_1 > \rho.\)

Proof of Lemma 15: Since \(z_2\) is the endpoint of an active interval, either there exists an inactive interval \((z_2, z_2 + \varepsilon)\) or there is a sequence of maximal inactive intervals \([x_n, x_n']\) with \(x_n' \to z_2, x_n \geq z_2.\) If \(B'(z_2) > B(z_2),\) define the cost function \(B^*_2(\cdot)\) with the initial condition \(B^*_2(z_2) = B(z_2),\) then we have \(B'(z_2) - B^*_2(z_2) > 0\) by Lemma 5. Hence \(B(\cdot) > B^*_2(\cdot) > 0\) in a neighborhood \((z_2, z_2 + \varepsilon).\) This is a contradiction.

When we combine Lemma 15 with Lemma 5, we know that \(B(\cdot)\) cannot cross \(B^*_2(\cdot)\) unless \(B'(\cdot)\) is above \(B(\cdot).\) Once \(B(\cdot)\) crosses \(B^*_2(\cdot),\) Lemma 7 then tells us \(B(\cdot)\) will stay above \(B^*_2(\cdot)\) leading to an active interval, until \(B'(\cdot)\) ends the active interval at the place it crosses \(B(\cdot)\) from above. This is essentially the way a maximal active interval is determined.

### 3.6 Uniqueness under Symmetry and Monotonicity

In the last subsection, we know that the collection of active and inactive intervals are uniquely determined even if we have multiple equilibria. Here we want to show that there can only be one equilibrium which has the monotonicity and symmetry property.

Lemma 16 There can only be one equilibrium which satisfies the monotonicity and symmetry property.

Proof. To show that the bidding strategy \(b(\cdot)\) of a regular buyer is uniquely determined, it is sufficient to show that it is uniquely determined in all active or inactive intervals. In active intervals, we must have \(b(\cdot) = B(\cdot),\) hence it is unique in active intervals. Let \([z_1, z_2]\) be any inactive interval. We have either \(z_1 = \rho,\) or \(b(z_1) = B(z_1).\) If \(z_1 = \rho,\) then we must have \(b(\cdot) = b(\cdot)\) in \([z_1, z_2].\) If \(b(z) = B(z),\) let \(B^*_{z_1}(\cdot)\) be defined by the initial condition \(B^*_{z_1}(z_1) = B(z_1),\) and we must have \(b(\cdot) = B^*_{z_1}(\cdot)\) in \([z_1, z_2].\) Hence we have shown that \(b(\cdot)\) is uniquely determined in inactive intervals as well. Thus the bidding strategy of a regular buyer is uniquely determined. Now we want to show that \(H(\cdot)\) is also uniquely determined. Since the active intervals are uniquely determined, the maximum bid \(b^*_s\) of the speculator is uniquely determined. Hence we have the boundary condition \(H(b^*_s) = 1\) for \(H(\cdot)\). The function \(H(\cdot)\) satisfies the differential equation stated in Lemma 10 in any active interval. The differential equation is the same for any equilibrium. The function \(H(\cdot)\) stays constant on inactive intervals. These put together implies that \(H(\cdot)\) must be uniquely determined, as long as the active and inactive intervals are uniquely determined. The proof is complete.
4 Equilibrium Solution

We have the equilibrium solution in Theorem 2 when the speculator is inactive. The equilibrium bidding strategy of a regular buyer is simply given by $b_{\rho}(\cdot)$. The other extreme is when the speculator is fully active: the support $S_{H}$ is equal to the support of the equilibrium bid distribution of the regular buyers. We first provide a simple characterization and solution of equilibrium in this case, then a general equilibrium formula is given for the case with a finite number of maximal active intervals. We provide the equilibrium solution in this section under the assumption that the equilibrium strategy is increasing and symmetric. Section 6 proves that the equilibrium must have the symmetry and monotonicity property.

Once we have determined the active intervals, in equilibrium, we must have $B(\cdot) = B(\cdot)$ over an active interval. For an inactive interval, the equilibrium $\tilde{B}(\cdot)$ is the endogenous cost function defined at the end point $z$ of each active interval, with the initial condition $B_{z}^{\ast}(z) = B(z)$. This is how the equilibrium solution is obtained.

In equilibrium, typically the speculator is active in a collection of disjoint intervals. We now offer an intuitive explanation of this equilibrium behavior. Assume that initially there is no speculator. According to the standard first-price symmetric auction, bidders are bidding $b_{\rho}(\cdot)$ in equilibrium. The speculator does not enter bidding if there is no bid such that the revenue from resale $B(\cdot)$ exceeds the cost of winning $b_{\rho}(\cdot)$. If there is a bid yielding positive profit, the speculator will enter. Once the speculator becomes active in bidding, the regular buyers will respond to the entry, and bid $B(\cdot)$ which reduces the speculator profit to 0. Thus in active intervals, regular buyers bid the revenue function. An active interval starts at a point where the cost function crosses the revenue function from below. It ends when the function $B^{\ast}(\cdot)$ crosses $B(\cdot)$ from above. At the end of an active interval, a regular buyer can raise the bid to $B^{\ast}(\cdot)$ which makes the speculator inactive as the new cost exceeds the revenue. Given the new endogenous cost function, the speculator may find another higher profitable bid. This will bring in a new active interval. So it may continue. During the inactive intervals, the bidding function is the cost function. An example in Part II shows that the speculator in equilibrium bids (with positive probability) on two disjoint intervals, separated by an inactive interval. This pattern of behavior should be kept in mind as we analyze the equilibrium in the following two sections.

4.1 Equilibrium With a Fully Active Speculator

A simple example of a model with a fully active speculator is the model with one regular buyer and one speculator. When $N = 1$, the speculator is fully active with no reservation price. Let $\tilde{\rho} = B(\beta) = r(\beta)(1 - F(r(\beta)))$. With a reservation price $\rho$, the speculator is fully active in equilibrium, if $\rho < \tilde{\rho}$, and inactive if $\rho > \tilde{\rho}$. The equilibrium bidding strategy of the regular buyer is given by $B(v) = B(v)$ for $v > \eta(\rho), B(v) = \rho$ for $v \in [\rho, \eta(\rho)]$. The bid distribution $H(b)$ is given by $H(b) = F(\eta(b))$ for $b \in (\rho, B(\beta)], H(b) = F(\eta(\rho))$ for all $b \leq \rho$.

When $N > 1$, the following condition will be shown to be both necessary and sufficient for a fully active speculator when $\rho = 0$:

$$B^{\ast}(v) > B(v) \text{ for all } v \in (0, \beta) \text{ except a countable closed set.}$$

(28)

Note that for $N = 1$, we always have $B^{\ast}(v) = B^{\ast}(v) > B(v)$, when $v > 0$, and condition (28) is automatically satisfied.

---

13An interesting comparison can be made between this model one speculator and one regular buyer and the equilibrium in the Wilson’s Drainage Tract Common Value Model in which there is one neighbor firm and one non-neighbor firm. The equilibrium bidding of our model is similar to that of a common value model with the common value defined by the resale revenue. In the Wilson’s Drainage model, the neighbor firm bids instead the amount $\frac{1}{B(\nu)} \int_{0}^{\nu} x dF(x)$. 

---
Theorem 3 Let \( N > 1, \rho = 0 \). Then assumption \((28)\) is necessary and sufficient for the speculator to be fully active in equilibrium. Under assumption \((28)\), the equilibrium is given by

\[
\bar{B}(v) = B(v),
\]

and

\[
\bar{H}(b) = \exp \left( - \int_{b}^{B(\beta)} L(y) \ln F(\eta(y)) \right), \quad b \in [0, B(\beta)].
\]  

(29)

Proof. If the speculator is fully active, Lemma 11 says that condition \((28)\) holds. If the condition holds, we will show that the strategy profile \((\bar{B}(\cdot), \bar{H}(\cdot))\) is an equilibrium and the speculator is fully active with \(S_H = [0, B(\beta)]\). Hence \((28)\) is a necessary and sufficient condition. Note that the speculator gets zero profit in equilibrium with all \(b \in [0, B(\beta)]\), which is optimal. The first-order condition of a regular bidder, by Lemma 10, is

\[
\frac{H'(b)}{H(b)} = L(b) \ln F(\eta(b)),
\]

which is satisfied by the function \(\bar{H}(\cdot)\) given in the lemma. This proves the optimality of the bid \(B(v)\) for all \(v\). ■

For \(N > 1, \rho > 0\), the speculator cannot be fully active on the whole range \([\rho, B(\beta)]\) of the support of the regular buyer’s bid distribution, but will be fully active beyond some bid. Let \(\bar{\rho}\) the smallest \(\rho\) satisfying the following equation

\[
B(\beta) - b_{\rho}(\beta) = 0.
\]

For any \(\rho < \bar{\rho}\), the speculator is active in equilibrium, and there is a unique \(v(\rho)\) which is the starting point of the active value interval. It is the smallest \(v \geq \rho\) satisfying the equation

\[
B(v) - b_{\rho}(v) = 0.
\]

Let \(a(\rho) = B(v(\rho))\) be the beginning of the active bids. When \(N = 1\), we have \(a(\rho) = \rho, v(\rho) = \eta(\rho)\). For \(N > 1, v(\rho) > \rho\), and the speculator is not active in \([\rho, a(\rho)]\), but fully active after \(a(\rho)\).

Theorem 4 Let \(N > 1, \rho > 0\) and assume \((28)\). Then the speculator is active in equilibrium if and only if \(\rho < \bar{\rho}\). The equilibrium bidding strategy of a regular buyer is given by

\[
\bar{B}(v) = \begin{cases} 
 b_{\rho}(v) & \text{if } v \in [\rho, v(\rho)] \\
 B(v) & \text{if } v \geq v(\rho).
\end{cases}
\]

The equilibrium bid distribution of the speculator is given by

\[
\bar{H}(b) = \begin{cases} 
 \exp \left( - \int_{b}^{B(\beta)} L(y) \ln F(\eta(y)) \right) & \text{for } b \geq a(\rho), \\
 H(a(\rho)) & \text{if } b \leq a(\rho).
\end{cases}
\]

Proof. By Lemma 7, we must have \(B(v) > b_{\rho}(v)\) for all \(v > v(\rho)\). By assumption we also have \(B'(v) > B(v)\). The same arguments in the proof of Theorem 3 then shows that the speculator is fully active over the interval \((\rho, B(\beta))\). This implies that the equilibrium bidding strategy for a regular buyer with \(v \geq v(\rho)\) is \(B(v)\). For \(v < v(\rho)\), the speculator is inactive over \((0, a(\rho))\), hence the equilibrium bidding strategy is given by \(b_{\rho}(v)\) just as if there is no speculator. The solution of \(\bar{H}(b)\) is the same as in Theorem 3 for \(b \geq a(\rho)\), and is constant below \(a(\rho)\). The proof is complete. ■
4.2 Finite Number of Active Intervals

We now make the following assumption which will give us a finite number of maximal active intervals.

(FC) There is a finite number of solutions in the equation $B^c(.) = B(.)$.

The generic property of this assumption will be investigated in Part II of the paper. When $F(.)$ is analytic, assumption (FC) always holds. Hence (FC) is essentially satisfied by any computable function $F(.)$. From (FC), we can show that the number of maximal active intervals is finite. There is also an algorithm which allows us to construct all the maximal active intervals.

An Informal description of the algorithm is as follows. The start of the first active interval $z_1$ is the first place at which $B(.)$ crosses $b_p(.)$ from below, and the end point $v_1$ is the first place after $z_1$ at which $B^c(.)$ crosses $B(.)$ from above. At $v_1$, we now replace $b_p(.)$ by the endogenous cost function $B^*_k(.)$ defined at $v_1$, and the process is repeated cursively.

