A Portfolio Theory of International Capital Flows[1]

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Abstract

The substantial growth in gross external portfolio assets and liabilities has led to renewed interest in portfolio-based models of capital flows and the current account. This paper develops a simple and tractable model of net capital flows in which time-varying gross country portfolios are an essential element in current account imbalances. The main constituents of country portfolios in the model are nominal bond assets and liabilities. Under very weak conditions, the world wealth distribution is stationary. Stationarity is generated by movements in bond risk-premia such that the return on a debtor country's gross liabilities is less than the return on its gross assets.

1 Introduction

The last decade has seen an unprecedented increase in two-way financial flows between countries. For both developed and to a lesser extent developing countries, there has been a substantial increase in the ratio of both gross external assets and liabilities to GDP, at the same time as an increased dispersion in absolute current account positions across countries (e.g. Lane and Milesi-Ferretti, 2006). Obstfeld (2004) suggests that this phenomenon of 'financial globalization' increasingly calls into question the standard intertemporal view of the current account, in which external imbalances are financed by one-way movement in real bonds. In an environment with large holdings of gross assets and liabilities denominated in different currencies, asset classes, or maturity structures, the interpretation of the current account based on a simple measurement of net-foreign assets may be quite misleading. Financial globalization may facilitate larger current account positions than would be consistent with capital markets based on one-way capital flows[1].

If financial globalization had advanced to the stage where international financial markets were effectively complete, then the analysis of the current account would be relevant only as an accounting entity, since (correctly measured) ratios of national wealth across countries would not change over time. But empirical evidence on the failure of international risk sharing (e.g. Obstfeld and Rogoff 2000), as well as the substantial time variation in relative measures of national wealth, suggests that international financial markets are incomplete. The full understanding of the phenomenon of financial globalization therefore requires the combination of models of portfolio diversification with dynamic general equilibrium models of the current account under incomplete markets. This raises considerable technical difficulties however, as models of multi-agent dynamic general equilibrium economies with endogenous portfolio structures are very difficult to solve and analyze.

This paper develops a simple, analytically tractable general equilibrium model of portfolio choice in a two country economy with incomplete markets and trade in nominal bonds. The model is designed to address exactly the relationship between gross external asset and liability positions and the determination of the current account. The analysis highlights endogenous portfolio composition as essential in facilitating international net capital flows between countries. Movements in net foreign assets are generated by gross assets and gross liabilities moving in the same direction, giving rise to time-varying portfolios and asset returns. The key results of the paper may be summarized as follows:

1. The model implies a close connection between changes in gross portfolio positions and net capital flows. Current account movements are a direct consequence of time variation in

[1] Greenspan (2005) argues for a rationalization of the US current account balance along these lines.

a country's gross assets and liabilities.

2. Endogenous variation in gross portfolio positions and real returns ensure a stationary world wealth distribution. The key mechanism ensuring stationarity is that asset returns move so as to reduce the cost of borrowing for debtor countries.

3. There is a complementarity between trade in nominal bonds and trade in real (non-contingent) bonds. Nominal bond trade increases the gains from trade in real bonds.

4. There is a substitutability between trade in nominal bonds and trade in equity (claims on technologies). The more nominal bonds facilitate international risk-sharing, the smaller will be the portfolio shares of foreign equity.

The central feature of the model is the ability of nominal bonds to share country specific risk in an incomplete markets environment. We start with a stochastic, continuous time framework with country-specific technology shocks. If financial markets consisted only of a real risk-free bond, as in the standard current account model, then there would be no gains from trade between countries at all. Trade in nominal bonds allows for countries to to share risk by holding a diversified portfolio of domestic and foreign currency bonds. But because risk sharing is limited, country specific shocks cause movements in relative national wealth levels across countries. This causes time-variation in bond returns and portfolio shares. Movements in portfolio holdings, or gross positions, are in turn associated with net capital flows between countries. Thus, current account movements are inherently tied to the adjustment of national portfolios and two-way capital flows. For instance, a country experiencing net capital inflows may be simultaneously issuing bonds denominated in its own currency, but purchasing bonds denominated in the foreign currency.

We derive a novel condition for stationarity in the world distribution of wealth in the presence of nominal bond trade. So long as nominal bonds allow any cross country risk-sharing, the world wealth distribution is stationary. Moreover, there is a simple and highly intuitive explanation of the stationarity result: asset returns tend to move to the disadvantage of creditor countries and to the advantage of debtor countries. This ensures that as a country's relative wealth position deteriorates, its cost of borrowing also falls, encouraging it to invest in its domestic technology, and increase its expected growth rate. More specifically, we find that debtor countries face a lower return on their gross liabilities than they receive on their gross assets, while creditor countries face the opposite situation. In this way stationarity in the wealth distribution is tied directly to time-variation in asset returns and portfolio composition.

The results are illustrated in a baseline version of the model in which there is trade in nominal bonds and a real risk-free bond. But we show that the stationarity results apply in a number of different extensions of the model, restricting trade to nominal bonds only, to trade in just one country's nominal bond, or in environment extended to allow for trade in both nominal bonds and equity.

We find that nominal bonds act as a complement to trade in a risk-free real bond. The model is structured so that all shocks to a country's technology are permanent. If asset trade were permitted only in a risk-free real bond, there would be no gains from trade. But when nominal bonds may also be traded, the endogenous time-variation in the world risk free interest rate generates also gains from trade in a risk-free real bond. This represents a second channel linking gross country portfolio holdings and two way financial trade on the one hand, and standard intertemporal borrowing and lending through real non-contingent bonds on the other hand.

By contrast, nominal bonds represent a substitute for trade in equity. In the baseline model, we assume no direct trade in claims to the economy's production technology (equity). Unrestricted equity trade would imply complete markets. In an extension however, we allow limited equity trade. But even then, agents may hold only a small share of foreign equity. The reason is that the risk-sharing through nominal bonds may remove the need for trade in equity. Thus, the presence of nominal bond trade may imply a home bias in equity holdings.

The paper is related to growing recent literature on modeling portfolio choice in dynamic general equilibrium environments. Svennsson (1989) and Bachetta and Wincoop () specifically analyze the role of nominal bonds using two period models. More recently, in infinite horizon frameworks, Engel and Matsumoto (2006), Heathcote and Perri (2004), and Kollman (2006) develop alternative explanations for home bias in equity holdings within a complete markets framework. Pavlova and Rigobon (2003) construct a continuous-time stochastic model of exchange rates, and focus on aspects of asset pricing and the international transmission of stock prices.

In an incomplete markets environment Heaton and Lucas (1996) and Krusell and Smith (1998) develop numerical methods for analyzing asset pricing and risk sharing. Kubler and Schmedders (2003, 2005) prove the existence of a stationary equilibrium in asset pricing models with incomplete markets and collateral constraints, and propose a numerical algorithm to obtain optimal policy rules including portfolio decisions. More recently, Evans and Hnatkovska (2005) and Hnatkovska (2006) use a numerical method to solve an incomplete markets model in an open economy environment, but their local approximation does not allow them to explore the issue of stationarity[2].

Our model is substantially simpler than most of the previous literature in that we obtain a fully analytical characterization of asset returns, portfolio choice, and the distribution of

[2] Other recent papers based on approximation methods are Devereux and Sutherland, (2006), (2007), and Tille and Van Wincoop (2007).

wealth in an incomplete markets environment, without resort to approximation methods. To our knowledge, this is unique in the literature. The key assumptions that allow us to achieve this are a) log utility, and b) linear stochastic technologies. In essence, the model is a multi agent version of Merton (1971) with restrictions on asset trade[3].

The source of stationarity in wealth in this paper differs from that of previous literature. In a version of the neoclassical growth model with idiosyncratic endowment risk, Aiyagari (1994) shows that precautionary savings can support a stationary wealth distribution, as long as agents are not too patient (see also Carroll 2002, and Krussell and Smith 1998). With precautionary saving, stationarity is ensured by poor agents saving more and wealthy agents saving less, while all saving is done in the form of an aggregate risk free asset. In our model, stationarity is associated with aggregate shocks, which change the composition of real returns. But the presence of nominal bonds is also critical[4]. In order to ensure a stationary wealth distribution, agents must continually adjust not only their aggregate savings, but also the portfolio composition of their savings[5].

