Chapter 7. Simultaneous-Move Games: Mixed Strategies

- Now we return to simultaneous-move games.

- We resolve the issue of non-existence of Nash equilibrium in pure strategies through intentional mixing.

- Real life examples of mixing: sports, free riding, tax audit.
7.1 What is a mixed strategy

- A penalty taker with a powerful left-sided shot.

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<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
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<tbody>
<tr>
<td>Kicker</td>
<td></td>
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<tr>
<td>Left</td>
<td>1, 0</td>
<td>0.4, 0.6</td>
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<td>Right</td>
<td>0, 1</td>
<td>1, 0</td>
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- This is a zero-sum game, but unlike Matching Pennies, it is asymmetric.
There is no Nash equilibrium in pure strategies.

Each strategy of each player is rationalizable.

The only way to win, or equivalently not to lose, is to keep the opponent guessing by mixing between your own pure strategies.

Mixing is therefore making your own choice of strategy intentionally random from perspective of opponent.
• A mixed strategy of a player assigns a probability of playing each pure strategy, such that sum of assigned probabilities is equal to 1 over all the strategies of the player.
  
  – Mixing doesn’t mean assigning same probability to each pure strategy.
  
  – With only two pure strategies, a mixed strategy can be represented by one number between 0 and 1.
  
  – A pure strategy is a degenerate mixed strategy, with 1 assigned to the pure strategy and 0 to all other pure strategies.
• Expected payoff in the presence of mixing.
  
  – Expected payoff to a player is probability-weighted sum of the player’s payoffs, with probabilities determined by corresponding mixing.
  
  – Mixing is independent across players.
  
  – Example: expected payoff to Kicker from using mixed strategy of \textit{Left} with probability $p$ and \textit{Right} with $1 - p$ against Keeper’s mixed strategy of \textit{Left} with probability $q$ and \textit{Right} with probability $1 - q$ is given by following:

$$pq \cdot 1 + p(1 - q) \cdot 0.4 + (1 - p)q \cdot 0 + (1 - p)(1 - q) \cdot 1.$$
7.2 Mixing moves

- Benefit of mixing.

  - Suppose that Keeper plays a mixed strategy of Left with probability \( q \) and Right with probability \( 1 - q \).

  - Expected payoff to Kicker is \( q \cdot 1 + (1 - q) \cdot 0.4 \) from Left, \( q \cdot 0 + (1 - q) \cdot 1 \) from Right, and \( p \) times the first payoff plus \( (1 - p) \) times the second from a mixed strategy of Left with probability \( p \) and Right with probability \( 1 - p \).
- Kicker’s best response is $p = 1$ if $q > 3/8$ and $p = 0$ if $q < 3/8$.

- If $q = 3/8$, Kicker is indifferent between Left and Right, and expected payoff from best response is minimized.

- Benefit of mixing to Keeper: since this is a zero-sum game, Keeper should mix between Left and Right with exactly $q = 3/8$, and thus increases guaranteed payoff from 0 to 3/8.
Figure 1. Benefit of mixing to Keeper.
• Kicker should mix with $p$ of \textit{Left} and $1 - p$ of \textit{Right} in such a way that Keeper is indifferent between \textit{Left} and \textit{Right}.

  - Equating expected payoff of Keeper $p \cdot 0 + (1 - p) \cdot 1$ from \textit{Left} to expected payoff $p \cdot 0.6 + (1 - p) \cdot 0$ from \textit{Right} gives $p = 5/8$.

  - Benefit of mixing to Kicker: by choosing $p = 5/8$, Kicker increases guaranteed payoff from 0.4 (from choosing $p = 1$) to 5/8.
• Nash equilibrium in mixed strategies.
  – With only two pure strategies, a mixed strategy can be represented by one number between 0 and 1, so mixed strategies are special case of continuous strategies.
  – Nash equilibrium can be found by intersection of best response functions.
Figure 2. Nash equilibrium in Penalty Kick.
7.3 Nash equilibrium as a system of beliefs and responses

- In a Nash equilibrium, each player plays a best response against the equilibrium strategies of other players.
  
  – Recall two features of Nash equilibrium: non-cooperative and correct beliefs.

  – Nash equilibrium in mixed strategies requires players to have correct beliefs in that they know equilibrium mixtures of all players.
7.4 Mixing in non-zero-sum games

- Free-rider problem in crime-reporting.
  - Setup: two witnesses of crime decide whether to report it or not; reporting it costs $c$ individually; each witness receives $b > c$ if at least one of them reports it.
  - Game table.

