2 Money-in-the-Utility Function

2.1 Introduction

The neoclassical growth model, due to Ramsey (1928) and Solow (1956), provides the basic framework for much of modern macroeconomics. Solow's growth model has just three key ingredients: a production function allowing for smooth substitutability between labor and capital in the production of output, a capital accumulation process in which a fixed fraction of output is devoted to investment each period, and a labor supply process in which the quantity of labor input grows at an exogenously given rate. Solow showed that such an economy would converge to a steady-state growth path along which output, the capital stock, and the effective supply of labor all grew at the same rate.

When the assumption of a fixed savings rate is replaced by a model of forward-looking households choosing savings and labor supply to maximize lifetime utility, the Solow model becomes the foundation for dynamic stochastic models of the business cycle. Productivity shocks or other real disturbances affect output and savings behavior, with the resultant effect on capital accumulation propagating the effects of the original shock over time in ways that can mimic some features of actual business cycles (see Cooley 1995).

The neoclassical growth model is a model of a nonmonetary economy, and while goods are exchanged and transactions must be taking place, there is no medium of exchange—that is, no "money"—that is used to facilitate these transactions. Nor is there an asset, like money, that has a zero nominal rate of return and is therefore dominated in rate of return by other interest-bearing assets. To employ the neoclassical framework to analyze monetary issues, a role for money must be specified so that the agents will wish to hold positive quantities of money. A positive demand for money is necessary if, in equilibrium, money is to have positive value.[1]

A fundamental question in monetary economics is the following: How should we model the demand for money? How do real economies differ from Arrow-Debreu economies in ways that give rise to a positive value for money? Three general approaches to incorporating money into general equilibrium models have been followed: (1) assume that money yields direct utility by incorporating money balances directly into the utility functions of the agents of the model (Sidrauski 1967); (2) impose transactions costs of some form that give rise to a demand for money, either

1. This is just another way of saying that we would like the money price of goods to be bounded. If the price of goods in terms of money is denoted by P, then 1 unit of money will purchase $1/P$ units of goods. If money has positive value, $1/P > 0$ and P is bounded ($0 < P < \infty$). Bewley (1983) refers to the issue of why money has positive value as the *Hahn problem* (Hahn 1965).

by making asset exchanges costly (Baumol 1952; Tobin 1956), requiring that money be used for certain types of transactions (Clower 1967), assuming that time and money can be combined to produce transaction services that are necessary for obtaining consumption goods, or assuming that direct barter of commodities is costly (Kiyotaki and Wright 1989); or (3) treat money like any other asset used to transfer resources intertemporally (Samuelson 1958). All involve shortcuts in one form or another; some aspects of the economic environment are simply specified exogenously in order to introduce a role for money. This can be a useful device, allowing one to focus attention on questions of primary interest without being unduly distracted by secondary issues. But our confidence in the ability of a model to answer the questions we bring to it is reduced if those aspects that are simply specified exogenously appear to be critical to the issue of focus. An important consideration in evaluating different approaches will be to determine whether conclusions generalize beyond the specific model or are dependent on the exact manner in which a role for money has been introduced. We will see examples of results that are robust, such as the connection between money growth and inflation, and others that are sensitive to the specification of money's role, such as the impact of inflation on the steady-state capital stock.

In this chapter, we develop the first of these three approaches by incorporating into the basic neoclassical model agents whose utility depends directly on their consumption of goods *and* their holdings of money.[2] Given suitable restrictions on the utility function, such an approach can guarantee that, in equilibrium, agents choose to hold positive amounts of money so that money will be positively valued. The money-in-the-utility function, or MIU, model we begin with is originally due to Sidrauski (1967), and it has been used widely to study a variety of issues in monetary economics.[3] The model can be used to examine the critical issues in monetary economics—the relationship between money and prices, the effects of inflation on equilibrium, and the optimal rate of inflation. To better understand the role of money in such a framework, a log-linear approximation for which analytic solutions can be derived is also studied. This allows us to calculate the macro time series behavior that the model implies. We can then determine whether the model is capable of generating the type of time series behavior we actually observe in macro-

2. The second approach, focusing on the transactions role of money, will be discussed in chapter 3. The third approach has been developed primarily within the context of overlapping-generation models; see Sargent (1987) or Champ and Freeman (1994).