The model implies that large debtor countries tend to pay lower rates of return on their gross external liabilities than they earn on their gross external assets. Recent papers have found evidence for this from examination of the US external portfolio[6]. Gourinchas and Rey (2005) construct a data set on US external assets and liabilities over the period 1952-2004. They find that the US earned a higher rate of return on external assets than it paid on liabilities over the whole sample. Most of this excess return is due to a 'return discount', implying that there is a higher return on assets than on liabilities within each asset class. Moroever, the excess return on assets is higher in the second part of the sample,as the US NFA position turned negative. Using a VAR approach, Gourinchas and Rey (2006) find that a rise in the US trade deficit generates a predictable fall in the return on US external liabilities, through a predictable depreciation in the exchange rate. Similarly, Lane and Milesi Ferretti (2005), using a different data set, show that the ratio of US net foreign assets to GDP has a significantly positive effect on the excess return received by the rest of the world on US investments.

The paper is structured as follows.

[3] This makes the model amenable to a large number of applications, although it does restrict its applicability in explaining some puzzles, such as those related to asset pricing.

[4] With only a real risk free bond, the wealth distribution in our model is degenerate.

[5] An alternative mechanism for ensuring a stationary wealth distribution is through endogenous movements in the terms of trade. See Acemoglu and Ventura (2003) and Cole and Obstfeld (1999). A range of alternative mechanisms for ensuring stationarity in small open economy models are discussed in Schmitt-Grohe and Uribe (2004).

[6] Note that our model implies that movements in a country's net foreign assets must be large enough to impact on world capital markets. This makes the US case particularly appropriate.

2 The Model

We take a one-good two-country model of a world economy[7]. In each country there is a risky linear technology which uses capital and generates expected instantaneous return α_i with standard deviation σ_i, where $i = h$ or f, signifying the 'home' or 'foreign' country. Capital can be turned into consumption without any cost. The return on technology i (in terms of the homogeneous good) is given by:

$$\frac{dQ_i}{Q_i} = \alpha_i dt + \sigma_i dB_i, \tag{1}$$

for $i = h$ or f, where dB_i is the increment to a standard Weiner process. For simplicity, we assume that the returns on the two technologies are independent, so that

$$\lim_{\Delta t \to 0} \frac{Cov_t\left(\Delta B_h(t+\Delta t), \Delta B_f(t+\Delta t)\right)}{\Delta t} = 0.$$

To study the dynamics of the world wealth distribution, we assume that financial markets are incomplete. For the moment, assume that residents of one country cannot directly purchase shares in the technology of the other country (we relax this assumption in a later section). Both nominal and real risk-free bonds can be traded between the countries however.

Nominal bonds may be denominated in home or foreign currency. Although nominal bonds are risk-free in nominal terms, their real returns are subject to inflation risk. Let inflation in country i be represented as[8]:

$$\frac{dP_i}{P_i} = \Pi_i dt + v_i dM_i.$$

Thus, inflation has mean Π_i and standard deviation v_i, $i = h$ and f. dM_i represents the increment to a standard Weiner process. The monetary policy followed by country i is represented by the parameters Π_i and v_i, and the covariance of dM_i with dB_i. We let

$$\lim_{\Delta t \to 0} \frac{Cov_t\left(\Delta M_i(t+\Delta t), \Delta B_i(t+\Delta t)\right)}{\Delta t} = \lambda_i, \tag{2}$$

[7] It is possible to extend the model to many goods, and incorporate real exchange rate dynamics. However, the one-good framework is adequate for an analysis of the central issue we are concerned with here; that is, the interaction of portfolio choice and the current account.

[8] We do not explicitly model a source of demand for money. As in Woodford (2003), we can think of the model as representing a 'cashless economy'. What matters is that there is an asset whose payoff depends on the price level, and monetary policy can generate a particular distribution for the price level.

and

$$\lim_{\Delta t \to 0} \frac{Cov_t\left(\Delta M_i(t+\Delta t), \Delta M_j(t+\Delta t)\right)}{\Delta t} = 0. \tag{3}$$

for $i \neq j$. Equation (3) here says that inflation shocks are independent across countries. This is not critical, but simplifies the algebra.

The covariance term λ_i in equation (2) is a key parameter. It describes the cyclical characteristics of the inflation rate, and hence of the real return on nominal bonds. The only restriction we impose on this parameter is that $|\lambda_i| \neq 0$, and $-1 < \lambda_i < 1$. The central results of the paper regarding the stationarity of the world wealth distribution apply whether λ_i is positive or negative.

Let the instantaneous nominal return on currency i bonds be $\widehat{R}_i$. Then the real return on bond i is

$$(R_i - \Pi_i)dt - v_i dM_i,$$

where $R_i = \widehat{R}_i + v_i^2$ is an adjusted nominal interest rate[9]. This will be determined endogenously as part of the world bond market equilibrium[10].

The real risk-free bond has instantaneous return r, which is also determined by world bond market equilibrium. Agents in each economy divide their wealth across holdings of the domestic technology, the real risk-free bond and the two separate nominal bonds. C_h denotes consumption of the representative home household.

The budget constraint for the home country may then be written as:

$$\begin{aligned} dW_h &= W_h\left[\omega_T^h(\alpha_h - r) + \omega_h^h(R_h - \Pi_h - r) + \omega_f^h(R_f - \Pi_f - r) + r\right]dt \\ &\quad -C_h dt + W_h\left(\omega_T^h \sigma_h dB_h - \omega_h^h v_h dM_h - \omega_f^h v_f dM_f\right), \end{aligned} \tag{4}$$

where ω_T^h, ω_h^h, and ω_f^h are the portfolio shares, respectively, of the domestic technology, home currency nominal bonds, and foreign currency nominal bonds. Hence, $1 - \omega_T^h - \omega_h^h - \omega_f^h$ represents the share of the real risk-free bond.

Each country is populated by a continuum of identical agents. Preferences are identical

[9] Since we have the single-good world and PPP holds, then the rate of the change in the exchange rate S $(= \frac{P_h}{P_f})$ is just determined residually by:

$$d\ln S = \left(\Pi_h - \frac{1}{2}v_h^2 - \Pi_f + \frac{1}{2}v_f^2\right)dt + v_h dM_h - v_f dM_f.$$

In the symmetric case studied below, the drift term in the exchange rate is zero.

[10] In this nominal bond equilibrium, long-term bonds are redundant assets and are derivatives of instantaneous nominal bonds. Therefore, given the equilibrium path of instantaneous nominal interest rates, longer-term nominal interest rates are derived completely by arbitrage pricing.

across countries, and given by:

$$E_0 \int_0^{\infty} \exp(-\rho t) \ln C_i(t) dt, \tag{5}$$

where ρ is the rate of time preference.

From a welfare perspective, with preferences given by (5), the relevant measure of expected consumption growth in any equilibrium is the *risk-adjusted growth rate*, given by:

$$\lim_{\Delta t \to 0} E_t \left[\frac{\Delta \ln C_i(t+\Delta t)}{\Delta t} \right] = \lim_{\Delta t \to 0} \frac{E_t \left(\frac{\Delta C_i(t+\Delta t)}{C_i(t)} \right) - \frac{1}{2} Var_t \left(\frac{\Delta C_i(t+\Delta t)}{C_i(t)} \right)}{\Delta t}.$$

2.1 Optimal Consumption and Portfolio Rules

To highlight the role of nominal bonds in fostering intertemporal trade, we will focus on the case where countries have identical drift and diffusion parameters, so that, $\alpha_h = \alpha_f = \alpha$, $\sigma_h = \sigma_f = \sigma$, $\Pi_h = \Pi_f = \Pi$, $\nu_h = \nu_f = \nu$, and $\lambda_h = \lambda_f = \lambda$[11].