Formally, we can find the active intervals as follows. Assume $B(.) > b_p(.)$ at some point so that the speculator is active in equilibrium. Define $z_1$ as

$$z_1 = \max \{ z : B(x) \leq b_p(x) \text{ for all } x \leq z \}.$$  

By assumption, $z_1 < \beta$. We can show that $B^c(.) > B(.)$ in a neighborhood in a neighborhood $(z_1, z + \varepsilon)$. Let

$$v_1 = \max \{ v : B^c(x) > B(x) \text{ for all } x \in (z_1, v) \}.$$  

If $v_1 = \beta$, there is only one active interval $[z_1, v_1]$.

Recursively, given $(z_k, v_k), v_k < \beta$, let $B^*_k(.)$ be the endogenous cost function defined by the initial condition $B^*_k(v_k) = B(v_k)$. Either there is no $x$ such that $B(x) > B^*_k(x)$, and we have no more active intervals. Or such an $x$ exists, and we let

$$z_{k+1} = \max \{ z : B(x) \leq B^*_k(x) \text{ for all } x \leq z \}. \quad (31)$$  

By assumption, $z_k < \beta$, we can again define

$$v_{k+1} = \max \{ v : B^c(x) > B(x) \text{ for all } x \in (z_{k+1}, v) \}. \quad (32)$$  

This process will end because there are only a finite number of intersections $B^c(x) = B(x)$. The intervals $[z_k, v_k], k = 1, 2, ..., m$ will be the list of all maximal active intervals. Once this is shown, we can describe the equilibrium solution as follows.

Define

$$B^*(.) = b_p(.) \text{ in } [\rho, v_1),$$
$$B^*_k(.) \text{ in } [v_k, v_{k+1}), k < m,$$
$$B_m(.) \text{ in } [v_m, \beta], \text{ if needed.}$$

This uniquely defines $B^*(.)$, which is continuous everywhere, except possibly at $v_k$, where we may have an up-jump, and we only have right-continuity at such points. However, the function $\max\{B(.) , B^*(.)\}$ is a continuous function everywhere. Define

$$\bar{L}(b) = L(b) \text{ when } b \in [z_k, v_k] \text{ for some } k,$$
$$= 0 \text{ elsewhere.}$$

From Lemma 12, we know that in an inactive interval, we have $B(.) \leq B^*_2(.)$, but in an active interval, Lemma 14 says that the ranking is reversed. In either the active or inactive intervals, $\bar{B}(.)$ is the larger of the two. We have the following equilibrium solution which contains Theorem 2 as a special case. We use $b_s, b$ for the maximum bids of the speculator and the regular buyers respectively. Note that when $b_s = 0$ if and only if the speculator is inactive in equilibrium.
**Theorem 5**  Let $N > 1$, and assume (FC). Either the speculator is inactive in equilibrium, or there exists a finite sequence of ordered maximal active disjoint intervals $[z_k, v_k], k = 1, 2, \ldots, m$, defined recursively as in (31), (32). The equilibrium is given by

$$
\tilde{B}(.) = \max\{B(.), B^*(.)\} \text{ in } [\rho, \beta],
$$

$$
\tilde{H}(b) = \exp \left( - \int_{b}^{b_s} L(y) d \ln F(\eta(b)) \right) \text{ in } [\rho, \tilde{b}_s],
$$

where $\tilde{b}_s = B(v_m)$ is the maximum active strategy. Moreover the equilibrium has the following first-order stochastic dominance property\(^{14}\)

$$
\tilde{H}(b) > F(\phi(b)) \text{ for all } b < \tilde{b}.
$$

**Proof.** Let $v_0 = \rho, B_0^*(.) = b_\rho(.)$. First we show that $z_k, v_k$ are well-defined. After $z_k$ is defined, Lemma 5 says that we must have $B^c(z_k) \geq B(z_k)$. If $B^c(.) < B(.)$ in a neighborhood $(z_k, z_k + \varepsilon)$, then it also says that $B(.)$ cannot cross $B^c(.)$ at $z_k$, hence we must have $B^c(.) > B(.)$ in a neighborhood $(z_k, z_k + \varepsilon)$. This allows us to define $v_k$. Now we want to show that $[z_k, v_k]$ is an active interval, while $[v_{k-1}, z_k]$ is an inactive interval. By the choice of $z_k$, we know that $B(.) \leq B^c(.)$ in $[v_{k-1}, z_k]$. Hence we know that $[v_{k-1}, z_k]$ is an inactive interval. In the interval $(z_k, v_k)$, by our choice of $v_k$, we have $B^c(.) > B(.)$, hence Lemma 7 tells us we have $B(.) > B^c(.)$ in the interval. The interval $[z_k, v_k]$ is active when we show that the strategy profile specified in the theorem is an equilibrium strategy. Since the speculator is not active in $[v_{k-1}, z_k]$, $B^c_{k-1}(.)$ is an equilibrium strategy for a regular buyer. The speculator makes zero profit bidding in the interval $[a_k, b_k]$, and non-positive profit elsewhere, hence this is optimal for her. For the regular buyers, the first-order condition of equilibrium is satisfied because of the way $\tilde{H}(.)$ is defined. The supermodularity property insures that $B(.)$ is an optimal strategy for a regular buyer in $[z_k, v_k]$. This proves that the given strategy profile is an equilibrium strategy. To show (35), consider a maximal active interval $[a_k, b_k] = [B(z_k), B(v_k)]$, we have

$$
\ln \tilde{H}(b_k) - \ln \tilde{H}(a_k) = \int_{a_k}^{b_k} L(b) d \ln F(\eta(b)) = \int_{z_k}^{v_k} L(v) d \ln F(v)
$$

$$
< \int_{z_k}^{v_k} d \ln F(v) = \ln F(\eta(b_k)) - \ln F(\eta(a_k)).
$$

Over a maximal inactive interval $[v_k, z_{k+1}]$, we also have

$$
\ln \tilde{H}(a_{k+1}) - \ln \tilde{H}(b_k) < \ln F(\eta(a_{k+1})) - \ln F(\eta(b_k)),
$$

as the left-hand side is always $0$. Hence for all $b < b_s$, we must have

$$
\ln \tilde{H}(b_s) - \ln \tilde{H}(b) < \ln F(\eta(b_s)) - \ln F(\eta(b)) < - \ln F(\eta(b)),
$$

and we have

$$
\ln \tilde{H}(b) > \ln F(\eta(b)),
$$

or

$$
\tilde{H}(b) > F(\eta(b)).
$$

Rewrite in the variable $v$,

$$
\tilde{H}(B(v)) > F(v) \text{ for all } v < \eta(b_s).
$$

Note that we have $B(v) = \tilde{B}(v)$ over active intervals, and $\tilde{H}(.)$ is constant over inactive intervals. We have $\tilde{H}(B(v)) = \tilde{H}(\tilde{B}(v))$, and this translates into

$$
\tilde{H}(\tilde{B}(v)) > F(v) \text{ for all } v < \eta(b_s),
$$

which is equivalent to

$$
\tilde{H}(b) > F(\phi(b)) \text{ for } b < b_s,
$$

which is also valid for $b < \tilde{b}$, as $\tilde{H}(b) = 1$ when $b \geq \tilde{b}_s$. The proof is complete.\(^{14}\)

\(^{14}\)Virag (2011) has shown a similar result for the case of two types of bidders: strong and weak.
Remark 2 When there are infinitely many active intervals, the recursive construction does not work well. A modified more general method of determining all the maximal active intervals is presented in Part II of the paper. That general method then gives us a constructive proof of the existence of equilibrium with all the results presented in Theorem 5, except that we may have a countably infinite number of maximal active intervals.

We now give an easily computable revenue formula. Let \((z_k, v_k), k = 1, 2, ..., m\) be the active intervals, then \([v_{k-1}, z_k], k = 1, 2, ..., m\), and \([v_m, \beta]\) are the inactive intervals. Let \(\beta = z_{m+1}\).

Let \(R(N, \rho)\) denote the revenue when there are \(N\) regular buyers, and the reservation price is \(\rho\).

Theorem 6 When \(N > 1\), the revenue is given by

\[
R(N, \rho) = \sum_{k=1}^{m} \int_{z_k}^{v_k} B(v)\bar{H}(v)(1 + \frac{1}{N}\mathcal{L}(v))dF^N(v) + \sum_{k=1}^{m+1} \bar{H}(v_{k-1})\int_{z_{k-1}}^{z_k} B^*(v)dF^N(v)
\]

where

\[
\bar{H}(v) = \exp\left(-\int_{v}^{\beta} \mathcal{L}(x)d\ln F(x)\right).
\]

When \(N = 1\), the revenue for \(\rho \leq \bar{\rho}\) is given by

\[
\rho(F(\eta(\rho)) - F(\rho)) + \int_{\eta(\rho)}^{1} B(b)dF^2(v).
\]

4.3 An illustrative Example of Active Speculation

Distribution functions that yield an active speculator in equilibrium are the ones that allow the optimal revenue \(B(v)\) of selling to \(N\) regular buyers with value distribution \(F(.)|v\) to be higher at some point \(v\) than the expected revenue \(b_0(v)\) of the joint distribution of \(N - 1\) regular buyers with the same value distribution.

A simple example of such a case can be given in discrete distributions. Suppose that the use value is either \(v = 0\) with probability 0.5, and 1 with probability 0.5. The expected value of \(F(.)\) is 0.5. However, the speculator can set the optimal reservation price \(r = 1\), with the optimal revenue of selling to two regular buyers given by \(1 - 0.5^2 = 0.75 > 0.5\). Therefore when there are two regular buyers with this discrete distribution, the speculator must be active in equilibrium.

This idea can be applied to a continuous distribution through approximation as follows. Let

\[F(v) = 0.5 + 0.5v^n\ \text{over}[0, 1].\]

With two regular buyers, we have

\[b_0(1) = 1 - \int_{0}^{1} 0.5(1 + x^n)dx = 0.5 - \frac{0.5}{n + 1}.\]

We have \(r(1) = (n + 1)^{-\frac{1}{n}} \to 1\) as \(n \to \infty\), and

\[B(1) = 0.5 \int_{(n+1)^{-\frac{1}{n}}}^{1} ((n+1)x^n - 1)(1 + x^n)dx.
\]

Since

\[\lim_{n \to \infty} B(1) = \lim_{n \to \infty} 0.5(1 - (n + 1)^{-\frac{n+1}{n}} + \frac{n + 1}{2n + 1} (1 - r^{2n})) = 0.75.
\]
Hence for large \( n \), the speculator must be active in equilibrium. In the limit there is an atom at the end point \( v = 1 \), hence the speculator becomes active for large \( n \). The idea of an atom at the end was used in the example of Garrett and Tröger (2006).

Here we will give another more interesting example of active speculation based on the existence of an atom at 0. The example will also be used to illustrate many results in this section. Suppose \( F_0(v) = v^2 \). There are two regular buyers, and one speculator. It is shown earlier that the speculator is inactive in equilibrium in this case. Consider the change from \( F_0(v) = v^2 \) to the value distribution \( F_1(v) = 0.5(v^2 + 1) \) over \([0, 1]\), with an atom at 0 of size 0.5. This change is a down the rank in the first-order stochastic dominance. The atom means that with probability 0.5, a regular buyer has 0 value. Let \( \rho = 0 \). The cost function \( b_0(.) \) is now lower than before as

\[
b_0(.) = \frac{1}{F_1(v)} \int_0^v x dF_1(x) < \frac{1}{F_0(v)} \int_0^v x dF_0(x).
\]

However the revenue function \( B(.) \) is higher as the upper range of the value has a higher density than before. This will be shown below. The change will allow the revenue function to overtake the cost function, leading to active speculation.