3. Patinkin (1965, chapter 4) provides an earlier discussion of an MIU model, although he does not integrate capital accumulation into his model. However, the first order condition for optimal money holdings that he presents (see his equation 1, page 89) is equivalent to the one we will derive in the next section.

economic data, as well as assess the quantitative effects of inflation on the real economy.

2.2 The Basic MIU Model

To develop the basic MIU approach, we will ignore uncertainty and any labor-leisure choice, focusing instead on the implications of the model for money demand, the value of money, and the costs of inflation. Suppose that the utility function of the representative household takes the form

$$U_t = u(c_t, z_t),$$

where z_t is the flow of services yielded by money holdings and c_t is time t per capita consumption. Utility is assumed to be increasing in both arguments, strictly concave and continuously differentiable. The demand for monetary services will always be positive if we assume that $\lim_{z\to 0} u_z(c,z) = \infty$ for all c, where $u_z = \partial u(c,z)/\partial z$.

What constitutes z_t? If we wish to maintain the assumption of rational economic agents, then presumably what enters the utility function cannot just be the number of dollars (or yen or marks) that the individual holds. What should matter is the command over goods that are represented by those dollar holdings, or some measure of the transaction services, expressed in terms of goods, that money yields. In other words, z should be related to something like the number of dollars, M, times their price $(1/P)$ in terms of goods: $M(1/P) = M/P$. If the service flow is proportional to the real value of the stock of money, then we can set z equal to real per capita money holdings:

$$z_t = \frac{M_t}{P_t N_t} \equiv m_t.$$

To ensure that a monetary equilibrium exists, it is often assumed that, for all c, there exists a finite $\bar{m} > 0$ such that $u_m(c,m) \le 0$ for all $m > \bar{m}$. This means that the marginal utility of money eventually becomes negative for sufficiently high money balances. The role of this assumption will be made clear when the existence of a steady state is discussed later. It is, however, not necessary for the existence of equilibrium, and some common functional forms often employed for the utility function (and that will be used later in this chapter) do not satisfy this condition.[4]

4. For example, $u(c,m) = \log c + b \log m$ does not exhibit this property since $u_m = b/m > 0$ for all finite m.

Assuming that money enters the utility function is often criticized on the grounds that money itself is intrinsically useless (as with a paper currency), and that it is only through its use in facilitating transactions that it yields valued services. Approaches that emphasize the transactions role of money will be discussed in chapter 3, but as will be shown there, models in which money helps to reduce the time needed to purchase consumption goods can also be represented by the money in the utility function approach adopted in this chapter.[5]

The representative household is viewed as choosing time paths for consumption and real money balances subject to budget constraints to be specified below, with total utility given by

$$W = \sum_{t=0}^{\infty} \beta^t u(c_t, m_t), \tag{2.1}$$

where $0 < \beta < 1$ is a subjective rate of discount.

Equation (2.1) implies a much stronger notion of the utility provided by holding money than simply that the household would prefer having more money than less money. If the marginal utility of money is positive, then (2.1) implies that, *holding constant the path of real consumption for all t*, the individual's utility is increased by an increase in money holdings. That is, even though the money holdings are never used to purchase consumption, they yield utility. This should seem strange; we usually think the demand for money is instrumental in that we hold money to engage in transactions leading to the purchase of the goods and services that actually yield utility. All this is just a reminder that putting money in the utility function may be a useful shortcut for ensuring that there is a demand for money, but it is just a shortcut.[6]

To complete the specification of the model, assume that households can hold money, that bonds pay a nominal interest rate i_t, and physical capital. Physical capital produces output according to a standard neoclassical production function. Given its current income, its assets, and any net transfers received from the government (τ_t), the household allocates its resources between consumption, gross investment in physical capital, and gross accumulation of real money balances and bonds.

5. Brock (1974), for example, develops two simple transactions stories that can be represented by putting money directly in the utility function. See also Feenstra (1986).