With logarithmic utility, home country consumers follow the myopic consumption rule:

$$C = \rho W.$$

The optimal portfolio rules may be obtained as the solution to:

$$\begin{bmatrix} \omega_T^h \\ \omega_h^h \\ \omega_f^h \end{bmatrix} = \begin{bmatrix} \sigma^2 & -\lambda\sigma v & 0 \\ -\lambda\sigma v & v^2 & 0 \\ 0 & 0 & v^2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha - r \\ R_h - \Pi - r \\ R_f - \Pi - r \end{bmatrix} \tag{6}$$

A similar set of conditions hold for the foreign country. In some cases, we make the following parameter assumptions: $\sigma^2 + \lambda v \sigma > 0$ and $v^2 + \lambda v \sigma > 0$. These conditions ensure that the behavior of portfolio demands satisfy regularity properties. In particular, the first condition ensures that a rise in the real risk-free rate reduces the demand for home currency nominal bonds, while the second ensures that a rise in the real risk-free rate reduces the demand for shares in the domestic technology. These conditions are not necessary for the key stability results developed below, but they make the exposition substantially easier.

[11] None of the results are qualitatively different when drift and diffussion parameters differ across countries. We discuss this in Section 4 below.

3 Alternative Financial Market Configurations

To provide a reference point, we first characterize an equilibrium under two extreme asset market structures; where only real risk-free bonds are traded, and when there is free trade in claims on each countries technology. Then we analyze the equilibrium with nominal bond trade.

3.1 Trade in a Real Risk-Free Bond

With only trade in a real risk-free bond, there are no gains from trade between countries at all. Real risk-free bonds will not be traded, since the autarky risk-free rate on the real bond in each country is identical, given by $r = \alpha - \sigma^2$. In this case, $\omega_T^i = 1$. Each country's wealth is equal to its physical capital stock. A home technology shock dB_h will increase home consumption and wealth in the same proportion, leaving the real risk-free rate unchanged. The shock will have no affect at all on the foreign economy. In this case,the risk-adjusted consumption growth rate is $\alpha - \rho - \frac{1}{2}\sigma^2$.

3.2 Complete Financial Markets

If shares in each country's technology were freely tradable across countries, financial markets would be effectively complete. Trade in risk-free real or nominal bonds would then be redundant. The equilibrium share of each technology will be $\omega_T^i = \frac{1}{2}$, and the equilibrium risk-free rate on the real bond will be $r = \alpha - \frac{1}{2}\sigma^2$. Risk-pooling under complete markets implies a higher risk-free interest rate than in autarky. In this case, all technology shocks are equally shared among home and foreign consumption. The risk-adjusted consumption growth rate is then $\alpha - \rho - \frac{1}{4}\sigma^2$. Complete markets improve welfare by raising the risk-adjusted consumption growth rate.

3.3 Nominal Bond Trading Equilibrium

Now allow for trade in nominal bonds. Since the real returns on home and foreign currency nominal bonds are risky, and covary in different ways with the home and foreign technologies, the two countries will have different demands for nominal bonds. For instance, in the home economy under autarky, the equilibrium nominal interest rate on home currency bonds is $R_h^a = \Pi + \alpha - \sigma^2 - \lambda v \sigma$. This includes a risk premium term $\lambda v \sigma$. When $\lambda < 0$, the home nominal bond is a bad hedge against technology risk, and must have a return higher than the risk-free rate on the real bond, adjusted for inflation. The home country autarky equilibrium interest rate on foreign currency bonds is $R_f^a = \Pi + \alpha - \sigma^2$. Since the foreign price level is independent of home technology shocks, it is a better hedge against consumption risk when $\lambda < 0$, and therefore carries a lower autarky return than the home currency bond. The autarky returns on nominal bonds in the foreign country is just the mirror image of that in the home country.

When $\lambda > 0$ of course, the opposite reasoning applies. But again however, the autarky return on nominal bonds will differ across countries. This implies that there are gains from trade in nominal bonds. At any moment in time, an equilibrium in the market for home and foreign currency bonds determines the nominal rates of return R_h and R_f. Nominal bond market clearing conditions are given as:

$$\omega_h^h W_h + \omega_h^f W_f = 0, \tag{7}$$

$$\omega_f^h W_h + \omega_f^f W_f = 0. \tag{8}$$

These equations just say that the sum of bond demands must add up to the world (zero) bond supply.

We know that without trade in nominal bonds, there is no trade in the real risk-free bond. But as is established below, nominal bond trade will endogenously generate gains from trade in the real bond. Therefore, we take account of the market clearing condition in the real bond as:

$$(\omega_T^h + \omega_h^h + \omega_f^h - 1)W_h + (\omega_T^f + \omega_h^f + \omega_f^f - 1)W_f = 0. \tag{9}$$

Using (6), these three conditions may be solved for R_h, R_f, and r. Define $\theta = \frac{W_f}{W_h + W_f}$ as the ratio of foreign wealth to world wealth. The solution has a recursive structure. Given the consumption rule, equilibrium returns and portfolio holdings depend on θ. We may then write the implicit solution for nominal interest rates and the world risk-free rate as $R_h(\theta)$, $R_f(\theta)$, $r(\theta)$. When markets are incomplete, θ will be time-varying, and therefore so are rates of return and portfolio shares. The dynamics of θ may be constructed from the wealth dynamics (4) and the equivalent process for the foreign country.

3.3.1 Equilibrium Returns and Portfolio holdings with nominal bond trade

Equations (6)-(9) are all linear, and can be solved for the equilibrium returns as:

$$R_h(\theta) = \bar{R} - \frac{\sigma\lambda(1-\lambda^2)(1-2\theta)[v + \sigma\lambda(1-2\theta)]}{2\Delta}, \tag{10}$$

$$R_f(\theta) = \bar{R} + \frac{\sigma\lambda(1-\lambda^2)(1-2\theta)[v + \sigma\lambda(1-2(1-\theta))]}{2\Delta}, \tag{11}$$

$$r(\theta) = \bar{r} - \frac{\sigma^2\lambda^2(1-\lambda^2)(1-2\theta)^2}{2\Delta}, \tag{12}$$

where $\bar{R} = \alpha + \Pi - \sigma^2 + \frac{1}{2}\left(\lambda^2\sigma^2 - \lambda\sigma v\right)$, $\bar{r} = \alpha - \sigma^2 + \frac{1}{2}\lambda^2\sigma^2$, and $\Delta = (1-\lambda^2(1-2\theta(1-\theta))) > 0$. Here $\bar{R}$ and $\bar{r}$ are the nominal returns on the home or foreign bond and the return on the real risk-free bond, respectively, at the point of equal national wealth levels (i.e. $\theta = 0.5$).

Using (10)-(12) in (6), we may derive the equilibrium portfolio holdings under nominal bond trade. The shares of wealth held in the home country technology, the home currency nominal bond, and the foreign currency nominal bond are written as:

$$\omega_T^h = \frac{1-\lambda^2(1-\theta)}{\Delta}, \tag{13}$$

$$\omega_h^h = \frac{\sigma\theta\lambda(1-\lambda^2(1-\theta))}{v\Delta}, \tag{14}$$

$$\omega_f^h = -\frac{\sigma\theta\lambda(1-\lambda^2\theta)}{v\Delta}. \tag{15}$$

From (13)-(15), the share of wealth invested in the real risk-free bond is:

$$1 - \omega_T^h - \omega_h^h - \omega_f^h = \frac{\theta\lambda^2(1-2\theta)(v+\sigma\lambda)}{v\Delta}. \tag{16}$$

It is clear that these portfolio shares will be time-varying in response to changes in θ, even though returns on the real technologies and trend inflation rates are identical across countries. Note also that when $\theta = 0.5$, $\omega_T^h = 1$, and $\omega_h^h = -\omega_f^h = \frac{\sigma\lambda}{2v}$.