**Example 2** There are two regular buyers, and one speculator. Let the regular buyer’s use value distribution be

\[
F(v) = 0.5(1 + v^2) \text{ over } [0, 1].
\]

We have

\[
b_0(v) = \frac{2v^3}{3(1 + v^2)}.
\]

The revenue without the speculator or without resale is

\[
\int_0^1 b_0(x) dF^2(x) = 2 \cdot 0.5^2 \int_0^1 \frac{2x^3}{3(1 + x^2)}(1 + x^2)2x dx
\]

\[
= 2 \cdot 0.5^2 \int_0^1 \frac{4x^4}{3} dx = 0.1333333.
\]

The virtual value is

\[
J(x, \beta) = x - \frac{1 - 0.8(1 + x^2)}{1.6x}.
\]

The optimal reservation price for revenue is given by \( r(v) = \frac{v}{\sqrt{3}} \). We have

\[
B(v) = \frac{4v^3}{(1 + v^2)^2} \left[ \frac{1}{3} + \frac{1}{15} \left( 2 + \frac{1}{3\sqrt{3}} \right)v^2 \right],
\]

\[
B'(v) = \frac{v}{1 + v^2} \left[ \frac{1}{3\sqrt{3}} + \frac{1}{3} \left( 2 + \frac{1}{3\sqrt{3}} \right)v^2 \right],
\]

\[
B''(v) = \frac{v}{1 + v^2} \left[ \frac{2}{\sqrt{3}} - 1 + \frac{1}{3} \left( 1 + \frac{2}{3\sqrt{3}} \right)v^2 \right].
\]

We have

\[
B(v) - b_0(v) = \frac{4v^3}{1 + v^2} \left[ \frac{1}{135} \sqrt{3} - \frac{1}{30} v^2 + \frac{1}{9} \sqrt{3} - \frac{1}{6} \right] > 0
\]

when \( v \leq 1 \). Hence \( B(.) > b_0(.) \) in \([0, 1]\). This however does not mean that the speculator is fully active. To determine active or inactive intervals, we first solve

\[
B''(v) - B(v) = \frac{v}{(1 + v^2)^2} \left[ \left( \frac{8}{9\sqrt{3}} - \frac{2}{3} \right)v^2 + \left( \frac{2}{15\sqrt{3}} - \frac{1}{5} \right) v^4 + \frac{2}{\sqrt{3}} - 1 \right] = 0.
\]
We have a unique positive solution $v_1 = 0.8120631$, and $B(v_1) = 0.2246755$. Since $B^c(.) > B(.)$ and $B(.) > b_0(.)$ in $(0, v_1)$, we know $(0, v_1)$ is an active value interval, and $(v_1, 1)$ is an inactive interval as $B(.) < b_0(.)$ in this range. The speculator is active in the bid interval $(0, B(v_1))$, and $B(v_1) = b_s$ is the maximum equilibrium bid of the speculator. The endogenous cost function at $v_1$ is

$$B_1^*(v) = \frac{0.01582887 + \frac{\bar{e}}{3} v^3}{1 + v^2}.$$ 

In equilibrium, a regular buyer bids $B(.)$ in $[0, v_1]$, and bids $B_1^*(.)$ in $[v_1, 1]$. The bid distribution of the speculator, expressed as a function of $v$, is

$$\mathcal{H}(v) = \exp \left( -\int_0^v \frac{B^c(x) - B(x)}{B^c(x) - B(x)} f(x) \, dx \right)$$

$$= \exp \left( -\int_0^{0.8120631} \left( \frac{\frac{1}{3} - \frac{1}{3}}{3} \right) v^2 + \left( \frac{\frac{2}{2} \sqrt{3} - \frac{1}{8}}{1 + 2v^2} \right) x^4 + \frac{\frac{1}{15} + \frac{2}{15}}{1 + x^2} \right) \text{.}$$

The revenue can be computed as follows

$$R = 0.5^2 \int_0^{0.8120631} B(x) F(x) f(x) \mathcal{H}(x) (2 + \frac{B_1^*(x) - B(x)}{B_1^*(x) - B(x)}) \, dx + 0.5^2 \int_{0.8120631}^1 2B_1^*(x)(1 + x^2) \, dx \text{.}$$

$$= 0.1405397 \text{.}$$

The revenue is higher than that without resale.
In the following graph, $B(\cdot), B^v(\cdot)$ in $(v_1, 1)$ are very close, but the difference can be seen when magnified.

$B(\cdot)$ blue line, in the middle, $B^v(\cdot)$ black dash line on top, $b_0$ red dot, at bottom, $B^v_{v_0}(\cdot)$ in $(v_0, 1)$ red line slightly above $B(\cdot)$, but almost equal
5 Optimal Revenue with Speculative Resale

In this section, we will extend the optimal auction analysis of Myerson (1981) to our model of speculative resale. For this purpose, we shall first extend the revenue formula of Myerson (1981) to our model. The optimal auction result will follow easily from the formula as in the case of Myerson (1981). Bulow and Klemperer (1996) argue that focusing on more participation is more effective in raising revenue than the choice of an optimal reservation price. We will show that their argument also applies in auctions with speculative resale. For this purpose, we need to show first that adding one more regular buyer will indeed raise the original seller’s revenue, and we show that the Bulow-Klemperer argument is even more persuasive as the revenue increase from one more regular buyer can be even greater, compared to the revenue increase from setting the optimal reservation price.

5.1 Myerson Revenue Formula

In this model of speculative resale, we will establish the revenue formula similar to that of Myerson (1981) with some natural adjustments needed. Although regular buyers make payments to both the auctioneer and the speculator, all revenue ends up in the hands of the auctioneer as the speculator makes zero profit in equilibrium. The winning probability is the sum of the winning probability in the first-stage auction and the winning probability in the resale auction. Let \( r \) be the optimal reservation price set by the speculator facing a buyer with value distribution \( F(.) \) according to the Myerson (1981) theory. When a regular buyer has value \( v \) higher than \( r \), then \( v \) is always higher than the reservation price set by the winning speculator during resale. There will be resale to the buyer if he has the highest value among the losers. This means the regular buyer will get it back even if he loses in the first-stage auction. Hence the regular buyer has the winning probability \( F^{N-1}(v) \) the same as that of a regular buyer in the auction with resale, if the equilibrium is symmetric (all regular bidders bid the same way). If the regular buyer has value below \( r \), then the speculator may set a reservation price above his value in the resale market, and as a result fail to sell to the buyer even if he has the highest value among all the losing bidders. In this case the winning probability will be of the form \( t(v)F(v)^{N-1} \) for a regular bidder with value \( v \). The term \( t(v) \) depends only on \( F(.) \) and the bid distribution \( H(.) \) of the speculator. We have \( t(v) = 1 \) when \( v \geq r \), and \( t(v) < 1 \) when \( v < r \). Computing the revenue using this information about the winning probabilities gives us a formula similar to that of Myerson (1981). The virtual value, \( J(v) \) is now discounted (keeping the same sign) by \( t(v) \), which is equal to one when \( v \geq r \). We have the following formula for the revenue of an auction with speculative resale

\[
\int_{\rho}^{\beta} t(v)J(v)dF^{N}(v).
\]

Since the virtual value is positive above \( r \) and negative below \( r \), we immediately have three implications from this revenue formula. Firstly speculative resale would not change the optimal reservation price \( r \). Secondly, resale would not affect the optimal revenue as \( t(v) = 1 \) for \( v \geq r \). Thirdly, since \( t(v) < 1, J(v) < 0 \) when \( v < r \), we have \( t(v)J(v) = J(v) \) for all \( v \geq r \), and \( t(v)J(v) > J(v) \) for \( v < r \). Thus we conclude that for all reservation price set by the auctioneer, the revenue is strictly higher in auctions with resale than without resale, as long as the speculator is active in equilibrium\(^{15}\).

The winning probability of a regular buyer is now the sum of the probability of winning from the original seller and that of winning from the speculator. Since the speculator makes zero profit, the total contribution to revenue by a regular buyer to the speculator and the original seller can be added up to get the total revenue of the original seller. The probability of winning is now a fraction of the winning probability\(^{15}\).

\(^{15}\) GT(supp) have shown with a different proof that with zero reservation price, the revenue is higher when speculators are active in equilibrium. This is a special case of our result which holds for all reservation price. Our proof is much simpler and more intuitive. Zhang and Wang (2013) also looked at the optimal auction with speculative resale with one speculator and one regular buyer, and showed that Myerson’s optimal revenue is achieved. They also considered more general resale markets than ours.
in the auction without resale model. To express this discount, let $\hat{H}(\cdot), \hat{B}(\cdot)$ be the equilibrium strategy profile. Since we have $B(\cdot) = \hat{B}(\cdot)$ in active intervals, and $H(\cdot)$ is constant in inactive intervals, we have $\mathcal{H}(v) = H(B(v)) = \hat{H}(B(v))$. When $N = 1$, let $\hat{\rho} = B(\beta)$. When $N > 1$, $\hat{\rho}$ is defined to be the smallest $\rho \geq 0$ satisfying the following equation

$$B(v) \leq b_{\rho}(v) \text{ for all } v \geq \rho.$$  \hfill (36)

Let $\rho(\rho)$ be the starting point of the first maximal active interval and $a(\rho) = B(v(\rho))$. We define

$$t(x) = \begin{cases} \hat{H}(\rho(\rho)) = \hat{H}(\rho(x)) & \text{if } \rho \leq x \leq r(\rho(\rho)) \\ \hat{H}(r^{-1}(x)), & \text{if } r(\rho(\rho)) \leq x < r(\beta), \\ 1, & \text{if } x \geq r(\beta). \end{cases}$$

When $N = 1$, we have the simpler expression

$$t(x) = \begin{cases} F(\rho(\rho)) & \text{if } \rho \leq x \leq r(\rho(\rho)) \\ F(r^{-1}(x)), & \text{if } r(\rho(\rho)) \leq x < r(\beta), \\ 1, & \text{if } x \geq r(\beta). \end{cases}$$

We will show that the winning probability of a regular buyer with value $v$ is given by $t(v)F^{N-1}(v)$.

**Theorem 7** Let $\rho \geq 0$ be the reservation price. We have the following equilibrium revenue formula

$$\int_{\rho}^{\beta} t(x)J(x, \beta)dF^{N}(x).$$  \hfill (37)

**Proof.** First let $N > 1$. We want to show that the equilibrium winning probability of a regular buyer with value $x$ is $t(x)F^{N-1}(x)$. In the equilibrium, we first show that a regular buyer with value $x \geq r(\beta)$ has winning probability $F^{N-1}(x)$. When the speculator wins, in the resale auction, the optimal reservation price is set below $r(\beta)$, and the regular buyer will win as long as he has the highest value among the regular buyers. Therefore his winning probability is $F^{N-1}(x)$. For $x \in [r(\rho(\rho)), r(\beta))$, the winning probability is $\mathcal{H}(r^{-1}(x))F^{N-1}(x)$ because he can win in the resale market only if the speculator wins by bidding $b \leq \hat{B}(r^{-1}(x))$. The probability of the speculator bidding below $\hat{B}(r^{-1}(x))$ is $\mathcal{H}(\hat{B}(r^{-1}(x))) = \mathcal{H}(r^{-1}(x)) = t(x)$. Hence the winning probability is given by $t(x)F^{N-1}(x)$. For $x \in [\rho, r(\rho(\rho))]$, the winning probability is the same as the winning probability in the first-stage auction, because when a regular loses the first-stage auction, a winning speculator will set a reservation price for him to accept. Hence the winning probability is $\mathcal{H}(a(\rho))F^{N-1}(x) = t(x)F^{N-1}(x)$. By the Myerson (1981) argument, we must have the following expected contribution of revenue from a regular buyer with value $v$

$$F^{N-1}(v)t(v)v - \int_{\rho}^{v} F^{N-1}(x)t(x)dx.$$  

Hence the expected revenue of the auction with resale is given by

$$N \int_{\rho}^{\beta} \left( F^{N-1}(v)t(v)v - \int_{\rho}^{v} F^{N-1}(x)t(x)dx \right) dF(v)$$

$$= \int_{\rho}^{\beta} t(v)v dF^{N}(v) - N \int_{\rho}^{\beta} \left( \int_{\rho}^{v} F^{N-1}(x)t(x)dx \right) dF(v)$$

$$= \int_{\rho}^{\beta} t(v)v dF^{N}(v) - N \int_{\rho}^{\beta} t(x)F^{N-1}(x) \left( \int_{x}^{\beta} dF(v) \right) dx$$

$$= \int_{\rho}^{\beta} t(v)v dF^{N}(v) - N \int_{\rho}^{\beta} t(x)(1 - F(x))F^{N-1}(x)dx.$$
\[
\int_{\rho}^{\beta} t(x)xdF^N(x) - \int_{\rho}^{\beta} t(x)(1 - F(x))dF^N(x) = \int_{\rho}^{\beta} t(x)J(x, \beta)dF^N(x).
\]

This proves the theorem. \[\blacksquare\]

From the revenue formula, we can derive many important properties. Since \(J(x, \beta)\) has the same sign as \(t(x)J(x, \beta)\), the same argument in Myerson (1981) implies that the revenue is maximized in the same way. For the case of \(N = 1\), both terms in (37) are increasing in \(\rho \leq r(\rho)\), hence the revenue is optimal at \(\rho = r(\beta)\) as well.