6. In some environments, money might yield utility, even if never actually spent, if it is held for insurance purchases. For example, Imrohoroğlu (1992) studies a model in which agents can insure against income fluctuations only by holding "money."

If the rate of depreciation of physical capital is δ, the aggregate economy-wide budget constraint of the household sector takes the form

$$Y_t + \tau_t N_t + (1-\delta)K_{t-1} + \frac{(1+i_{t-1})B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} = C_t + K_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} \tag{2.2}$$

where Y_t is aggregate output, K_{t-1} is the aggregate stock of capital at the start of period t, and $\tau_t N_t$ is the aggregate real value of any lump-sum transfers (taxes if negative).

The timing implicit in this specification of the MIU model assumes that it is the household's real money holdings at the end of the period, M_t/P_t, *after* having purchased consumption goods, that yield utility. Carlstrom and Fuerst (2001) have criticized this timing assumption, arguing that the appropriate way to model the utility from money is to assume that money balances available *before* going to purchase consumption goods yield utility. As they demonstrate, alternative timing assumptions can affect the correct definition of the opportunity cost of holding money and whether multiple real equilibria can be ruled out. Because it is standard in the MIU approach to assume that it is end-of-period money holdings that yield utility, we will continue to maintain that assumption in our development of the MIU model.[7]

The aggregate production function relates output Y_t to the available capital stock K_{t-1} and employment N_t: $Y_t = F(K_{t-1}, N_t)$. Assuming that this production function is linear homogeneous with constant returns to scale, output per capita at time t will be a function of the per capita capital stock:[8]

$$y_t = f\left(\frac{k_{t-1}}{1+n}\right), \tag{2.3}$$

where n is the population growth rate (assumed to be constant). Note the assumption that output is produced in period t using capital carried over from period $t-1$. The production function is assumed to be continuously differentiable and to satisfy the usual Inada conditions ($f_k \geq 0, f_{kk} \leq 0, \lim_{k\to 0} f_k(k) = \infty, \lim_{k\to\infty} f_k(k) = 0$).

7. Problems 1 and 2 at the end of the chapter ask you to derive the first order conditions for money holdings under an alternative timing assumption.

8. That is, if $Y_t = F(K_{t-1}, N_t)$, where Y is output, K is the capital stock, and N_t is labor input, and $F(\lambda K, \lambda N) = \lambda F(K, N) = \lambda Y$, we can write $Y_t/N_t \equiv y_t = F(K_{t-1}, N_t)/N_t = F(K_{t-1}/N_t, 1) \equiv f(k_{t-1}/(1+n))$, where $n = (N_t - N_{t-1})/N_{t-1}$ is the constant labor-force growth rate. In general, a lowercase letter will denote the per capita value of the corresponding uppercase variable.

Dividing both sides of the budget constraint (2.2) by the population N_t, the per capita version becomes

$$\omega_t \equiv f\left(\frac{k_{t-1}}{1+n}\right) + \tau_t + \left(\frac{1-\delta}{1+n}\right)k_{t-1} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+\pi_t)(1+n)}$$
$$= c_t + k_t + m_t + b_t, \tag{2.4}$$

where τ_t is the rate of inflation, $b_t = B_t/D_tN_t$, and $m_t = M_t/P_tN_t$.

The household's problem is to choose paths for c_t, k_t, b_t, and m_t to maximize (2.1) subject to (2.4). This is a problem in dynamic optimization, and it is convenient to formulate the problem in terms of the value function. The value function gives the maximized value of utility the household can achieve by behaving optimally, given its current state.[9] The state variable for the problem is the household's initial resources ω_t. The value function, defined as the present discounted value of utility if the household optimally chooses consumption, capital holdings, bond holdings, and money balances, is given by

$$V(\omega_t) = \max\{u(c_t, m_t) + \beta V(\omega_{t+1})\}, \tag{2.5}$$

where the maximization is subject to the budget constraint (2.4) and

$$\omega_{t+1} = \frac{f(k_t)}{1+n} + \tau_{t+1} + \left(\frac{1-\delta}{1+n}\right)k_t + \frac{(1+i_t)b_t + m_t}{(1+\pi_{t+1})(1+n)}.$$