We summarize the results of (10)-(16) in the following propositions:

Proposition 1 *In the equilibrium with trade in nominal bonds,*

a) The risk-free real interest rate lies between the autarky level and the complete markets level,

b) For $\lambda < 0$, each country holds a short position in its own-currency nominal bonds, and a long position in the other currency nominal bonds.

c) The home country holds a positive (negative) share in real risk-free bonds for $\theta < 0.5$ ($\theta > 0.5$),

d) Let $\rho(\theta) = R_h(\theta) - R_f(\theta)$ be defined as the risk-premium on home-currency relative to foreign-currency nominal bonds. Then when $\lambda < 0$, $\rho(\theta)$ is positive (negative) for $\theta < 0.5$, ($\theta > 0.5$).

e) The home country has a positive (negative) net foreign asset (NFA) position as $\theta < 0.5$, ($\theta > 0.5$).

When $\lambda > 0$, the opposite statements apply in parts b) and d) of the proposition. That is, the home country holds a long (short) position in home (foreign) currency bonds, and $\rho(\theta)$ is negative (positive) for $\theta < 0.5$ ($\theta > 0.5$).

Proof. a) Direct inspection establishes that

$$r(\theta) = r^A + \frac{\sigma^2\lambda^2\theta(2-\lambda^2)(1-\theta)}{\Delta} > r^A$$

$$r(\theta) = r^C - \frac{\sigma^2(1+2\lambda^2\theta(1-\theta))(1-\lambda^2)}{2\Delta} < r^C.$$

where $r^A = \alpha - \sigma^2$ and $r^C = \alpha - 0.5\sigma^2$ represent the autarky and complete markets real risk-free rate, respectively.

b) Follows directly from (14) and (15).

c) Follows directly from (16).

d) From (10) and (11), we have $\rho(\theta) = \frac{\sigma\lambda\nu(1-2\theta)(1-\lambda^2)}{\Delta} > 0$ (< 0) as $\theta < 0.5$ (> 0.5).

e) From (14)-(16), we have $1 - \omega_T^h = \frac{\theta\lambda^2(1-2\theta)}{\Delta} > 0$ (< 0) as $\theta < 0.5$ (> 0.5). ■

3.3.2 Portfolio Diversification

It is clear from the proposition that an equilibrium with nominal bond trade allows for resource transfers across countries. The reason is that home and foreign currency nominal bonds have different characteristics with respect to hedging consumption risk for home and foreign consumers. When $\lambda < 0$, the home currency bond tends to have a high real return when returns on the home technology are high, and it thus represents a relatively bad hedge against home consumption risk. But the foreign currency bond represents a relatively good hedge against home consumption risk. On the other hand, for the foreign household, in the case $\lambda < 0$, the home currency bond is a good hedge and the foreign currency bond a bad hedge. In an equilibrium with bond trade, home households will thus sell home currency bonds in return for foreign currency bonds, leading to the portfolio position described in (14) and (15). This portfolio position allows the home country to receive a relatively high portfolio return when there is a positive shock to the foreign technology, and vice versa.

First focus on the point $\theta = 0.5$ (we show later this is the long run modal point of θ), where each country will have a zero net external asset position. But the gross external assets will comprise a positive position in one currency's bond, balanced by a negative position in the other currency's bond. Thus, for the home country, in the case of $\theta = 0.5$, we have $\omega_h^h = \frac{\sigma\lambda}{2\nu}, \omega_f^h = -\frac{\sigma\lambda}{2\nu}$. The absolute positions are higher, the greater the volatility of the productivity shock, and lower, the greater is the volatility of prices.

The risk-sharing from portfolio diversification in nominal bonds reduces the volatility of consumption, and increases welfare. This is reflected in a higher real risk-free interest rate. At the point $\theta = 0.5$, the risk-free interest rate is $\bar{r} = \alpha - \sigma^2 + \frac{1}{2}\lambda^2\sigma^2$. This is closer to the complete markets value, the closer is λ to unity in absolute value, since the higher is $|\lambda|$, the better are nominal bonds as a hedge against consumption risk due to productivity shocks. The risk-adjusted consumption growth rate at $\theta = 0.5$ is written as $\alpha - \rho - \frac{1}{2}\sigma^2\left(1 - \frac{\lambda^2}{2}\right)$.When $\lambda = 0$, this is identical to that under autarky, while as $|\lambda| \to 1$, trading in nominal bonds alone attains the complete markets growth rate.

3.3.3 Capital Flows

We now wish to explore the implications of the model for capital flows. Net capital flows (or intertemporal trade) occur when the changes in a country's gross bond holdings do not sum to zero. The sum of wealth in the two countries is equal to the world capital stock, since capital is the only outside asset in the world economy. If portfolio diversification could sustain the complete markets allocation, then there would be no change in relative wealth across the two countries, and each country would maintain a constant share of the world capital stock. But because nominal bond trade cannot achieve the complete markets equilibrium, productivity shocks in one country will have a larger impact on that country's wealth than on the wealth of the other country. These changes in relative wealth levels give rise to net capital flows across countries.

Differentiating equations (13) - (15) at $\theta = 0.5$, we see that a rise in θ has the following effect on the home country's portfolio:

$$\left.\frac{d\omega_h^h}{d\theta}\right|_{\theta=0.5} = \frac{2\lambda\sigma}{v(2-\lambda^2)}, \tag{17}$$

and

$$\left.\frac{d\omega_f^h}{d\theta}\right|_{\theta=0.5} = -\frac{2\lambda\sigma(1-\lambda^2)}{v(2-\lambda^2)}. \tag{18}$$

When $\lambda < 0$, the first expression is negative, and the second is positive. Hence, beginning at $\theta = 0.5$, a rise in foreign relative wealth will be followed by a rise in home gross borrowing in home currency bonds, and a rise in gross lending in foreign currency bonds. In addition however, there is also a change in the holdings of *real* bonds. From (15) we have:

$$\left.\frac{d(1-\omega_h^h-\omega_f^h-\omega_T^h)}{d\theta}\right|_{\theta=0.5} = -\frac{2\lambda^2(v+\lambda\sigma)}{v(2-\lambda^2)}. \tag{19}$$

This is always negative, so that as the home country becomes an overall net debtor, it will issue sell real risk free bonds to the foreign country.

From Proposition 1, we know that the sum of (17)-(19) is less than zero. That is, while at $\theta = 0.5$, the home country has a diversified portfolio but a zero net external balance, as the foreign country becomes larger in terms of world wealth, the home country becomes a recipient of foreign capital inflows.

With trade in real bonds alone, there are no international capital flows at all. How does the presence of nominal bonds generate capital flows? The key feature is the interaction between changes in nominal bond returns and *gross* bond holdings.

From the solutions for R_h and R_f, we find that:

$$\left.\frac{dR_h}{d\theta}\right|_{\theta=0.5} = 2\frac{\lambda\sigma v(1-\lambda^2)}{2-\lambda^2},$$

and

$$\left.\frac{dR_f}{d\theta}\right|_{\theta=0.5} = -2\frac{\lambda\sigma v(1-\lambda^2)}{2-\lambda^2}.$$

The first expression is negative, while the second is positive, for $\lambda < 0$. Thus, a rise in the share of world wealth for the foreign country drives down the return on home currency bonds, while pushing up the return on foreign currency bonds. Intuitively, as the foreign country increases its wealth, its portfolio preferences dominate the global bond markets. It increases its demand for home currency bonds, while increasing its supply of foreign currency bonds. This is reflected in the movements in the returns on nominal bonds.