**Corollary 8** The optimal revenue is obtained by setting the reservation price \(\rho = r(\beta)\) and the revenue is the same as the optimal revenue without resale.

In fact, we can also show directly that at the reservation price \(\rho = r(\beta)\), the speculator is inactive in equilibrium, so that the revenue is the same as the the one without resale when \(\rho = r(\beta)\). This is another confirmation that the optimal revenue without resale is the same as that with resale.

**Corollary 9** The speculator is inactive at the reservation price \(\rho = r(\beta)\).

**Proof.** First we look at the case \(N = 1\). In this case, the regular buyer bids \(r(\beta)\), and the speculator does not participate because the cost of winning is at least \(r(\beta)\), but the revenue is \(B(\beta) < r(\beta)\). When \(N > 2\), \(\rho = r(\beta) \geq r(v)\), by Lemma 1, we have for \(v > 0\),

\[
B(v) < B^*(v) = v - \int_{r(v)}^{v} F^{N-1}(x|v)dx \leq v - \int_{\rho}^{v} F^{N-1}(x|v)dx = b_\rho(v).
\]

Hence by Theorem 2, the speculator is inactive in equilibrium. \[\blacksquare\]

Note that \(J(x, \beta) \geq 0\) for \(x \geq r(\beta)\), and in this case \(t(x) = 1\). When \(x < r(\beta)\), and the speculator is active in equilibrium, then \(B(b_s) > \rho\), and \(t(x) < 1\) for all \(x < B(b_s)\). Since \(J(x, \beta) < 0\) for such \(x\), the multiplicative factor increases the value of the revenue integral formula. Hence we know that the revenue is strictly higher when the speculator is active in equilibrium.

From Corollary 9, we know that \(\bar{\rho} \leq r(\beta)\). In fact, we must have \(\bar{\rho} < r(\beta)\) from (36). The number \(\bar{\rho}\) is essentially software-computable. The following is another important corollary of Theorem 7. It tells us precisely when the speculator is active in equilibrium, and that the revenue is higher with resale than without resale when they are active.

**Corollary 10** When the speculator is active in equilibrium, the revenue with resale is strictly higher than the auction without resale. When she is inactive in equilibrium, there is no difference in the revenue with or without resale.

Using the example in the last section, we can compute the revenue for \(\rho = 0\) as follows:

\[
t(v) = \mathcal{H}(\sqrt{3}v) = \exp\left( - \int_{\sqrt{3}v}^{0.5} \frac{B^c(x) - B(x) - 2x}{B^c(x) - B(x) + x^2}dx \right)
\]

The revenue using the Myerson formula is

\[
2 \ast \frac{1}{4} \left( \int_{0}^{\sqrt{3}v} (3x^2 - 1) \mathcal{H}(\sqrt{3}x)(1 + x^2)dx + \int_{\sqrt{3}v}^{0.8129931} (3x^2 - 1)(1 + x^2)dx \right) = 0.1405397.
\]

which confirms the number we computed earlier using a different method.
5.2 Revenue Monotonicity

Now we show that when we add one more regular buyer, the revenue will increase. First we need to prove a useful lemma. The proof is in section 7.

Lemma 17 If \( b_N(x) \leq b_{N+1}(x) \) for all \( x \leq v, v > \rho \), then \( b_N(v) < b_{N+1}(v) \).

We now show that the equilibrium bidding strategy is an increasing function of \( N \).

Theorem 11 Let \( \rho = 0 \), and \( b_N(.) \) be the equilibrium bidding strategy of a regular buyer when there are \( N \) of them, then we have

\[
b_{N+1}(v) > b_N(v) \quad \text{for all } v > 0.
\]

Proof. Let \( v_0 = \max \{ v : b_{N+1}(x) \geq b_N(x) \text{ for all } x \leq v \} \). First assume that \( \rho > 0 \). In some neighborhood \((\rho, \rho + \varepsilon)\), the speculator is inactive as \( B(\rho) < \rho \). Clearly we have \( b_{N+1}(.) > b_N(.) \) in some \((\rho, \rho + \varepsilon)\). Hence \( v_0 > \rho \). If we have \( v_0 < \beta \), by Lemma 17, we have \( b_{N+1}(v_0) > b_N(v_0) \). We can then find \( v' > v_0 \) such that \( b_{N+1}(x) \geq b_N(x) \) for all \( x \leq v' \), contradicting the definition of \( v_0 \). Therefore we have \( v_0 = \beta \). Now we can apply Lemma 17 to any \( v > \rho \), and we have \( b_{N+1}(v) > b_N(v) \) for all \( v > \rho \). We have proved the theorem for the case \( \rho > 0 \). We can take the limit \( \rho \to 0 \), and we get \( b_N(v) \leq b_{N+1}(v) \) for all \( v \leq \beta \). Now use the same arguments above, and we have \( b_N(v) > b_{N+1}(v) \) for all \( v > 0 \), and our proof is now complete. 

The increasing property of the equilibrium bidding strategy with respect to \( N \) is not quite sufficient on its own to insure the revenue monotonicity. The effect of \( N \) on \( H(.) \) is still a factor to take into account. However, we can still prove the revenue monotonicity of our model.

Theorem 12 Let \( R(N+1, \rho), R(N, \rho) \) be the revenues with \( N + 1, N \) regular buyers respectively. Then we have

\[
R(N, \rho) < R(N+1, \rho), \rho < \beta.
\]

We also have monotonicity in \( \rho \in [0, r(\beta)] \),

\[
R(N, \rho) < R(N, \rho') \quad \text{when } \rho < \rho' \leq r(\beta).
\]

Proof. We know from Theorem 11 that, letting \( \phi_{N+1}(.) = b_{N+1}^{-1}(.) \), \( \phi_N(.) = b_N^{-1}(.) \), we have

\[
\phi_{N+1}(b) < \phi_N(b), \quad b < b
\]

hence, from Theorem 5, we have

\[
F^{N+1}(\phi_{N+1}(b)) < F^N(\phi_N(b)) \tilde{H}(\phi_N(b)).
\]  

(38)

The inequality (38) is precisely the first-order stochastic dominance of the distribution of the maximum bid of the \( N + 1 \) regular buyers in the \( N + 1 \)-equilibrium over that of the maximum bid of the \( N \)-equilibrium of all buyers. This is sufficient to insure \( R(N+1, \rho) > R(N, \rho) \).

5.3 Bulow-Klemperer Result: competition vs. optimal reservation price

Now we show the Bulow-Klemperer result in the auctions with speculative resale. To our knowledge, this is the first extension of their result beyond the benchmark symmetric private value model without resale. The outcome of our model is in fact equivalent to the outcome of a common value auction with the common value defined by the resale value.

The following is the Bulow-Klemperer result for the auctions with speculative resale. We make use of their result, rather than providing a more direct proof.
Theorem 13 In our model of auctions with speculative resale, adding just one more bidder (and setting a zero reserve price) is always preferable to setting the optimal reserve price.

Proof. Let $R^*(N, \rho)$ be the revenue of the model with $N$ regular buyers and reservation price $\rho$, without resale. From Corollary 8, $R^*(N, r) = R(N, r)$. By the Bulow Klemperer result, we have $R^*(N + 1, 0) > R^*(N, r)$. By Corollary 10, we have $R(N + 1, 0) > R^*(N + 1, 0)$. Hence we have

$$R(N + 1, 0) > R^*(N + 1, 0) > R^*(N, r) = R(N, r),$$

and the Theorem is proved. ■

In the symmetric auction with no speculator, let $R^*(N + 1, 0) - R^*(N, 0)$ be the revenue increase from one additional buyer, and $R^*(N, r) - R^*(N, 0)$ be the revenue increase from setting the right reservation price. The Bulow-Klemperer result says that

$$D^* = R^*(N + 1, 0) - R^*(N, r) = (R^*(N + 1, 0) - R^*(N, 0)) - (R^*(N, r) - R^*(N, 0)) > 0.$$

When there is a speculator added to the $N$ regular buyers, we use the corresponding notations $R(N + 1, 0) - R(N, 0)$, $R(N, r) - R(N, 0)$ to denote the two revenue increases. Theorem 13 says that

$$D = R(N + 1, 0) - R(N, r) = ((N + 1, 0) - R(N, 0)) - (R(N, r) - R(N, 0)) > 0.$$

The following result says that the difference $D$ is greater than $D^*$. Therefore the Bulow-Klemperer argument is even more persuasive, as the increase in revenue from adding a regular buyer is even greater if the speculator is active in equilibrium

Corollary 14 We have $D \geq D^*$, and $D > D^*$ holds when the speculator remains active after adding one more regular buyer.

Proof. From Corollary 8, we know that $R(N, r) = R^*(N, r)$. Hence we have

$$D - D^* = R(N + 1, 0) - R^*(N + 1, 0) \geq 0,$$

and strict inequality holds when the speculator is active with $N + 1$ regular buyers and $\rho = 0$. ■

With speculative resale, it is still possible to obtain the optimal revenue without resale as in Myerson (1981). The original seller can set the same optimal reservation price $\rho = r(\beta)$ without resale. With this reservation price, the speculator is not active in equilibrium, and hence the equilibrium is the same as if there is no speculator and no resale. The revenue is the same as that of the optimal revenue without resale. Garratt, Tröger and Zheng (2009) have shown that optimal revenue can be achieved through a second price auction with resale. In the first-price auction with resale, the equilibrium is unique, while in second-price auctions, there are many different equilibria. Our result has the additional benefit that the equilibrium is uniquely determined, and the same reservation price applies to all buyers.