Using (2.4) to express k_t as $\omega_t - c_t - m_t - b_t$ and making use of the definition of ω_{t+1}, (2.5) can be written as

$$V(\omega_t) = \max\Bigg\{u(c_t, m_t) + \beta V\left(\frac{f(\omega_t - c_t - m_t - b_t)}{1+n} + \tau_{t+1} + \left(\frac{1-\delta}{1+n}\right)(\omega_t - c_t - m_t - b_t) + \frac{(1+i_t)b_t + m_t}{(1+\pi_{t+1})(1+n)}\right)\Bigg\}$$

with the maximization problem now an unconstrained one over c_t, b_t, and m_t. The first order necessary conditions for this problem are

$$u_c(c_t, m_t) - \frac{\beta}{1+n}[f_k(k_t) + 1 - \delta]V_\omega(\omega_{t+1}) = 0 \tag{2.6}$$

9. For introductions to dynamic optimization designed for economists see Sargent (1987), Lucas and Stokey (1989), Dixit (1990), Chiang (1992), Obstfeld and Rogoff (1996), or Ljungquist and Sargent (2000).

$$\frac{1+i_t}{(1+\pi_{t+1})(1+n)} - \left[\frac{f_k(k_t) + 1 - \delta}{1+n}\right] = 0 \tag{2.7}$$

$$u_m(c_t, m_t) - \beta\left[\frac{f_k(k_t) + 1 - \delta}{1+n}\right]V_\omega(\omega_{t+1}) + \frac{\beta V_\omega(\omega_{t+1})}{(1+\pi_{t+1})(1+n)} = 0 \tag{2.8}$$

together with the transversality conditions

$$\lim_{t\to\infty} \beta^t \lambda_t x_t = 0, \quad \text{for } x = k, b, m, \tag{2.9}$$

where λ_t is the marginal utility of period t consumption. The envelope theorem implies

$$\lambda_t = u_c(c_t, m_t) = V_\omega(\omega_t). \tag{2.10}$$

The first order conditions have straightforward interpretations. Since initial resources ω_t must be divided between consumption, capital, bonds, and money balances, each use must yield the same marginal benefit at an optimum allocation. Using (2.6) and (2.10), (2.8) can be written as

$$u_m(c_t, m_t) + \frac{\beta u_c(c_{t+1}, m_{t+1})}{(1+\pi_{t+1})(1+n)} = u_c(c_t, m_t), \tag{2.11}$$

which states that the marginal benefit of adding to money holdings at time t must equal the marginal utility of consumption at time t. The marginal benefit of additional money holdings has two components. First, money directly yields utility u_m. Second, real money balances at time t add $1/(1+\pi_{t+1})(1+n)$ to real per capita resources at time $t+1$; this addition to ω_{t+1} is worth $V_\omega(\omega_{t+1})$ at $t+1$, or $\beta V_\omega(\omega_{t+1})$ at time t. Thus, the total marginal benefit of money at time t is $u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1})/(1+\pi_{t+1})(1+n)$.

From (2.6), (2.7), and (2.11),

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = 1 - \left[\frac{1}{(1+\pi_{t+1})(1+n)}\right]\frac{\beta u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)}$$
$$= 1 - \frac{1}{(1+r_t)(1+\pi_{t+1})}$$
$$= \frac{i_t}{1+i_t} \equiv \Upsilon_t, \tag{2.12}$$

where $1 + r_t \equiv f_k(k_t) + 1 - \delta$ is the real return on capital. To interpret (2.12), consider a very simple choice problem in which the agent must pick x and z to maximize

$u(x, z)$ subject to a budget constraint of the form $x + pz = y$, where p is the relative price of z. The first order conditions imply $u_z/u_x = p$. Comparing this to (2.12) shows that Υ has the interpretation of the relative price of real money balances in terms of the consumption good. The marginal rate of substitution between money and consumption is set equal to the price, or opportunity cost, of holding money. The opportunity cost of holding money is directly related to the nominal rate of interest. The household could hold one unit less of money, purchasing instead a bond yielding a nominal return of i; the real value of this payment is $i/(1+\pi)$, and since it is received in period $t+1$, its present value is $i/(1+r)(1+\pi) = i/(1+i)$.[10] Because money is assumed to pay no rate of interest, the opportunity cost of holding money is affected both by the real return on capital and by the rate of inflation. If the price level is constant (so $\pi = 0$), then the forgone earnings from holding money rather than capital are determined by the real return to capital. If the price level is rising ($\pi > 0$), this causes the real value of money to decline, adding to the opportunity cost of holding money.