The gross portfolio position, when combined with the evolution of returns that are driven by relative wealth dynamics, allows for gains from intertemporal trade in the economy with nominal bonds, even though there are no gains when only real bonds can be traded. To see the intuition, take the position $\theta = 0.5$, where the two countries have exactly equal net wealth, and given the symmetry in the model, the current account of each country is zero. Say that there is a rise in W_f, driven for instance by a positive technology shock in the foreign country. This will raise θ. If there were trade only in a real risk-free bond, this would simply permanently increase the foreign country's expected consumption, and have no impact at all on the home country. But with trade in nominal bonds, the rise in θ leads to a fall in R_h and rise in R_f (in the case $\lambda < 0$). This reduces the effective cost of borrowing for the home country, leading it to a higher net foreign debt, higher investment in the domestic technology, and a higher level of wealth and consumption. In this manner, the original positive technology shock in the foreign economy is shared by the home economy. Moreover, we see that there is an essential interrelationship between *net* *capital flows* and *gross portfolio holdings*. As the home country receives capital inflows when $\theta > 0.5$, it simultaneously increases its borrowing in its own currency, and lending in the foreign currency. This levered portfolio ensures that its overall cost of borrowing is lowered, facilitating net capital inflows.

3.3.4 Complementarity between Nominal Bonds and Real Bonds

These results also reveal an interesting feature of the coexistence of real and nominal bonds. Without nominal bonds, there are no gains from trade in risk-free bonds. But in the nominal bond trading equilibrium, a country experiencing net capital inflows will borrow partly in real bonds. Why is it that risk-free bonds are traded *simultaneously* with nominal bonds? The answer is that, given incomplete markets, consumption growth is not equalized across countries, and the implicit equilibrium risk-free real rates would differ across countries, if there were no trade in real risk free bonds. This gives rise to gains from pure intertemporal trade, as in conventional models of the current account. That is, when $\theta > 0.5$, the foreign country is temporarily wealthier than the home country. Without trade in a risk-free bond, it would have a lower risk-free rate than the home country, indicating that it would wish to save some of the temporarily higher wealth if an international risk-free bond market were opened. Likewise, in the same circumstance, the home country would wish to borrow as its wealth is temporarily below its long run mean. Thus, there are gains from pure borrowing and lending, when nominal bonds are traded.

One element is missing from the above logic, however. In the example, how do we know that the higher foreign country wealth is only temporary? For this to be true, movements in θ away from $\theta = 0.5$ must be self-correcting. We address this in the next section.

4 Endogenous θ: A Stationarity Result

So far, we have described θ as a shift variable. But the evolution of θ is determined by endogenous movements in relative wealth levels, driven by productivity and price level shocks in each country. A fundamental question is whether the wealth distribution is stable. Thus, while a shock which generates a rise in θ will lead the foreign country to accumulate net claims on the home country, will the rise in θ be self-correcting? For this to be the case, it must be that home wealth grows faster than foreign wealth, when $\theta > 0.5$.

Using Ito's lemma and equation (4), we may write the diffusion process governing θ as:

$$d\theta = \theta\,(1-\theta)\,F(\theta)dt + \theta(1-\theta)G(\theta)dB, \tag{20}$$

where the functional forms of $F(\theta)$, $G(\theta)$, and dB are described in the Appendix. The asymptotic distribution of θ must satisfy either; (a) $\theta \to 1$, (b) $\theta \to 0$, or (c) θ follows a stable distribution in $(0,1)$. Given the form of (20), clearly $\theta = 1$ and $\theta = 0$ are absorbing states. But the following proposition establishes the conditions under which (c) will apply.

Proposition 2 *For $\lambda \neq 0$, θ has a symmetric ergodic distribution in $(0,1)$ centered at $\theta = \frac{1}{2}$.*
Proof. *See Appendix.* ■

The content of this proposition is illustrated through the effect of θ on risk-adjusted growth rates of wealth. The risk-adjusted growth rate for country i as:

$$g_i(\theta) = \lim_{\Delta t \to 0} E_t \left[\frac{\Delta \ln W_i(t+\Delta t)}{\Delta t} \right] = \lim_{\Delta t \to 0} \frac{E_t\left(\frac{\Delta W_i(t+\Delta t)}{W_i(t)}\right) - \frac{1}{2} Var_t \left(\frac{\Delta W_i(t+\Delta t)}{W_i(t)}\right)}{\Delta t}.$$

Then, θ has an ergodic distribution if it cannot access the boundaries 0 or 1. Defining the difference between the foreign and home risk-adjusted growth rate as $\delta(\theta) = g_f(\theta) - g_h(\theta)$, this property holds if the probability of reaching either is zero. For the lower bound, this is the case if $\delta(0) > 0$. Likewise, the probability of reaching the upper bound is zero if $\delta(1) < 0$. This just says that as the home country gets arbitrarily wealthy, relative to the foreign country, the foreign country's risk-adjusted growth rate exceeds that of the home country. Likewise, if the foreign country's wealth increases arbitrarily relative to that of the home country, then the home risk-adjusted growth rate will exceed that of the foreign country. Proposition 2 establishes that, for $\lambda \neq 0$, this property always holds.

We may show this directly by computing $\delta(\theta)$. The Appendix shows that $\delta(\theta)$ may be written as:

$$\delta(\theta) = \frac{\sigma^2 \lambda^2 (1-2\theta)\left[1 - \frac{3}{2}\lambda^2(1-\frac{2}{3}\lambda^2)\right]}{\left[1-\lambda^2\left(1-2\theta(1-\theta)\right)\right]^2}. \tag{21}$$

The denominator is always positive, and since $1 > |\lambda| \neq 0$, the numerator is positive (negative) for $\theta < 0.5$ (> 0.5). Moreover, this satisfies the conditions

$$\delta(0) = \frac{\sigma^2 \lambda^2 \left[1 - \frac{3}{2}\lambda^2(1-\frac{2}{3}\lambda^2)\right]}{\left[1-\lambda^2\left(1-2\theta(1-\theta)\right)\right]^2} > 0,$$

$$\delta(1) = -\frac{\sigma^2 \lambda^2 \left[1 - \frac{3}{2}\lambda^2(1-\frac{2}{3}\lambda^2)\right]}{\left[1-\lambda^2\left(1-2\theta(1-\theta)\right)\right]^2} < 0,$$

and $\delta(0.5) = 0$.

Hence, for $\theta > 0.5$, when the foreign country is relatively wealthy, the home risk-adjusted growth rate exceeds that of the foreign country, and θ falls. The same dynamics occur in reverse when $\theta < 0.5$. The expressions also make clear that the distribution of θ is symmetric. Thus, θ converges towards 0.5 from either direction.

Stationarity is ensured whether λ is positive or negative. In either case, agents can make use of nominal bonds to hedge internationally against consumption risk, holding short the home (foreign) currency bond if $\lambda < 0$, and conversely if $\lambda > 0$. But if $\lambda = 0$, then nominal bond returns are independent of consumption risk in either country, and they will not be held in equilibrium (i.e. $\omega_j^i = 0$, for all i and j). In this case, the stationarity result fails.

The stationarity of the world wealth distribution is linked to the composition of gross portfolio holdings and the dynamics of nominal returns. Take the case $\lambda < 0$. Then when $\theta > 0.5$, the foreign country's demand for home currency bonds and supply of foreign currency bonds pushes down R_h and pushes up R_f, leading to a lower cost of borrowing for the home country. The home currency bond returns (the home country's liabilities) will approach $\alpha + \Pi - \sigma^2$, and foreign currency bond returns (the home country's assets) approach $\alpha + \Pi - \sigma^2 - v\lambda\sigma$, as $\theta \to 1$. Since the expected return on the domestic technology exceeds that on its nominal asset portfolio, this increases the risk-adjusted expected growth rate for the home country, relative to the foreign country. As a result, θ is driven back towards 0.5 again. In effect, it is the levered portfolio composition and its implication for the net borrowing costs for the debtor country as the wealth distribution evolves that ensures the stability of the wealth distribution itself. Current account imbalances are naturally self-correcting when agents hold an optimal currency portfolio of international debt.

While this interpretation is based on a negative value of λ, this is not necessary for the stability result. If $\lambda > 0$, then the equivalent stabilizing force takes place, but now with the foreign country holding positive (negative) amounts of foreign (home) currency bonds. Stability is ensured because it is always the case that countries hold a gross portfolio such that their cost of borrowing falls as the rest of the world gets wealthier.