<table>
<thead>
<tr>
<th>Witness 1</th>
<th>Report</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Report</td>
<td>$b-c, b-c$</td>
<td>$b-c, b$</td>
</tr>
<tr>
<td>Don’t</td>
<td>$b, b-c$</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
- This is the same as Hawk-Dove, with two Nash equilibria in pure strategies.

\[
\begin{array}{c|cc}
\text{Witness 2} & \text{Report} & \text{Don't} \\
\hline
\text{Report} & b - c, b - c & *b - c, b^* \\
\text{Don't} & *b, b - c^* & 0, 0 \\
\end{array}
\]

- There is a mixed strategy equilibrium.
- Principle of making your opponent indifferent: your opponent is willing to mix between two pure strategies only when you mix your own two pure strategies in a way such that your opponent is indifferent.

- Apply the principle: if Witness 2 chooses Report with probability $q$ (and Don’t with probability $1 - q$), then expected payoff of Witness 1 is $(b - c)$ from Report, and $qb$ from Don’t, so $q = (b - c)/b$; symmetrically, Witness 1 has to choose Report with probability $p = (b - c)/b$ to make Witness 2 indifferent.
Figure 3. Three Nash equilibria in Reporting a Crime.
7.5 General discussion of mixed-strategy equilibria

- Weak sense of equilibrium.
  - Nash equilibrium in pure strategies is typically strict in that each player plays the unique best response to equilibrium strategies of opponents.
  - Principle of making your opponent indifferent means that each player’s equilibrium mix is not a strict best response to opponent’s equilibrium mix.
• Counterintuitive comparative statics.

– Penalty Kick again: a smaller $\alpha$ means an improvement in Keeper’s skill in reducing Kicker’s advantage.

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<td>$\alpha, 1 - \alpha$</td>
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– Mixed-strategy Nash equilibrium depends on $\alpha$, and can be found by applying the principle of making your opponent indifferent.
- To make Kicker indifferent, Keeper chooses \textit{Left} with $q$ such that $q \cdot 1 + (1 - q) \cdot \alpha = q \cdot 0 + (1 - q) \cdot 1$, which gives $q = (1 - \alpha) / (2 - \alpha)$.

- To make Keeper indifferent, Kicker chooses \textit{Left} with $p$ such that $p \cdot 0 + (1 - p) \cdot 1 = p \cdot (1 - \alpha) + (1 - p) \cdot 0$, which gives $p = 1 / (2 - \alpha)$.

- When $\alpha$ decreases, Keeper gets better with \textit{Right} but in equilibrium achieves a higher payoff by using \textit{Right} less often.
7.6 Mixing when one player has three or more pure strategies

- Penalty Kick with Panenka.

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Keeper still has only two pure strategies so we continue to represent any mixed strategy with probability $p$ that Keeper chooses *Left*. 
Figure 4. Penalty Kick with Panenka.
- To minimize Kicker’s payoff from best response, Keeper has to keep Kicker indifferent between Left and Middle: equating \( q \cdot 1 + (1 - q) \cdot 0.4 \) to \( q \cdot 0.6 + (1 - q) \cdot 0.8 \) gives Keeper’s equilibrium mix \( q = 0.5 \).

- Kicker does not use Right in equilibrium.

- In order for Keeper to mix, Kicker needs to choose Left with probability \( p_l \) and Middle with probability \( 1 - p_l \) to make Keeper indifferent: equating \( p_l \cdot 0 + (1 - p_l) \cdot 0.4 \) to \( p_l \cdot 0.6 + (1 - p_l) \cdot 0.2 \) gives Kicker’s equilibrium mix \( p_l = 0.25 \).
• Modified principle of making your opponent indifferent when at least one player has more than two pure strategies: in any mixed-strategy equilibrium, a player has to be indifferent among all pure strategies used in equilibrium, and prefers any of them to the ones unused in equilibrium.

– When only one player has three or more strategies in a zero-sum game, only two will be used in equilibrium, and which two is determined by the opponent mixing to minimize the former’s best payoff.
Figure 5. Penalty Kick: Panenka is used with Right.
Figure 6. Penalty Kick: Panenka is never used.
7.6 Mixing when both players have three or more strategies

- Only strategies that survive iterated elimination of never best responses can be used in equilibrium.

- So we apply modified principle of making your opponent indifferent to reduced game.

- Again, some pure strategies may not be used in equilibrium, but mixing among three or even more pure strategies can occur in equilibrium.

- Verifying equilibrium is easier than finding equilibrium.
• 3-by-3 Penalty Kick.

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• Kicker and Keeper both choosing each strategy with equal probability is a Nash equilibrium.