Equation (2.6) for capital holdings has a similar interpretation; the net marginal return from holding additional capital, $\beta[f_k(k_t) + (1-\delta)]V_\omega(\omega_{t+1})/(1+n)$, must equal the marginal utility of consumption.

Equation (2.7) links the nominal return on bonds, inflation, and the real return on capital. It can be written as

$$1 + i_t = [1 + f_k(k_t) - \delta](1 + \pi_{t+1}) = (1 + r_t)(1 + \pi_{t+1}). \tag{2.13}$$

This relationship between real and nominal rates of interest is called the *Fisher relationship* after Irving Fisher (1896). It expresses the gross nominal rate of interest as equal to the gross real return on capital times 1 plus the expected rate of inflation. If we note that $(1+x)(1+y) \approx 1 + x + y$ when x and y are small, (2.13) is often written as

$$i_t = r_t + \pi_{t+1}.$$

Equations (2.6) to (2.8), together with the budget constraint (2.4), characterize the household's choice of consumption, money, bond, and capital holdings at each point in time. Equilibrium also requires that the nominal demand for money equals the nominal supply of money (assumed to be exogenous). In addition, since we have

10. Suppose households gain utility from the real money balances they have at the start of period t rather than the balances they hold at the end of the period, as we have been assuming. Then the marginal rate of substitution between money and consumption will be set equal to i_t (see Lucas 1982; Carlstrom and Fuerst 2001). See also problem 1 at the end of this chapter.

assumed that all households are identical, the stock of bonds must equal zero in equilibrium.

In deriving the first order conditions for the household's problem, we could have, equivalently, assumed that the household rented its capital to firms, receiving a rental rate of r_k, and sold its labor services at a wage rate of w. Household income would then be $r_k k + w$ (expressed on a per-capita basis). With competitive firms hiring capital and labor in perfectly competitive factor markets under constant returns to scale, $r_k = f'(k)$ and $w = f(k) - kf'(k)$, so household income would be $r_k k + w = f_k(k)k + [f(k) - kf_k(k)] = f(k)$, as in (2.4).[11]

While we could use this system to study analytically the dynamic behavior of the economy (for example, see Sidrauski 1967, Fischer 1979b, Blanchard and Fischer 1989), we will instead focus first on the properties of the steady-state equilibrium. And, since our main focus here is not on the exogenous growth generated by population growth, it will provide some slight simplification to set $n = 0$ in the following. After we have examined the steady state, we will study the dynamic properties by examining the time-series behavior of macroeconomic variables implied by a stochastic version of the model, a version that also includes uncertainty, a labor-leisure choice, and variable employment.

2.2.1 Steady-State Equilibrium

Consider the properties of this economy when it is in a steady-state equilibrium with $n = 0$ and the nominal supply of money is growing at the rate θ. Let the superscript ss denote values evaluated at the steady state. The steady-state values of consumption, the capital stock, real money balances, inflation, and the nominal interest rate must satisfy the first order necessary conditions for the household's decision problem given by (2.6)–(2.8), the economy-wide budget constraint, and the specification of the exogenous growth rate of M. These conditions can be written as

$$u_c(c^{ss}, m^{ss}) - \beta[f_k(k^{ss}) + 1 - \delta]V_\omega(\omega^{ss}) = 0 \tag{2.14}$$

$$\frac{1 + i^{ss}}{1 + \theta} - [f_k(k^{ss}) + 1 - \delta] = 0 \tag{2.15}$$

$$u_m(c^{ss}, m) - \beta[f_k(k^{ss}) + 1 - \delta]V_\omega(\omega^{ss}) + \frac{\beta V_\omega(\omega^{ss})}{1 + \theta} = 0 \tag{2.16}$$