If $\lambda = 0$, the portfolio composition is indeterminate, since bonds can then play no role as a hedge against technology risk. In fact, agents will hold no bonds at all. Since technologies are identical, there can be no gains from trade in international bonds at all. Any innovations to wealth are permanent. Clearly then the wealth distribution will not be stationary. In fact, θ will be characterized by hysteresis. Technology shocks will give rise to an expected permanent increase in wealth without international asset trade at all.

4.1 Figures

As demonstrated in the Appendix, the model allows for an explicit solution for the distribution of wealth. Figure 1 illustrates the distribution of $\ln(\frac{W_f}{W_h}) = \ln(\frac{\theta}{1-\theta})$ for different values of

λ with $\sigma = v = 0.02$ [12]. In each case, the distribution has zero mean. But the λ parameter matters substantially for the shape of the distribution. For high absolute values of λ, the distribution is tightly centered around zero. But as λ falls in absolute value, the distribution becomes substantially spread out. This means that the speed of convergence in the wealth distribution depends critically on the size of λ. For high absolute values of λ, convergence is much faster.

Figure 2 and 3 illustrate the implications of the model for the return on real risk-free bonds, and nominal bonds, respectively, for different values of θ. Here we use a value of $\lambda = -0.4$[13]. The real risk-free rate is highest at a value of $\theta = 0.5$. Figure 4 and 5 illustrate the net nominal and real bond positions, and gross home and foreign bond positions, respectively, for various values of θ.

4.2 The relationship between returns and net foreign assets

Proposition 2 relies on the property of the model that debtor countries will face lower real returns on their nominal liabilities than they receive on their nominal assets. Empirical evidence for this relationship is presented by Gourinchas and Rey () and Lane and Milesi Ferretti (). Gourinchas and Rey (2006) find that a rise in the US trade deficit generates a predictable fall in the return on US external liabilities, through a predictable depreciation in the exchange rate. Lane and Milesi Ferretti show that the ratio of US net foreign assets to GDP has a significantly positive effect on the excess return received by the rest of the world on US investments.

In our model, the excess return on home currency bonds is given by $\rho(\theta) = -\frac{\sigma\lambda\nu(1-2\theta)(1-\lambda^2)}{\Delta}$. The ratio of home country net foreign assets to GDP is $NFA = \frac{(1-\theta)(1-2\theta)\theta\lambda^2}{1-\lambda^2(1-\theta)}$. For $\lambda < 0$, there will be a negative relationship between the two. Figure 6 illustrates a time series as a function of random fluctuations in θ. Clearly we see that a higher NFA will be associated with a higher excess return on home currency bonds[14].

[12] The reason for adopting $\ln \frac{W_f}{W_h}$ instead of $\frac{W_f}{W_h+W_f}$ is that the former definition can illuminate the behavior at tails of wealth distribution with a support of $(-\infty, +\infty)$ rather than $(0, 1)$.

[13] Devereux and Saito (2006) estimate values of λ between -0.7 and -0.1. Kydland and Prescott (1995) and Cooley and Hansen (1995), for US data, estimate λ approximately -0.6.

[14] This example is based on $\sigma = v = 0.1$, and $\lambda = -.4$. Given the simplying assumptions of the model (e.g. log utility etc.), it is difficult to replicate the size of the excess returns seen in the data, however.

5 Extensions of the basic model

5.1 No Trade in Real Bonds

In the exposition of the model, we assumed that countries could trade in both real and nominal bonds. In reality, almost all international bond trade is carried out with nominal bonds. If we restrict the model so that only nominal bonds are traded, the essential results are unchanged. To solve the model in this case, we impose (7) and (8) in combination with the restrictions of zero supply or real bonds within each country; so that $\omega_T^h + \omega_h^h + \omega_f^h = 1$, and $\omega_T^f + \omega_h^f + \omega_f^f = 1$. The solutions for portfolio holdings are written as:

$$\omega_h^h = \frac{\theta\lambda\sigma(v^2 + 2\sigma^2 + 2\lambda\sigma v + \lambda\sigma v(1-2\theta))}{v\left(v^2 + 2\sigma^2 + 2\lambda\sigma v - \sigma^2\lambda^2(1-4\theta(1-\theta))\right)}, \tag{22}$$

and

$$\omega_f^h = -\frac{\theta\lambda\sigma(v^2 + 2\sigma^2 + 2\lambda\sigma v - \lambda\sigma v(1-2\theta))}{v\left(v^2 + 2\sigma^2 + 2\lambda\sigma v - \sigma^2\lambda^2(1-4\theta(1-\theta))\right)}. \tag{23}$$

The parameter assumptions made already imply that $\omega_h^h < 0$ (> 0), as $\lambda < 0$, (> 0), and $\omega_f^h > 0$ (< 0), as $\lambda < 0$, (> 0). Thus, part b) of Proposition 1 applies as before. Then adding (22) and (23) together we get

$$\omega_h^h + \omega_f^h = -\frac{2\theta(2\theta-1)\lambda^2\sigma^2}{v^2 + 2\sigma^2 + 2\lambda\sigma v - \sigma^2\lambda^2 + 4\sigma^2\lambda^2\theta(1-\theta)}, \tag{24}$$

which establishes the equivalent of part e) of proposition 1. The risk premium on home currency bonds may now be written as:

$$\rho(\theta) = -\frac{(v^2 + 2\sigma^2 + 2\lambda\sigma v - \sigma^2\lambda^2)(1-2\theta)\lambda\sigma v}{v^2 + 2\sigma^2 + 2\lambda\sigma v - \sigma^2\lambda^2 + 4\sigma^2\lambda^2\theta(1-\theta)}. \tag{25}$$

Again, this is negative (positive) as $\lambda < 0$, $(\lambda > 0)$, for $\theta > 0.5$, and vice versa. Hence, part c) of Proposition 1 holds as before. It is possible also to show that part a) for proposition 1 holds for the individual risk-free rates of return on real bonds within each country.

The only difference between this case and the benchmark model above is that all trade must be intermediated by nominal bonds. As a country experiences capital inflows, these must be all financed by issuing domestic currency bonds (for $\lambda < 0$), but hedged by also purchasing foreign currency bonds. Again, the risk-premium evolves so that the return on gross liabilities of a debtor country are below the return it receives on its gross assets.

Since the movements in the risk-premium are qualitatively as before, the stationarity result of Proposition 2 holds in the same way as before. The Appendix shows that $\delta(\theta)$ for this case may be written as:

$$\delta(\theta) = \frac{\lambda^2\sigma^2(1-2\theta)(v^2+2\lambda\sigma v+2\sigma^2-\sigma^2\lambda^2)(v^2+2\lambda\sigma v+2\sigma^2)}{(4\sigma^2\theta\lambda^2(\theta-1)+\sigma^2\lambda^2-2\lambda\sigma v-2\sigma^2-v^2)^2}.$$

This satisfies $\delta(0.5) = 0$, and $\delta(\theta) > 0$ (< 0) for $\theta < 0.5$ (> 0.5), as long as $|\lambda| \neq 0$. Thus, as before, the relatively poorer country will be a net foreign debtor, but grows faster than the richer country, ensuring a stable distribution of world wealth.

5.2 Substitutability between Nominal Bonds and Equity Trade

This subsection relaxes the assumption that shares in the national production technologies are non-tradable across countries. We extend the model to allow for trade in bonds and partial trade in shares in production technologies (or equities). If we kept the model as before, and equities were freely traded, then financial markets would be complete, and each country would hold a perfectly diversified equity portfolio with half their portfolio in home equity and half in foreign equity. But it is reasonable to assume that there is some part of each country's production technology that is not traded internationally[15]. Assume now that there are two linear production technologies in each country. As before, the technology described by (1) is not tradable internationally. But there another technology characterized as

$$\frac{dQ_i^E}{Q_i^E} = \beta dt + \epsilon dB_i^E, \tag{26}$$

for $i = h$ or f, where dB_i^E is an increment to the standard Weiner process uncorrelated with dB_i, and correlated with dM_i with the coefficient λ [16].