Although the active participation of the speculator increases the original seller’s revenue with the same reservation price, the optimal revenue possible is actually to discourage the active participation of the speculator by making the reservation price sufficiently high. To illustrate this somewhat paradoxical point, we will use a simple uniform distribution example with one speculator, one regular buyer, hence $N = 1, F(v) = v$. If the reservation price is 0, the speculator is active at all bids. Without the participation of the speculator, the revenue would be zero. With the speculator participation, the original seller’s revenue is

$$\int_0^4 bd(4b)^2 = 32 \int_0^4 b^2 db = \frac{1}{6}.$$

Hence the speculator participation increases the revenue. To compute the revenue for a general reservation $\rho \in [0, 0.25)$, by Theorem 6, the revenue is

$$12\rho^3 + 32 \int_\rho^4 b^2 db = \frac{4}{3} \rho^3 + \frac{1}{6}.$$
This revenue is increasing in $[0, \frac{1}{2}]$. When $\rho > 0.25$, the speculator is inactive in equilibrium. The regular buyer bids $\rho$ when his value is above $\rho$. The revenue is $\rho(1 - \rho)$ which is increasing in $[0.25, 0.5]$. The maximum is obtained at $\rho = 0.5$, with the optimal revenue $\frac{1}{4}$. This is also the optimal revenue without the speculator.

Alternatively, we can compute the revenue according to (37),

$$t(x) = \begin{cases} 4\rho & \text{for } x \leq 2\rho, \\ 2x & \text{for } 2\rho \leq x \leq \frac{1}{2}, \\ 1 & \text{for } x \geq \frac{1}{2}. \end{cases}$$

We obtain the same revenue

$$4\rho \int_{\rho}^{2\rho} (2x - 1)dx + \int_{2\rho}^{\frac{1}{2}} 2x(2x - 1)dx + \int_{\frac{1}{2}}^{1} (2x - 1)dx = \frac{4}{3}\rho^3 + \frac{1}{6}$$

for $\rho \leq 0.25$.

6 Uniqueness of Equilibrium

We want to show that in equilibrium all regular buyers use the same monotone bidding strategy, so that this assumption we adopt in earlier sections is not a restrictive assumption to our results.

6.1 Supermodularity Property

In this section, we examine the supermodularity property of the auctions with speculative resale model when strategies are not monotonic and different among the buyers. The results will apply to a model with no speculator (set $H(0) = 1$) and with different value distributions $F_i(.)$ for different bidders. As explained by Milgrom (2004, Chapter 4, section 4.1), the proper version of this property insures that the first-order condition of a bid in the maximization of payoffs is sufficient to guarantee the optimal property. It is also important for the increasing property of the equilibrium strategies. We will prove the symmetry property and increasing property for the equilibrium strategy profile. The proofs are offered in the next section. The version for the symmetric case has been discussed in section 2.2.

Given any profile of possibly non-symmetric and non-monotone bidding strategies $\sigma$, we can define the payoff function $u_i(v_i, b, \sigma)$ as in (1). We assume that $b_i(\cdot), H(\cdot)$ are continuously differentiable. This will insure that the payoff function has left and right derivatives with respect to $b$. Even though we assume symmetric regular bidders in the model, Theorem 15 holds more generally with asymmetric bidders. The proof we provide does not require $F(.)$ to be the same for all buyers. Let $\frac{\partial}{\partial b_i}$ denote the partial derivative from the right. The following condition determines whether you get the strong or weak form of supermodularity:

$$\text{There exists some } j \neq i \text{ such that } v_i \leq w_j(b) < \beta. \quad (39)$$

**Theorem 15** Let $N > 1$, and $\sigma$ be a profile of possibly non-symmetric and non-monotone bidding strategies. Assume that regular buyer $i$ with value $v_i$ bids $b \in (\max_{j \neq i} b_j, \max_{j \neq i} b_j), H(b) > 0$. If there exists at least one regular buyer $j \neq i$ such that $v_i \leq w_j(b) < \beta$, then we have

$$\frac{\partial}{\partial b_i} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) > 0. \quad (40)$$

If condition (39) fails, then we have

$$\frac{\partial}{\partial b_i} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) \geq 0. \quad (41)$$

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16 Only piecewise continuous differentiability assumption is needed here. Even if they are continuously differentiable, the revenue function need not be and are only piecewise continuously differentiable. See also footnote 9.
Note that in section 2.2, we have seen why the condition (39) is needed for the strong form of supermodularity. The condition simply says that if there is payoff from selling the object after winning it in the first-stage auction, then we have the strong form of supermodularity. Otherwise, if there is only payoff from buying in the resale market after losing the first-stage auction, and zero payoff after winning the object, then we only get the weak form.

The supermodularity property we have in Theorem 15 is sufficient for our equilibrium analysis. We have explained in section 2.2 that it is strong enough to guarantee that the first-order condition is sufficient for optimality. In the next section, we will show that it is also strong enough to imply that equilibrium strategies must be increasing.

Let $Q_{1i}(v_i, b)$ be the probability of the event $E_{1i}(v_i, b)$ that regular buyer $i$ wins the first-stage auction bidding $b$, but fails to sell in the resale stage. We prove the following lemma.

**Lemma 18** The functions $\pi_{1i}(v_i, b, \sigma), \pi_{wi}(v_i, b, \sigma)$ are differentiable with respect to $v_i$, and we have

$$\frac{\partial}{\partial v_i} \left[ \pi_{1i}(v_i, b, \sigma_{-i}) + \pi_{wi}(v_i, b, \sigma_{-i}) \right] = Q_{1i}(v_i, b).$$

The proof of Theorem 15 is simpler when the strategies are monotone, and $w_j(b) > v_i$ for all $j \neq i$. In this case there is zero payoff for a regular buyer $i$ after losing the first-stage auction. It is then a simple matter to show that $\frac{\partial}{\partial v_i} Q_{1i}(v_i, b) > 0$. This proves (40) for the case when there is zero payoff from losing the first-stage auction.

When there is payoff after losing the first-stage auction and buying from the winner during the resale, additional arguments are needed. Let $Q_{2i}(v_i, b)$ be the probability of the event $E_{2i}(v_i, b)$ that regular buyer $i$ loses the first-stage auction, but then wins it back during resale. We prove the following lemma.

**Lemma 19** The function $\pi_{2i}(v_i, b, \sigma)$ is differentiable with respect to $v_i$, and we have

$$\frac{\partial}{\partial v_i} \pi_{2i}(v_i, b, \sigma) = Q_{2i}(v_i, b).$$

When $b$ is higher, $Q_{2i}(v_i, b)$ becomes smaller, but it is compensated by the increase in $Q_{2i}(v_i, b)$. Note that when $b' > b$, $E_{1i}(v_i, b) \subset E_{1i}(v_i, b')$, but $E_{2i}(v_i, b') \not\subset E_{2i}(v_i, b)$. We have the following lemma.

**Lemma 20** We have

$$\frac{\partial}{\partial b} [Q_{1i}(v_i, b) + Q_{2i}(v_i, b)] \geq 0.$$

Now we are ready to provide the final proof. Assume condition (1) holds. From Lemma 19, It is sufficient to show

$$\frac{\partial}{\partial b} [Q_{1i}(v_i, b) + Q_{2i}(v_i, b)] > 0.$$

From Lemma 20, we know that

$$\frac{\partial}{\partial b} [Q_{1i}(v_i, b) + Q_{2i}(v_i, b)] \geq 0.$$

To show it is strictly positive, note that the increase of $g_j(v_i, b)$ to $g_j(v_i, b')$ leads to an additional increase in probability in $Q_{1i}(v_i, b)$ which has no counterpart in $Q_{2i}(v_i, b)$. With $\frac{\partial}{\partial b} P_j(v_i, b) > 0$ when $b_j < \tilde{b}_j$ for at least one $j$, we conclude that $\frac{\partial}{\partial b} [Q_{1i}(v_i, b) + Q_{2i}(v_i, b)] > 0$. 

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Now consider the other extreme case when \( w_j(b) < v_i \) for all \( j \neq i \). Then \( \pi_{wi}(v_i, b, \sigma) = 0 \). Hence we have

\[
\frac{\partial}{\partial b_i} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \frac{\partial}{\partial b_i} [Q_{1i}(v_i, b) + Q_{2i}(v_i, b)] \geq 0.
\]

We have proved Theorem 15.

Intuitively, we can think of \( E_{1i}(v_i, b) \) as a circular region with a hole (a donut). When \( b \) is raised to \( b' \), and condition (39) holds, the outer boundary expands, while the inner boundary shrinks. The shrinking in the inner hole compensates the loss of probability in \( E_{2i}(v_i, b) \). But the expansion of the outer boundary leads to an additional increase in probability. When (39) fails, the outer boundary does not expand, and we have (39). This is the intuition behind the two forms of supermodularity property. See the details of the proof in section 7.1.

### 6.2 Monotonicity and Symmetry

We will prove both properties together in our treatment. We will establish the result in several steps. First we shall deal with the symmetry property of the symmetric auction allowing resale, but with no speculator. This is not a trivial issue, as we know that in a symmetric second-price auction with resale but with no speculator, there are many non-symmetric equilibria. We need to show that this would not happen in a symmetric first-price auction with resale with no speculator. When we add the speculator, the symmetry property also holds, and the proof is very similar to the case with no speculator.

**Lemma 21** The equilibrium in the auction with resale model with no speculator must have the symmetry and increasing property for all regular bidders.

Now we state the following major result on the increasing symmetry property for the model with a speculator.

**Theorem 16** The equilibrium strategy of the regular buyers in our model must be symmetric and increasing.

The arguments are presented in section 7. Intuitively, the proof goes as follows. We show that if all regular buyers have the same maximum bid, then a regular buyer with value \( \beta \) must bid \( b_i \) and we have increasing property for the equilibrium strategies near \( \beta \). Then we show that the differential equations satisfied by the inverse equilibrium strategies force all of them to have the symmetry property near the maximum bid. The symmetry and increasing property can be further extended downward to \( \rho \). To show that the maximum bid must be the same for all regular buyers, we note that if there are two different maximum equilibrium bids, the buyer with the highest maximum bid faces less or the same degree of competition than the buyer with the lower maximum bid, leading to a contradiction. This essentially means that the former can lower the bid to increase his or her payoff, violating the equilibrium condition.
7 Proofs

7.1 Proof for the supermodularity Property

Proof of Theorem 15:

Note that when \( b_j(.) \) is not monotonic, the probability distribution \( G_j(.b) \) of the updated belief about a losing bidder \( j \) need not be regular even if \( F(.) \) is regular. Moreover, the support of the distribution is not a connected interval. The density of \( G_j(.b) \) now is \( > 0 \), but may have kinks. Myerson (1981) has shown that the optimal auction mechanism result allows non-regular distributions. Furthermore, the analysis can be adapted to deal with non-connected intervals and kinks as well. The virtual value \( J(x,w) \) needs to be modified so that it is weakly monotone and the allocation is based on this monotone virtual value which we should denote by the same notation in this proof. The same proof that Myerson (1981) gives for the case of non-regular distributions applies in our case.

When regular buyer \( i \) with the value \( v_i \) bids \( b \), there are payoffs from (i) winning the first-stage auction; (ii) resale to some regular buyer \( k \) after winning the auction; (iii) losing the auction to some speculator or regular buyer \( j \neq i \), then buys the object from winner. Let \( \pi_{1i}(v_i,b), \pi_{wi}(v_i,b), \pi_{ii}(v_i,b) \) be the payoffs from each part respectively. We have

\[
u_i(v_i,b) = \pi_{1i}(v_i,b) + \pi_{wi}(v_i,b) + \pi_{ii}(v_i,b).
\]

Let \( K_1 = \{j \neq i : v_i \leq w_j(b) < \beta\} \). When condition (39) holds, \( K_1 \) is not empty.