11. This follows from Euler's theorem: If the aggregate constant-returns-to-scale production function is $F(N, K)$, then $F(N, K) = F_N N + F_K K$. In per capita terms, this becomes $f(k) = F_N + F_K k = w + rk$ if labor and capital are paid their marginal products.

$$f(k^{ss}) + \tau^{ss} + (1-\delta)k^{ss} + \frac{m^{ss}}{1+\theta} = c^{ss} + k^{ss} + m^{ss}, \tag{2.17}$$

where $\omega^{ss} = f(k^{ss}) + \tau^{ss} + (1-\delta)k^{ss} + m^{ss}/(1+\pi)$. In (2.14)–(2.17), use has been made of the fact that, in the equilibrium of this representative agent model, $b = 0$. We have also used the result that the rate of inflation is determined by the growth rate of the nominal quantity of money. This is simply an implication of the steady-state property that real, per capita money holdings are constant in the steady state, and a constant value of real money balances requires prices to change at the same rate as the nominal stock of money, so $\pi^{ss} = \theta$.[12]

Notice that in (2.14)–(2.17), money appears only in the form of m, real money balances. Thus, any change in the nominal quantity of money that is matched by a proportional change in the price level, leaving m unchanged, has no effect on the economy's real equilibrium. This is described by saying that the model exhibits the *neutrality of money*. If prices do not adjust immediately in response to a change in M, then a model might display nonneutrality with respect to changes in M in the short run but still exhibit monetary neutrality in the long run once all prices have adjusted. In fact, this will be the case with the models used in chapters 5–11 to examine issues related to short-run monetary policy.

Using (2.10), (2.14) implies that $1 = \beta[f_k(k^{ss}) + 1 - \delta]$, or

$$f_k(k^{ss}) = \frac{1}{\beta} - 1 + \delta. \tag{2.18}$$

This equation defines the steady-state capital-labor ratio k^{ss}. If the production function is Cobb-Douglas, say $f(k) = k^{\alpha}$, then $f_k(k) = \alpha k^{\alpha-1}$ and we have

$$k^{ss} = \left(\frac{\alpha\beta}{1+\beta(\delta-1)}\right)^{\frac{1}{1-\alpha}}. \tag{2.19}$$

What is particularly relevant for our purposes is the implication from (2.18) that the steady-state capital-labor ratio is independent of 1) all parameters of the utility function other than the subjective discount rate β and 2) the steady-state rate of inflation π^{ss}. In fact, k^{ss} depends only on the production function, the depreciation rate, and the discount rate. It is independent of the rate of inflation.

Because changes in the nominal quantity of money are engineered in this model by making lump-sum transfers to the public, the real value of these transfers is equal

12. If the population is growing at the rate n, then $1 + \pi^{ss} = (1+\theta^{ss})/(1+n)$.

to $(M_t - M_{t-1})/P_t$. Hence, steady-state transfers are given by $\tau^{ss} = m^{ss} - m^{ss}/(1+\pi^{ss}) = \theta m^{ss}/(1+\theta)$, so the budget constraint (2.17) reduces to

$$c^{ss} = f(k^{ss}) - \delta k^{ss}. \tag{2.20}$$

The steady-state level of consumption per capita is completely determined once we know the level of steady-state capital. If we again assume that $f(k) = k^{\alpha}$, k^{ss} is given by (2.19) and

$$c^{ss} = \left(\frac{\alpha\beta}{1+\beta(\delta-1)}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha\beta}{1+\beta(\delta-1)}\right)^{\frac{1}{1-\alpha}}.$$

Steady-state consumption per capita depends on the parameters of the production function (α), the rate of depreciation (δ), and the subjective rate of time discount (β).

This model exhibits a property called the *superneutrality of money*; the steady-state values of the capital-labor ratio, consumption, and output are all independent of the rate of inflation. That is, not only is money neutral, so that proportional changes in the *level* of nominal money balances and prices have no real effects, but changes in the *rate of growth* of nominal money also have no effect either on the steady-state capital stock or, therefore, on output or per capita consumption. Since the real rate of interest is equal to the marginal product of capital, it also is invariant across steady states that differ only in their rates of inflation.