Assume that shares in the technology described by (26) are tradable in each country. Now, in addition to investments in its own non-tradable technology ω_T^h (ω_T^f), the home (foreign) country can invest in its own tradable technology with a portfolio weight ω_{Th}^h (ω_{Tf}^f), and the tradable technology of the foreign (home) country with a portfolio weight ω_{Tf}^h (ω_{Th}^f).

Holding a share in technology in this model is equivalent to making a direct investment in the production technology. Thus, we impose a restriction that investors can not take a short position on the tradable technologies. Then, the following portfolio restrictions must be satisfied for both countries: $\omega_T^h + \omega_{Th}^h + \omega_{Tf}^h + \omega_h^h + \omega_f^h = 1$ with $\omega_T^h \geq 0$, $\omega_{Th}^h \geq 0$, and $\omega_{Tf}^h \geq 0$, and $\omega_T^f + \omega_{Tf}^f + \omega_{Th}^f + \omega_f^f + \omega_h^f = 1$ with $\omega_T^f \geq 0$, $\omega_{Tf}^f \geq 0$, and $\omega_{Th}^f \geq 0$.

To illustrate the properties of this extended model, we make the additional assumptions;

[15] Evidence for this is presented in Stulz and Warnock (2006).

[16] An indirect correlation between dB_i^E and dB_i does not show up at the variance-covariance matrix because it converges to zero as $\Delta t \to 0$ by a higher-order effect.

$\alpha = \beta$, and $\sigma = \epsilon = \nu$. These assumptions are not essential, but help to simplify the exposition.

The model may be solved in the same manner as before. The portfolio holdings and returns depend on the state variable θ. The model is still entirely symmetric, so that countries have zero NFA at $\theta = 0.5$. The nominal bond portfolio for the home country at $\theta = 0.5$, is given by $\omega_h^h = -\omega_h^f = \frac{\lambda}{2(3-4\lambda^2)}$. As in the previous case without direct trade in shares of the production technologies, the home (foreign) country still takes a short position in the home currency bond, and a long position in the foreign currency bond, when λ is negative, and $\lambda^2 \leq \frac{1}{2}$. The condition $\lambda^2 \leq \frac{1}{2}$ now defines the range of λ for which markets are effectively incomplete (see below).

At the point $\theta = 0.5$, shares in the home non-tradable technology, and the home and foreign tradable technolgies are given by $\omega_T^h = \frac{1-\lambda^2}{3-4\lambda^2}$, $\omega_{Th}^h = \frac{1-\lambda^2}{3-4\lambda^2}$, and $\omega_{Tf}^h = \frac{1-2\lambda^2}{3-4\lambda^2}$. These are all non-negative under the condition $\lambda^2 \leq \frac{1}{2}$. In addition,we confirm that $\omega_T^h + \omega_{Th}^h + \omega_{Tf}^h = 1$ in this case, so NFA is indeed equal to zero.

Under this parameterization and $\theta = 0.5$, the home and foreign bond market clearing condition determine equilibrium interest rates equal to $R_h = R_f = \alpha + \Pi - \frac{\left[2+\lambda(3-3\lambda-4\lambda^2)\right]\sigma^2}{6-8\lambda^2}$, while the corresponding risk-free rates (r_h and r_f) are equal to $\alpha - \frac{(2-3\lambda^2)\sigma^2}{6-8\lambda^2}$. At the lower limit of $\lambda^2 = 0$, nominal bonds play no role in hedging consumption risk, and an equilibrium is characterized by each country dividing its wealth equally over the three technologies (the domestic non-tradable and two tradable technologies). The equilibrium risk-free rate is then equal to $\alpha - \frac{1}{3}\sigma^2$. As long as λ is non-zero, nominal bond trading can still play an effective role in sharing country-specific shocks, even when there is tradable equity[17].

Again, θ has a stationary distribution in $(0, 1)$ centered at $\theta = 0.5$. As before, define $\delta(\theta) = g_f(\theta) - g_h(\theta)$. It is possible to show that $\delta(0.5) = 0$, and

$$\delta(1) = -\delta(0) = -\frac{(2 - 3\lambda^2)\sigma^2\lambda^2}{18\left[1 - 3\lambda^2(1 - 2/3\lambda^2)\right]} < 0,$$

as long as $0 < \lambda^2 \leq \frac{1}{2}$. Therefore, when a country's share of world wealth falls, its relative growth rate increases, ensuring stationarity of θ. Despite the ability to trade equity, the underlying force behind the stationarity condition is the presence of nominal bond trading, just as in the previous case.

We saw above that trade in nominal bonds was complementary to trade in real risk free bonds. But nominal bonds may substitutable for trade in equity. Even in the case

[17] At the upper limit of $\lambda^2 = \frac{1}{2}$, on the other hand, the equilibrium risk-free rate reaches $\alpha - \frac{1}{4}\sigma^2$, which is equivalent to the risk-free rate in the complete markets case where each country divides its wealth equally over the two domestic and two foreign technologies.

where equity is freely tradable, cross country equity holdings may be small. In particular, as $\lambda^2 \to \frac{1}{2}$, we find that $\omega^h_{Tf} \to 0$, $\omega^h_T \to \frac{1}{2}$, and $\omega^h_{Th} \to \frac{1}{2}$. Thus, as the nominal bond markets become more proficient at risk sharing, the direct holding of foreign equity goes to zero, and home agents hold 100 percent of the home technologies (both non-tradable and tradable). Thus, although direct trade in equity is possible, the portfolio equilibrium is characterized by complete *home bias* in equity holdings. The intuitive reason for this is that the bond portfolio held by residents of each country represents a perfect claim on the foreign technology in the case when $\lambda^2 \to \frac{1}{2}$. Thus, our initial assumption that there is no trade in equity becomes an equilibrium outcome, the better the risk-sharing characteristics of nominal bonds.

5.3 One-way Capital Flows

It is widely recognized that many countries can not or do not issue debt denominated in their own currency (e.g. Eichengreen and Hausmann 2000). In fact, much of the nominal debt traded internationally is denominated in US dollars. We now briefly look at another special case of the model which captures this phenomenon. We restrict all trade in nominal bonds to take place in the home currency only. Even if $\lambda < 0$, the foreign country cannot issue its own currency debt.

For simplicity assume that there is no trade in risk-free real bonds. Optimal portfolio rules (ω^h_T, ω^h_h, ω^f_T, and ω^f_h) are still determined by a version of equation (6) with $\lambda_h = \lambda < 0$ for the home country choice and $\lambda_f = 0$ for the foreign country. Then, two portfolio restrictions ($\omega^h_T + \omega^h_h = 1$, $\omega^f_T + \omega^f_h = 1$) and a bond market clearing ($(1-\theta)\omega^h_h + \theta\omega^f_h = 0$) may be used to determine the equilibrium nominal interest rates on home currency bonds (R_h), and the risk-free real interest rate on implicit (non-traded) real bonds (r_h and r_f). The home country's holding of home bonds is given by

$$\omega^h_h = \frac{\lambda\theta\upsilon\sigma}{2\lambda\theta\upsilon\sigma + \upsilon^2 + \sigma^2}. \tag{27}$$

As before, the home country has a negative position in home currency bonds, when $\lambda < 0$. The difference now however is that (27) represents both the *gross* and *net* bond position of the home country. When $\lambda < 0$, the home country *always* has a negative net foreign asset position. The international capital market is asymmetric in structure. To hedge against domestic consumption risk, the home country wishes to issue domestic denominated bonds. The foreign country is willing to purchase these bonds because their returns are uncorrelated with foreign technology shocks.