For \( j \in K_1 \), let the optimal reservation price in the resale market be denoted by \( \gamma_j(v_i,b) = \inf \{v_j : b_j(v_j) \leq b, J(v_j, w_j(b)) \geq v_i\} \). When \( b_j(.) \) is monotonic, \( \gamma_j(v_i,b) \) is simply the optimal reservation price for buyer \( j \) set by buyer \( i \) during the resale. For convenience, we also define \( \gamma_j(v_i,b) = v_i \) when \( j \notin K_1 \). We have

\[
\frac{\partial}{\partial b_j} \gamma_j(v_i,b) > 0 \quad \text{when} \quad j \in K_1.
\]

Let \( v_{-i} \) be the vector of realized values \( v_j, j \neq i \). Let \( b_s \) denote the realized bid of the speculator. Let \( E_{1i}(v_i,b) \) be the event that regular buyer \( i \) wins the first-stage auction, but fails to sell to the losing bidders. The event \( E_{1i}(v_i,b) \) is characterized (allowing exception for a set of measure 0) by the property that

\[
b_s < b, b_j(v_j) < b \quad \text{and} \quad v_j < \gamma_j(v_i,b), \quad \text{for all} \quad j \neq i.
\]

Hence we can write

\[
E_{1i}(v_i,b) = \{(b_s, v_{-i}) : \max(b_s, v_{-i}) < b, v_j < \gamma_j(v_i,b) \quad \text{for all} \quad j \neq i\}.
\]

Let \( Q_{1i}(v_i,b) \) be the probability of the event \( E_{1i}(v_i,b) \). Let \( P_j(v_i,b) \) be the probability of \( \{v_j : v_j < \gamma_j(v_i,b), b_j(v_j) < b\} \) for \( j \in K_1 \). Note that although \( \gamma_j(v_i,b) \) may be discontinuous, \( P_j(v_i,b) \) is a continuous function. In the resale, buyer \( i \) only sells to buyer \( j \in K_1 \) with \( v_j \geq \gamma_j(v_i,b) \).

We have

\[
Q_{1i}(v_i,b) = \tilde{H}(b) \prod_{j \notin \{i,K_1\}} Q_j(b) \prod_{j \in K_1} P_j(v_i,b).
\]

Proof of Lemma 18: The first part of the payoff is

\[
\pi_{1i}(v_i,b) = (v_i - b) \tilde{H}(b) \prod_{j \neq i} Q_j(b),
\]

and we have

\[
\frac{\partial}{\partial v_i} \pi_{1i}(v_i,b) = \tilde{H}(b) \prod_{j \neq i} Q_j(b).
\]

Let \( \pi_{wij} \) be the payoff from selling to buyers \( j \in K_1 \). When buyer \( j \) with value \( x \geq \gamma_j(v_i,b) \) wins the resale auction, the expected price it pays conditional on winning the resale auction depends only on \( x \), not on \( v_i \),
and is denoted by \( p_j(x) \). Let \( P_{ij}(x) \) be the cumulative (unconditional) probability of buyer \( i \) winning the first-stage auction, with the resale winner being buyer \( j \in K_1 \) with value \( v_j \leq x \). Then

\[
\pi_{wij} = \int_{\gamma_j(v, b)}^{w_j(b)} (p_j(x) - v_i) dP_{ij}(x).
\]

Since \( p(x) = v_i \) when \( x = \gamma_j(v, b) \), we have

\[
\frac{\partial}{\partial v_i} \pi_{wij} = P_{ij}(\gamma_j(v, b)) - P_{ij}(w_j(b)),
\]

and

\[
\frac{\partial}{\partial w_i} \pi_{wij}(v_i, b) = \sum_{j \not\in K_1} (P_{ij}(\gamma_j(v, b)) - P_{ij}(w_j(b)))
\]

\[
= \bar{H}(b) \left( \prod_{j \not\in \{i\} \cup K_1} Q_j(b) \prod_{j \in K_1} P_j(b) - \prod_{j \not\in i} Q_j(b) \right)
\]

\[
= Q_{1i}(v_i, b) - \frac{\partial}{\partial v_i} \pi_{1i}(v_i, b),
\]

and the proof is complete.

There is a simpler proof for the case when there is zero payoff after losing the auction, i.e. zero payoff from (iii). This is the case, for example, when \( b_j(.) \)'s are monotonic and \( K_1 \) contains every buyer except \( i \). We have \( H(b) > 0 \), \( Q_j(b) > 0 \) by assumption. Since \( \frac{\partial}{\partial v_i} \gamma_j(v, b) > 0 \), for \( j \in K_1 \), we have \( \frac{\partial}{\partial v_i} P_j(v, b) > 0 \) when \( j \in K_1 \). We conclude from Lemma 18 and (42) that

\[
\frac{\partial^2}{\partial b_i \partial v_i} u(v_i, b) = \frac{\partial^2}{\partial b_i \partial v_i} \left[ \pi_{1i}(v_i, b) + \pi_{wi}(v_i, b) \right]
\]

\[
= \frac{\partial}{\partial b_i} \left[ \bar{H}(b) \prod_{j \not\in \{i\} \cup K_1} Q_j(b) \prod_{j \in K_1} P_j(v, b) \right] > 0
\]

This gives the proof when there is zero payoff from (iii).

To deal with the case when there is payoff from (iii), we first prove Lemma 19. Let \( K_2 = \{ j \not\in i : w_j(b) < v_i \} \). If \( K_1 \) is empty, then \( K_2 \) is not empty from our assumptions. Let \( E_{2i}(v_i, b) \) be the event that buyer \( i \) loses the first-stage auction, but then wins it back during the resale. Let \( Q_{2i}(v_i, b) \) be the probability of \( E_{2i}(v_i, b) \). When regular buyer \( j \not\in i \) with \( v_j \) wins the first-stage auction bidding \( b' > b \), and sells to buyer \( i \) during resale, we can also define \( \gamma_i(v_j, b') \) in a similar way. Let \( \gamma_i(s, b') \) be the corresponding function for the speculator. The realized \( (b_s, v_{-i}) \) in \( E_{2i}(v_i, b) \) satisfies the following two properties:

If \( b_i > \max_{j \not\in i} b_j(v_j) \), then we have \( J(v_j, w_j(b_s)) \leq J(v_i, w_i(b_s)) \) for all \( j \not\in i \),

If \( b_j(v_j) = b' > \max \{ b_s : k \not\in i, j \} \), then \( J(v_k, w_k(b')) \leq J(v_i, w_i(b')) \) for \( k \not\in i, j \), and \( \gamma_i(v_j, b') \leq v_i \).

Proof of Lemma 19: Let \( y_j(v_i) = \sup \{ b' : b' \leq w_i(b_i) \} \) such that \( \gamma_i(v_j, b') \leq v_i \). Similarly, let \( y_k(v_j) = \sup \{ b' : b' \leq w_i(b_i) \} \) such that \( \gamma_i(s, b') \leq v_i \). Let \( \pi_{1ij} \) be the payoff of buying from regular bidders \( j \) in \( K_2 \) during resale after buyer \( i \) loses the first-stage auction. Let \( \pi_{1is} \) be the similar payoff of buying from the speculator during resale. When a buyer \( j \) with value \( v_j \) wins the first-stage auction bidding \( y = b_j(v_j) \), the expected price he gets conditional on selling to buyer \( i \) during resale depends only on \( y, v_j \) not on \( v_i \), and is
denoted by \( q(y, v_j) \). Let \( Q_{ij}(x) \) be the cumulative (unconditional) probability of buyer \( i \) losing the first-stage auction, but buying from buyer \( j \in K_2 \) who has a winning bid \( b_j(v_j) \leq x \). Then

\[
\pi_{ij} = \int_b^{y_j(v_i)} (v_i - q(x, v_j))dQ_{ij}(x).
\]

Since either we have \( y_j(v_i) = w_i(b_i) \) or we have \( q(x, v_j) = v_i \) when \( x = y_j(v_i) \), we must have

\[
\frac{\partial}{\partial v_i} \pi_{ij} = Q_{ij}(y_j(v_i)) - Q_{ij}(b).
\]

Similarly, let \( Q_{is}(x) \) be the corresponding cumulative (unconditional) probability of buying from the speculator whose winning bid is \( \leq x \), we have

\[
\frac{\partial}{\partial v_i} \pi_{wis} = Q_{is}(y_s(v_i)) - Q_{is}(b).
\]

Hence

\[
\frac{\partial}{\partial v_i} \pi_{ii}(v_i, b) = \frac{\partial}{\partial v_i} \pi_{wis} + \sum_{j \in K_2} \frac{\partial}{\partial v_i} \pi_{ij} = Q_{2i}(v_i, b).
\]

This completes the proof of the lemma.

When \( b \) is higher, \( Q_{2i}(v_i, b) \) becomes smaller, but it is compensated by the increase in \( Q_{2i}(v_i, b) \). For two subsets \( A, B \), the difference \( A \setminus B \) is the set of elements in \( A \) but not in \( B \). Note that when \( b' > b \), \( E_{1i}(v_i, b) \subset E_{1i}(v_i, b') \), but \( E_{2i}(v_i, b') \subset E_{2i}(v_i, b) \). We have the following lemma.

**Proof of Lemma 20:** It is sufficient to show that when \( b' > b \), we have

\[
E_{2i}(v_i, b) \setminus E_{2i}(v_i, b') \subset E_{1i}(v_i, b') \setminus E_{1i}(v_i, b).
\]

For any \((b_s, v_{-i}) \in E_{2i}(v_i, b) \setminus E_{2i}(v_i, b') \), The highest bid among \( b_s, b_j(v_j), j \neq i \) must be lower than \( b' \), hence buyer \( i \) is the winner with the bid \( b' \) when \((b_s, v_{-i}) \) is realized. If the speculator is the winner in \( E_{2i}(v_i, b) \), buyer \( i \) is the winner in the resale auction, hence for all \( j \neq i \), we have \( J(v_j, w_j(b_s)) \leq J(v_i, w_i(b_s)) \leq v_i \), which implies that \( v_j \leq \gamma_j(v_i, b_s) \leq \gamma_j(v_i, b') \). Therefore we have \((b_s, v_{-i}) \in E_{2i}(v_i, b') \) in this case. If a regular buyer \( j \) wins in the event \( E_{2i}(v_i, b) \), and \( y = b_j(v_j) < b' \), we also have \( J(v_k, w_k(b_s)) \leq J(v_i, w_i(b_s)) \leq v_i \) for \( k \neq i, j \), which implies that \( v_k \leq \gamma_k(v_i, y) \leq \gamma_k(v_i, b') \). For buyer \( j \), we have \( v_j \leq v_i \), hence \( v_j \leq \gamma_j(v_i, b') \). Thus in either case we have \((b_s, v_{-i}) \in E_{1i}(v_i, b') \). We must have \((b_s, v_{-i}) \notin E_{1i}(v_i, b) \) because it is in \( E_{2i}(v_i, b) \). The proof is complete.

### 7.2 Proof for the Increasing Symmetry Property

We apply the following standard result on the existence and uniqueness of the solution to a system of differential equations with the initial boundary conditions.

**Picard–Lindelöf Theorem:** Let \( b \) be a real variable, \( z \) be a vector, and \( h(b, z) \) be vector of functions continuous in \( b \), and Lipschitz continuous in \( z \). Then the system of differential equations \( y'(b) = h(b, y(b)) \) with the initial boundary condition \( y(b_0) = z_0 \) has a unique solution over \([b_0, b_1]\).
Proof of Lemma 21:
First assume that all regular bidders have the same maximum bid \( \bar{b} \) in equilibrium. Lemma 22 says that the inverse strategies must be strictly increasing near \( \bar{b} \). Lemma 23 says that the first order condition at \( \bar{b} \) is also the same as that of the auction without resale model for non-symmetric strategies. Lemma 24 proves the symmetry and increasing property under the assumption that all bidders have the same maximum bid in the support of the bid distributions. Lemma 25 then shows that in equilibrium, all bidders must have the same maximum bid in the support of the bid distributions, making use of Theorem 15 again.