An important distinction is that between changes in the *level* of the nominal supply of money and changes in the *rate of growth* of the nominal money supply. In all the models we will examine, the nominal money stock enters in the form M/P. Thus, proportional changes in the level of M and P, changes that leave M/P unaffected, have no real effects. A model displays the property of *superneutrality* if the real equilibrium is independent of the rate of growth of the nominal money supply. Thus, the Sidrauski MIU model possesses the properties of both neutrality and superneutrality.

To understand why superneutrality holds, note that from (2.10), $u_c = V_{\omega}(\omega_t)$, so using (2.6),

$$u_c(c_t, m_t) = \beta[f_k(k_t) + 1 - \delta]u_c(c_{t+1}, m_{t+1}),$$

or

$$\frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} = \frac{1/\beta}{f_k(k_t) + 1 - \delta}. \tag{2.21}$$

Recall from (2.18) that the right side of this expression is equal to 1 in the steady state. If $k < k^{ss}$ so that $f_k(k) > f_k(k^{ss})$, then the right side is smaller than 1, and the marginal utility of consumption will be declining over time. It will be optimal to postpone consumption to accumulate capital and have consumption grow over time (so u_c declines over time). As long as $f_k + 1 - \delta > 1/\beta$, this process continues, but as the capital stock grows, the marginal product of capital declines until eventually $f_k(k) + 1 - \delta = 1/\beta$. The converse holds if $k > k^{ss}$. Consumption remains constant only when $f_k + 1 - \delta = 1/\beta$. If an increase in the rate of inflation were to induce households to accumulate more capital, this would lower the marginal product of capital, leading to a situation in which $f_k + 1 - \delta < 1/\beta$. Households would then want their consumption path to decline over time, so they would immediately attempt to increase current consumption and reduce their holdings of capital. The value of k consistent with a steady state is independent of the rate of inflation.

What is affected by the rate of inflation? One thing we should expect is that the interest rate on any asset that pays off in units of money at some future date will be affected; the real value of those future units of money will be affected by inflation, and this will be reflected in the interest rate required to induce individuals to hold the asset, as shown by (2.13). To understand this equation, consider the nominal interest rate that an asset must yield if it is to give a real return of r_t in terms of the consumption good. That is, consider an asset that costs 1 unit of consumption in period t and yields $(1 + r_t)$ units of consumption at $t + 1$. In units of money, this asset costs P_t units of money at time t. Since the cost of each unit of consumption at $t + 1$ is P_{t+1} in terms of money, the asset must pay an amount equal to $(1 + r)P_{t+1}$. Thus, the nominal return is $[(1 + r_t)P_{t+1} - P_t]/P_t = (1 + r_t)(1 + \pi_{t+1}) - 1 \equiv i_t$. In the steady state, $1 + r^{ss} = 1/\beta$ and $\pi^{ss} = \theta$, so the steady-state nominal rate of interest is given by $[(1 + \theta)/\beta] - 1$ and varies (approximately) one for one with inflation.[13]

Existence of the Steady State To ensure that a steady-state monetary equilibrium exists, there must exist a positive but finite level of real money balances m^{ss} that satisfies (2.12), evaluated at the steady-state level of consumption. If utility is separable in consumption and money balances, say $u(c, m) = v(c) + \phi(m)$, this condition can be written as $\phi_m(m^{ss}) = \Upsilon^{ss} v_c(c^{ss})$. The right side of this expression is a positive constant; the left side approaches ∞ as $m \to 0$. If $\phi_m(m) \leq 0$ for all m greater than some finite level, a steady-state equilibrium with positive real money balances is guaranteed to exist. This was the role of the earlier assumption that the marginal

13. Outside of the steady state, the nominal rate can still be written as the sum of the expected real rate plus the expected rate of inflation, but there is no longer any presumption that short-run variations in inflation will leave the real rate unaffected.

utility of money eventually becomes negative. Note that this assumption is not necessary; $\phi(m) = \log m$ yields a positive solution to (2.12) as long as $\Upsilon^{ss} v_c(c^{ss}) > 0$. When utility is not separable, we can still write (2.12) as $u_m(c^{ss}, m^{ss}) = \Upsilon^{ss} u_c(c^{ss}, m^{ss})$. If $u_{cm} < 0$ so that the marginal utility of consumption decreases with increased holdings of money, both u_m and u_c decrease with m and the solution to (2.12) may not be unique; multiple steady-state equilibria may exist.[14]