To further gain insight into this example, restrict attention to the case $\upsilon = \sigma$. Then, the nominal interest rate on the home currency bond is:

$$R_h = \alpha + \pi - \frac{\sigma^2(1+\lambda)}{1+\lambda\theta}.$$

This is declining in θ, for $\lambda < 0$. Thus, the return on home net foreign liabilities falls as the foreign country gets relatively wealthier. This ensures that the same stationarity condition holds as before. We may calculate the $\delta(\theta) = g_f(\theta) - g_h(\theta)$ function as follows:

$$\delta(\theta) = \frac{1}{4}\sigma^2\lambda^2\frac{(1-\theta(2+\lambda\theta))}{(1+\lambda\theta)^2}. \tag{28}$$

This satisfies the conditions $\delta(0) > 0$, and $\delta(1) < 0$, for $|\lambda| \neq 1$. But unlike the symmetric economy, we now have $\delta(0.5) = -\frac{1}{4}\frac{\sigma^2\lambda^3}{(2+\lambda)^2}$, which is positive for $\lambda < 0$. Thus, the long run wealth share is not equalized across countries. We may use (28) to establish that the unconditional mode of θ is $\theta = \frac{-1+\sqrt{1+\lambda}}{\lambda}$, which exceeds 0.5 for $\lambda < 0$. If the home country is a net foreign debtor in its own currency, then the long run distribution of world wealth is skewed in favor of the foreign country. Moreover, the higher in absolute value is λ, the higher the foreign country's long run share of world wealth. Unlike the symmetric world economy where bonds of either currency can be traded internationally, in the case where only a single currency bond is acceptable, the debtor country achieves risk sharing only by accepting a lower and lower share of world wealth.

5.4 Welfare Evaluation

The model implies that an economy with nominal bonds achieves a degree of risk-sharing somewhere between that of capital market autarky (zero risk-sharing) and complete markets (full risk-sharing). Figure () presents an evaluation of welfare levels for different values of λ and different initial starting values of θ. Here we measure welfare by the unconditional mean of risk-adjusted wealth (or consumption) growth. It is clear that the risk sharing possibitities of nominal bonds are highly sensitive to the absolute value of λ. The larger is λ in absolute value, the closer is the mean risk-adjusted growth rate is to that of the complete markets.

Appendix

Process of Wealth Distribution θ

To obtain the process of wealth distribution θ $(=\frac{W_f}{W_h+W_f})$, we define $m_h(\theta) = \lim_{\Delta t\to 0}\frac{E_t\left[\frac{\Delta W_h(t+\Delta t)}{W_h(t)}\right]}{\Delta t}$, $m_f(\theta) = \lim_{\Delta t\to 0}\frac{E_t\left[\frac{\Delta W_f(t+\Delta t)}{W_f(t)}\right]}{\Delta t}$, $n_h(\theta) = \lim_{\Delta t\to 0}\frac{Var_t\left[\frac{\Delta W_h(t+\Delta t)}{W_h(t)}\right]}{\Delta t}$, $n_f(\theta) = \lim_{\Delta t\to 0}\frac{Var_t\left[\frac{\Delta W_f(t+\Delta t)}{W_f(t)}\right]}{\Delta t}$, and $n_{hf}(\theta) = \lim_{\Delta t\to 0}\frac{Cov_t\left[\frac{\Delta W_h(t+\Delta t)}{W_h(t)},\frac{\Delta W_f(t+\Delta t)}{W_f(t)}\right]}{\Delta t}$. Then, using Ito's lemma, we can derive the process of wealth distribution θ $(=\frac{W_f}{W_h+W_f})$ as

$$d\theta = \theta(1-\theta)\left[F(\theta)dt + G(\theta)dB\right], \tag{29}$$

where

$$F(\theta) = m_f(\theta) - m_h(\theta) - \theta n_f(\theta) + (1-\theta)n_h(\theta) + (2\theta-1)n_{hf}(\theta),$$
$$G(\theta) = \sqrt{n_h(\theta) + n_f(\theta) - 2n_{hf}(\theta)},$$

and

$$\begin{aligned} dB &= \frac{1}{G(\theta)}\left[\omega_T^f(\theta)\sigma dB_f - \omega_f^f(\theta)v dM_f - \omega_h^f(\theta)v dM_h\right] \\ &\quad -\frac{1}{G(\theta)}\left[\omega_T^h(\theta)\sigma dB_h - \omega_h^h(\theta)v dM_h - \omega_f^h(\theta)v dM_f\right]. \end{aligned}$$

$dB(t)$ is newly defined as the increment to a standard Brownian motion. Note here that

$$\lim_{\Delta t\to 0}\frac{E_t\left[\Delta B(t+\Delta t)\right]}{\Delta t} = 0, \quad \lim_{\Delta t\to 0}\frac{Var_t\left[\Delta B(t+\Delta t)\right]}{\Delta t} = 1.$$

Stationarity of Wealth Distribution θ

To make theorems 16 and 18 of Skorohod (1989) applicable, we consider the process of κ or $\ln\frac{\theta}{1-\theta}$ $(=\ln\frac{W_f}{W_h})$ instead of θ. The process of κ is derived as

$$d\kappa = \delta(\theta)dt + G(\theta)dB, \tag{30}$$

where $\theta = \frac{\exp(\kappa)}{1+\exp(\kappa)}$, and $\delta(\theta) = g_f(\theta) - g_h(\theta)$. As defined in the main text, $\delta(\theta)$ represents the difference in risk-adjusted wealth growth between the two countries. Given equilibrium asset pricing characterized by equations (10) through (**??**), $\delta(\theta)$ is computed as

$$\delta(\theta) = \frac{\sigma^2\lambda^2(1-2\theta)(2-3\lambda^2+\lambda^4)}{2\left[1-\lambda^2\left(1-2\theta(1-\theta)\right)\right]^2}. \tag{31}$$

We then introduce the following integrals:

$$\begin{aligned} I_1 &= \int_{-\infty}^{0} \exp\left[-\int_0^w c(u(v))dv\right] dw, \\ I_2 &= \int_0^{\infty} \exp\left[-\int_0^w c(u(v))dv\right] dw, \end{aligned}$$

and

$$M = \int_0^{\infty}\left[\frac{2}{G(u(w))^2}\exp\left[\int_0^w c(u(v))dv\right]\right] dw,$$

where

$$c(u(v)) = \frac{2\delta(u(v))}{G(u(v))^2}, \tag{32}$$

and

$$u(v) = \frac{\exp(v)}{1+\exp(v)}.$$

According to the above theorems of Skorohod (1989), if $I_1 = \infty$, $I_2 = \infty$, and $M < \infty$, then κ has a unique ergodic distribution in $(-\infty, +\infty)$; accordingly, θ has a unique ergodic distribution in $(0, 1)$.

A function $c(\ )$ characterized by equation (32) plays a key role in determining stationarity of κ. Saito (1997) demonstrates that if $c(0) > 0$ and $c(1) < 0$, then κ (θ) has a unique ergodic distribution under some regulatory conditions. The process of κ or equation (30) always satisfies $c(0) > 0$ and $c(1) < 0$, because from equation (31),

$$\delta(0) = \frac{\lambda^2\sigma^2\left[(\lambda^2-1)^2+(1-\lambda^2)\right]}{2(1-\lambda^2)^2} > 0,$$

and

$$\delta(1) = -\frac{\lambda^2\sigma^2\left[(\lambda^2-1)^2+(1-\lambda^2)\right]}{2(1-\lambda^2)^2} < 0,$$

given finite $G(0)$ and $G(1)$.

Density Function of Wealth Distribution $\ln \frac{W_f}{W_h}$

According to Gihman and Skorohod (1972), given the process of κ $(= \ln \frac{W_f}{W_h})$ or equation (30), a density function of κ is derived as

$$\frac{2\mu}{G(u(\kappa))^2} \exp\left[\int_0^{\kappa} c(u(v))dv\right],$$

where μ is chosen such that $\mu \int_0^{\infty} \left[\frac{2}{G(u(w))^2} \exp\left[\int_0^{w} c(u(v))dv\right]\right] dw = 1$. Figure 1 depicts density functions of κ or $\ln \frac{W_f}{W_h}$ for $\lambda = -0.9$, -0.8, -0.5, and -0.3 when $\sigma = v = 0.02$.