Lemma 22 Assume that all buyers have the same maximum bid \( \bar{b} \) in their support of the equilibrium bid distributions. In equilibrium strategies \( b_i(\cdot) \), we must have \( b_i(\beta) = \bar{b} \), and \( b_i(\cdot) \) is strictly increasing in a neighborhood of \( \beta \) with a well-defined inverse.

Proof. Suppose it is false. There is some \( v_0 < \beta \) such that \( b_i(v_0) = \bar{b} \). There is an interval \((v', v'')\) to the right of \( v_0 \), but nearby, such that \( b_i(\cdot) \) is strictly decreasing and we have \( v_i < w_j(b), \bar{b} < b \) for all \( v_i \in (v', v'') \), \( b \) near \( b_i(v') \). Condition (39) holds for such \( v_i, b \), hence the strong form of supermodularity in Theorem 15 implies that \( b_i(\cdot) \) must be increasing which is a contradiction. Hence we have shown that \( b_i(\beta) = \bar{b} \) for all \( i \). The same argument in the above lemma also shows that \( \max_{v \leq v_0} b_i(v) < \bar{b} \) if \( v_0 < \beta \). Hence \( b_i(\cdot) \) has a uniquely defined inverse function \( \phi_i(\cdot) \) in a neighborhood of \( \beta \).

We can now write \( Q_i(b) = F(\phi_i(b)) \) in a neighborhood of \( \beta \).

Lemma 23 Assume that all buyers have the same maximum bid \( \bar{b} \) in their support of the equilibrium bid distributions, then the first-order condition for equilibrium at \( \bar{b} \) is

\[
\frac{d}{db} \left( (\beta - b) \prod_{k \neq i} F(\phi_k(b)) \right) |_{b=\bar{b}} = 0,
\]

which is the same as if there is no resale. Furthermore \( \phi_i'(\bar{b}) = \phi_j'(\bar{b}) \) for all \( i, j \).

Proof. Let buyer \( i \) with use value \( \beta \) bids \( b < \bar{b} \). There is no resale if the buyer wins the auction, but there is resale when he loses the auction. After losing the auction to some buyer \( k \), he may buy it from buyer \( k \). Let \( \gamma_k(y) \) be the optimal reservation price set by the winning buyer \( k \) with value \( \phi_k^{-1}(y) \) after winning with the bid \( y \). Let \( P_k(y) \) be the probability of winning bid being below \( y \). The expected payoff of buyer \( i \) bidding \( b < \bar{b} \), losing to buyer \( k \), and buying it back, is given by

\[\int_b^{\bar{b}} (v_i - \gamma_k(y))dP_k(y).\] (43)

At \( y = \bar{b} \), we have \( \gamma_k(y) = v_i \). The derivative of (43) with respect to \( b \), evaluated at \( \bar{b} \), is 0. Since this is true for all winning buyer \( k \), the first-order condition for the auction with resale at \( \bar{b} \) is

\[
\frac{d}{db} \left( (\beta - b) \prod_{j \neq i} F(\phi_j(b)) \right) |_{b=\bar{b}} = 0.
\]

From the first-order condition, we get

\[
\sum_{j \neq i} \phi_j'(\bar{b}) = \frac{1}{f(\beta)(\beta - \bar{b})} \text{ for all } i,
\]

which implies that

\[
\phi_i'(\bar{b}) = \frac{1}{(N-1)f(\beta)(\beta - \bar{b})} \text{ for all } i.
\]
and the proof is complete. ■

The following is a more restrictive version of the symmetry result with no speculator. It will be used for the proof of the full version of the symmetry property. The idea of the proof will also be used in the symmetry result when there is a speculator.

Lemma 24 Assume that there is a speculator. In the auction with resale, assume that all buyers have the same maximum bid $b$ in their support of the equilibrium bid distributions. Then the equilibrium bidding strategy $b_i(.)$ must be symmetric and strictly increasing, and $b_i(v_i) < v_i$ for all $v_i > \rho$.

Proof. First we show that the maximum equilibrium bid $\tilde{b}$ is strictly less than $\beta$. If not, since it is a common bid by all buyers with the use value $\beta$, the equilibrium payoff of a buyer with the use value $\beta$ must be 0. By bidding slightly lower, such a buyer has a positive probability of winning at a price lower than his use value, thus the payoff will be higher than the equilibrium payoff, a contradiction. Hence we must have $\tilde{b} < \beta$ in equilibrium. Now we will show symmetry. We consider a regular bidder $i$ with use value $v_i$ bidding $b$ near $\tilde{b}$. Let $\tilde{\phi}(b)$ be the vector $(\phi_1(b), \phi_2(b), ..., \phi_N(b))$. The first-order conditions of the equilibrium of the model can be written as

$$\sum_{k \neq i} a_{ik}(b, \tilde{\phi}(b)) \phi_k'(b) = c_i(b, \tilde{\phi}(b)),$$

where $a_{ik}(b, \tilde{\phi}(b))$ and $c_i(b, \tilde{\phi}(b))$ are functions of $(b, \tilde{\phi}(b))$, continuous in $b$ and Lipschitz continuous in $\tilde{\phi}(b)$. These properties are insured by the piecewise $C^2$ smooth assumptions of $b_i(.)$, $F(.)$. Thus we can write the system of linear equations in $\phi_i'(b)$ as

$$A\phi'(b) = \tilde{c}.$$  \hspace{1cm} (44)

When $v_i = \beta, b = \tilde{b}$, from Lemma (23), we have

$$\phi_k'(\tilde{b}) = \frac{1}{(N-1)(\beta - b)f(\beta)}$$

for all $i$.

In other words, the matrix $A$ at $b = \tilde{b}$ can be written as an identity matrix and $c_i(\tilde{b}, \tilde{\phi}(\tilde{b})) = \frac{1}{(N-1)(\beta - b)f(\beta)}$. Hence for $b$ near $\tilde{b}$, by continuity, $A$ must be a singular matrix, and (44) can be solved uniquely for $\tilde{\phi}(b)$. Therefore we have a system of differential equations satisfying the conditions of the Picard–Lindelöf Theorem. The solution is unique with the initial condition $\phi_i(\tilde{b}) = \beta$ for all $i$. Since it is obvious that a symmetric increasing solution $b_i(.)$ satisfying the boundary conditions exists, the uniqueness of the solution implies we have the symmetric and increasing property in a neighborhood of $\tilde{b}$. In the next step, we want to show that the symmetric increasing property holds for all $b > \rho$. Assume this is not true, then we must have some $b_0 = b_i(v_0) > \rho$ such that the symmetric increasing property holds for all $b \geq b_0$, but fails in any neighborhood to the left of $b_0$. We claim that $v_0 > b_0$, otherwise bidder $i$ can bid slightly lower to get a positive payoff while in the solution the payoff is 0, violating the optimal property of the bid $b_0$. Therefore we have $\phi_i(b_0) > b_0$. Lemma 23 is valid at $b_0$ instead of $\tilde{b}$ with the same proof. Furthermore, Lemma 22 with $\tilde{b}$ replaced by $b_0$ also holds, meaning $b_i(.)$ must be increasing in a neighborhood $(v_0 - \varepsilon, v_0)$, as the stronger supermodularity holds for each regular buyer. For the same reason, we also know that $\max_{v \leq v'} b_i(.) < b_0$ if $v' < v_0$. Hence $b_i(.)$ has a uniquely defined inverse in the neighborhood of $v_0$. Repeating the arguments above, we can extend the definition below $b_0$, so that the symmetric increasing property must hold in a neighborhood of $b_0$. This is a contradiction, and the contradiction implies that we must have the symmetry for all $b > \rho$. The rest is simple and our proof for the Lemma is complete. ■

We now show that the maximum bid of all the buyers must be the same. This gives us the full symmetry result.

Lemma 25 Assume that there is a speculator. In the auction with resale, the maximum equilibrium bid of all buyers must be the same, and hence the equilibrium bidding strategies must be symmetric and increasing.
\textbf{Proof.} Assume that, by relabeling, buyer one has the largest maximum equilibrium bid \( \bar{b} \). There is at least another buyer, say buyer two, who also has the same maximum equilibrium bid. Let there be \( m \geq 2 \) buyers with the maximum equilibrium bid \( \bar{b} \), and let buyer \( m + 1 \) be the buyer with the next highest maximum equilibrium bid \( b_m < \bar{b} \). Using the same argument in Lemma 24, we can show that for each all \( b \), \( \phi_i(b), i = 1, 2, \ldots, m \) are defined over \([b_m, \bar{b}]\) and are all equal and \( \phi'_i(b) > 0 \) over the interval \([b_m, \bar{b}]\). Let \( \phi(b) \) denote \( \phi_i(b) \) for all \( i \leq m \). For \( b \leq \bar{b} \), when bidder \( i = 1 \) with use value \( \beta \) bids \( b \leq \bar{b} \), Lemma 23 gives us the first-order condition for the equilibrium bid at \( b = \bar{b} \)

\[(m - 1)f(\beta)\phi'(\bar{b})(\beta - \bar{b}) - 1 = 0.\]  

(45)

For bidder \( j = m + 1 \) with use value \( \beta \) bidding \( b \leq \bar{b} \), there is no resale after winning the auction. When he loses the auction, there may be resale. In the resale, the winner bidder \( k, m \geq k > 1 \) resells to other bidders, believing that buyers \( j \leq m, j \neq k \) has use value upper bound \( \phi(b) \), while buyers \( j \geq m + 1 \) are identical with the use value upper bound \( \beta \). The first-order derivative at \( \bar{b} \) is

\[mf(\beta)\phi'(\bar{b})(\beta - \bar{b}) - 1.\]  

(46)

Since the optimal bid of bidder \( j = m + 1 \) is below \( \bar{b} \), Theorem 15 (39) applies, and the derivative of the payoff (from the left) at \( \bar{b} \) is

\[mf(\beta)\phi'(\bar{b})(\beta - \bar{b}) - 1 < 0.\]  

(47)

Obviously, (45) and (47) are contradictory. This contradiction proves the Lemma. \( \blacksquare \)

Since the differential equations determining the equilibrium strategies of the auction with resale model are the same as those without resale for symmetric strategies, and the boundary conditions are the same, we immediately have \( b_\rho(.) \) as the unique equilibrium bidding strategy of the model.

Proof of Theorem 16:
The proof requires similar steps above. Let \( b_s = \inf\{b : H(b) > 0\} \). Note that regular buyers with value above \( b_s \) would not bid below \( b_s \) because the winning probability is zero in this case. Hence we only need to prove symmetry for all \( v \geq b_s \). Since we know that \( H(\rho) > 0 \), then we have symmetry increasing property for all \( v \geq \rho \).

Note that Lemma 22 holds without any change of the proof. We have a well-defined inverse bidding strategy for all \( b_\rho(.) \) near \( \beta \), when all regular buyers have the same maximum bid. We will be rather brief as the idea of the proof is the same, and the arguments only require minor modifications.

\textbf{Lemma 26} In the auction with resale with a speculator, assume that all regular buyers have the same maximum bid \( \bar{b} \) in their support of the equilibrium bid distributions, then the first-order condition for the equilibrium of a regular buyer at \( \bar{b} \) is

\[\phi'_i(\bar{b}) = \frac{1}{(N - 1)f(\beta)} \left( \frac{1}{\beta - \bar{b}} - H'(\bar{b}) \right) \text{ for all } i.\]  

(48)

Hence \( \phi'_i(\bar{b}) = \phi'_j(\bar{b}) \) for all \( i, j \).

\textbf{Proof.} The proof is the same as in the proof of Lemma 23, except that we now have an additional possibility that the speculator may be the winner in the first-stage auction. When the speculator wins, the proof is also similar. From the first order condition

\[\frac{d}{db} \left[ H(b)(\beta - b) \prod_{j \neq i} F(\phi_j(b)) \right]_{b=\bar{b}} = 0,\]

we have

\[f(\beta) \sum_{j \neq i} \phi'_j(\bar{b}) = \frac{1}{\beta - \bar{b}} - H'(\bar{b}),\]
hence
\[
\phi_i'(\bar{b}) = \frac{1}{(N-1)f(\beta)} \left( \frac{1}{\beta - b} - H'(b) \right) \quad \text{for all } i.
\]

Hence the lemma is proved. ■

Now we can establish the following symmetry property when all regular bidders bid the same maximum amount.

**Lemma 27.** Assume that there is a speculator in the auctions with resale, and assume that all regular buyers have the same maximum bid \( \bar{b} \) in their support of the equilibrium bid distributions. Then the equilibrium strategy must be symmetric and increasing for all \( b > \bar{b}_s \), and \( \phi(b) > b \) when \( b > \bar{b}_s \).

**Proof.** The proof is the same as in Lemma 24 except that we can only obtain symmetry for all \( b > \bar{b}_s \). The first-order condition takes a slightly different form having the same properties needed for the proof. ■

Next we show that the maximum bid must be the same among the regular bidders.

**Lemma 28.** In equilibrium, the support of all regular buyers must be the same, and hence the equilibrium bidding strategies must be symmetric and increasing.

**Proof.** The idea is similar to the proof of Lemma 25. Let \( m \) be the number of regular buyers (including the speculator) with the maximum bid \( \bar{b} \). There is at least one regular buyer with the maximum bid \( \bar{b} \). When there is no speculator active at the bid \( \bar{b} \), the proof is identical. If the speculator is active at the bid \( \bar{b} \), the first-order condition of the equilibrium bid at \( \bar{b} \) for such a regular buyer is
\[
(m - 2)f(\beta)\phi'(\bar{b}) = \frac{1}{\beta - b} - \frac{H'(\bar{b})}{H(\bar{b})}.
\]
When there is only one regular buyer with the maximum bid \( \bar{b} \), the first-order condition still holds with 0 for the left-hand side. For a regular bidder having a lower maximum bid, but with use value \( \beta \) bidding \( b < \bar{b} \), the derivative of the logarithm of payoff with respect to \( b \) at \( \bar{b} \) is
\[
(m - 1)f(\beta)\phi'(\bar{b}) + \frac{H'(\bar{b})}{H(\bar{b})} - \frac{1}{\beta - b}.
\]
The equilibrium bid is less than \( \bar{b} \). Again from Theorem 15, we must have
\[
(m - 1)f(\beta)\phi'(\bar{b}) < \frac{1}{\beta - b} - \frac{H'(\bar{b})}{H(\bar{b})}.
\]
The two statements (49) and (50) are clearly contradictory, and this proves the lemma. ■
7.3 Proofs for Section 3

Proof of Lemma 2: Rewrite (16) as
\[ B(v) = \frac{N}{F^N(v)} \int_{r(v)}^{v} [xf(x) + F(x) - F(v)]F^{N-1}(x)dx. \]

It is clear we must have \( B(v) \to 0 \). Taking the derivative of (51) with respect to \( v \), we get
\[
B'(v) = \frac{Nf(v)}{F(v)} \left( vF^{N-1}(v) - \int_{r(v)}^{v} F^{N-1}(x)dx \right) - \frac{Nf(v)}{F(v)}B(v)
\]
\[
= \frac{Nf(v)}{F(v)} \left[ v - \frac{1}{F^{N-1}(v)} \int_{r(v)}^{v} F^{N-1}(x)dx - B(v) \right] = \frac{Nf(v)}{F(v)} [B'(v) - B(v)].
\]

Lemma 1 implies that \( B'(v) > 0 \).

Proof of Lemma 5: For \( v \geq \rho \), we have
\[
B'(v) - B_z'(v) = \frac{f(v)}{F(v)} \left[ N(B'(v) - B(v)) - (N - 1)(v - B_z'(v)) \right]
\]
\[
= \frac{f(v)}{F(v)} \left[ Nv - \frac{N}{F^{N-1}(v)} \int_{r(v)}^{v} F^{N-1}(x)dx - (N - 1)(v - B_z'(v)) \right]
\]
\[
= \frac{f(v)}{F(v)} \left[ Nv - NB(v) + (N - 1)B_z(v) \right] = \frac{Nf(v)}{F(v)} \left[ Nv - NB(v) + (N - 1)B_z(v) \right] + \frac{N - 1}{N} B_z(v) - B(v).
\]

Proof of Lemma 6: Suppose \( B(.) = B_z(.) \) in some interval \([v_1, v_2]\). From Lemma 5, we get \( B'(.) = B_z(.) \) in the same interval. If \( B'(.) = B_z(.) \) in the interval, from Lemma 5, we get
\[
B'(v) - B_z'(v) = \frac{Nf(v)}{F(v)} [B_z'(v) - B(v)],
\]
which means that whenever \( B'(v) > B_z'(v) \), we must have \( B(.) < B_z(.) \) in a neighborhood \((v, v + \epsilon)\), a contradiction. Similarly, we cannot have \( B'(v) < B_z'(v) \). We conclude that \( B'(.) = B_z(.) \) in the interval, and we have \( B(.) = B_z(.) \). In the third case, if \( B(.) = B^z(.) \) in the interval, from Lemma 5, we get
\[
B'(v) - B_z'(v) = \frac{(N - 1)f(v)}{F(v)} [B_z'(v) - B(v)],
\]
and the same contradiction is obtained.

Proof of Lemma 7: To prove \((a)\), assume that \( B^z(.) > B(.) \) except a countable closed subset. We want to show that \( B(.) > B^z(.) \) except a countable closed subset. First we show that \( B(.) \leq B^z(.) \) in \((z, z + \epsilon)\) can not be true. Otherwise, Lemma 5 implies that \( B'(.) > B_z'(.) \) in \((z, z + \epsilon)\) except a countable closed subset. This further implies that \( B(.) > B_z(.) \) in \((z, z + \epsilon)\), which is a contradiction. Hence if it is not true that \( B(.) > B^z(.) \) except a countable closed subset, we can find an interval \((x, x + \epsilon)\) such that \( z < x \) and \( B^z(.) > B(.) \) in \((x, x + \epsilon)\). By Lemma 5 again, we must have \( B'(x) > B^z'(x) \), which implies \( B(.) > B^z(.) \) in some \((x, x + \epsilon')\), a contradiction. Now we want to prove the converse. The arguments are quite similar. Assume that \( B(.) > B^z(.) \) except a countable closed subset. We cannot have \( B^z(.) \leq B(.) \) in some \((z, z + \epsilon)\), otherwise we would have \( B'(.) < B^z'(.) \) in \((z, z + \epsilon)\) except a countable closed subset, which leads to the contradiction \( B(.) < B^z(.) \). Now we can find an interval \((x, x + \epsilon)\) such that \( z < x \) and \( B^z(.) > B(.) \) in \((x, x + \epsilon)\). By Lemma 5, we have \( B'(x) - B^z'(x) < 0 \) leading to the contradiction \( B(.) - B^z(.) < 0 \) in some \((x, x + \epsilon')\). All the arguments and Lemmata we use apply to \( B^z(.) = b(.) \) when \( z = \rho = 0 \). The proof of \((b)\) is similar, and \((c)\) is a consequence of Lemma 6.
7.4 Proofs for Section 4

Proof of Theorem 6: We have, for each active interval \((z_k, v_k)\), the revenue

\[
\int_{a_k}^{b_k} bd(F^N(\eta(b))\hat{H}(b)) = \int_{a_k}^{b_k} bF^N(\eta(b))\hat{H}'(b)db + \int_{a_k}^{b_k} b\hat{H}(b)dF^N(\eta(b))
\]

\[
= \int_{a_k}^{b_k} bF^N(\eta(b))\hat{H}(b)L(b)d\ln F(\eta(b)) + \int_{z_k}^{v_k} B(v)\hat{H}(v)dF^N(v)
\]

\[
= \int_{z_k}^{v_k} B(v)\hat{H}(v)F^{N-1}(v)f(v)\mathcal{L}(v)dv + N \int_{z_k}^{v_k} B(v)\hat{H}(v)F^{N-1}(v)f(v)dv,
\]

\[
= \int_{z_k}^{v_k} B(v)\hat{H}(v)(1 + \frac{1}{N}\mathcal{L}(v))dF^N(v).
\]

For each inactive interval, we have the revenue

\[
\hat{H}(b_{k-1}) \int_{b_{k-1}}^{a_k} bdF^N(\phi(b)) = \hat{H}(v_{k-1}) \int_{v_{k-1}}^{z_k} B^*(v)dF^N(v).
\]

Hence we just add up all the revenues. The case of \(N = 1\) is quite simple. The proof is complete.

7.5 Proofs for Section 5

Proof of Lemma 17

We call the equilibrium \(b_{N+1}(\cdot), b_N(\cdot)\) with \(N + 1\), \(N\) regular buyers \(N + 1\)-equilibrium, \(N\)-equilibrium respectively. Let \(B_{N+1}(\cdot), B_N(\cdot)\) be the respective revenue functions. Assume first that there is an active interval \((z, v)\) for some \(z < v\) in the \(N + 1\)-equilibrium. There are three cases we need to consider for the \(N\)-equilibrium: (1) there is also an active interval \((z', v)\) in the \(N\)-equilibrium, (2) there is an inactive interval \((z', v)\) in the \(N\)-equilibrium, (3) there is a sequence of active intervals \((z_n', z_n)\) such that \(z_n' \to v, z_n < v\). In case (1), we have \(b_N(v) = B_N(v) < B_{N+1}(v) = b_{N+1}(v)\), and the lemma is proved. In case (2), let \(z_0 = \max(z, z')\), and define \(B^{*}_{z_0}(:, N), B^{*}_{z_0}(:, N + 1)\) to be the endogenous cost functions for the \(N, N + 1\) equilibrium with the initial conditions

\[
B^{*}_{z_0}(z_0, N) = b_N(z_0), B^{*}_{z_0}(z_0, N + 1) = b_{N+1}(z_0), \tag{53}
\]

respectively. Since by assumption \(b_N(z_0) \leq b_{N+1}(z_0)\), we have \(B^{*}_{z_0}(v, N) < B^{*}_{z_0}(v, N + 1)\), hence

\[
b_N(v) = B^{*}_{z_0}(v, N) < B^{*}_{z_0}(v, N + 1) \leq B_{N+1}(v) = b_{N+1}(v), \tag{54}
\]

where the last inequality is by Lemma 7. The lemma is proved for case (2). For case (3), since \(b_N(v) = B_N(v)\), we have the same conclusion as in case (1). Now assume that there is an inactive interval \((z, v)\) in the \(N + 1\) -equilibrium. We need to consider the same three cases for the \(N\)-equilibrium. In case (1), Since \((z, v)\) is inactive in the \(N + 1\)-equilibrium, we have

\[
b_{N+1}(v) \geq B_{N+1}(v) > B_N(v) = b_N(v), \tag{55}
\]

and we are done. In case (2), let \(z_0 = \max(z, z')\), and define \(B^{*}_{z_0}(:, N), B^{*}_{z_0}(:, N + 1)\) with the initial conditions as in (53), then we have (54) as well, and we are done. For case (3), the inequalities in (55) hold as well, and we are done. Now consider the final possibility when there is a sequence of active intervals \((x'_n, x_n), x_n < v, x'_n \to v\) in the \(N + 1\)-equilibrium, then we have \(b_{N+1}(v) = B_{N+1}(v)\). In case (1), \(b_N(v) = B_N(v) < B_{N+1}(v) = b_{N+1}(v)\) and we are done. In case (2), the inequalities in (53) still apply, and we are done. In case (3), since \(b_N(v) = B_N(v)\), we have the same conclusion as in case (1). The proof is complete.
References