When $u(c, m) = v(c) + \phi(m)$, the dynamics of real balances around the steady state can be described easily by multiplying both sides of (2.11) by M_t and noting that $M_{t+1} = (1 + \theta)M_t$:

$$B(m_{t+1}) \equiv \frac{\beta}{1+\theta} v_c(c_{ss}) m_{t+1} = [v_c(c^{ss}) - \phi_m(m_t)] m_t \equiv A(m_t), \qquad (2.22)$$

which gives a difference equation in m. The properties of this equation have been examined by Brock (1974) and Obstfeld and Rogoff (1983, 1986). A steady-state value for m satisfies $B(m^{ss}) = A(m^{ss})$. The functions $B(m)$ and $A(m)$ are illustrated in figure 2.1. For the case drawn, $\lim_{m \to 0} \phi_m m = 0$ so there are two steady-state solutions to (2.22), one at m' and one at 0. Only one of these involves positive real money balances (and a positive value for money). If $\lim_{m \to 0} \phi_m m = \bar{m} > 0$, then $\lim_{m \to 0} A(m) < 0$ and there is only one solution. Paths for m_t originating to the right of m' involve $m_{t+s} \to \infty$ as $s \to \infty$. When $\theta \geq 0$ (nonnegative money growth), such explosive paths for m, involving a price level going to zero, violate the transversality condition that the discounted value of asset holdings must go to zero (see Obstfeld and Rogoff 1983, 1986).[15] When $\lim_{m \to 0} A(m) < 0$, paths originating to the left of m' converge to $m < 0$; but this is clearly not possible, since real balances cannot be negative. For the case drawn in figure 2.1, however, some paths originating to the left of m' converge to 0 without ever involving negative real balances. For example, a path that reaches m'' at which $A(m'') = 0$ then jumps to $m = 0$. Along such an equilibrium path, the price level is growing faster than the nominal money supply (so that m declines). Even if $\theta = 0$, so that the nominal money supply is constant, the equilibrium path would involve a speculative hyperinflation with the price level going to infinity.[16] Unfortunately, Obstfeld and Rogoff show that the conditions needed to

14. For more on the conditions necessary for the existence of monetary equilibria, see Brock (1974, 1975) and Bewley (1983).

15. Obstfeld and Rogoff (1986) show that any such equilibrium path with an implosive price level violates the transversality condition unless $\lim_{m \to \infty} \phi(m) = \infty$. This condition is implausible, as it would require that the utility yielded by money be unbounded.

16. The hyperinflation is labeled speculative since it is not driven by fundamentals, such as the growth rate of the nominal supply of money.

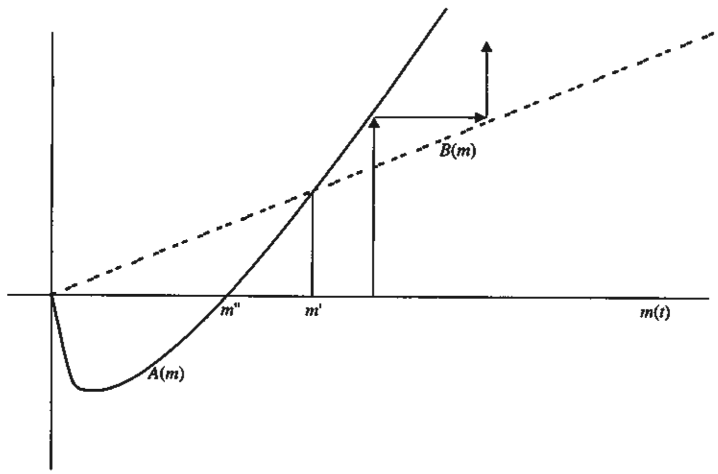

Figure 2.1
Steady-State Real Balances (Separable Utility)

ensure $\lim_{m\to 0} \phi_m m = \bar{m} > 0$ so that speculative hyperinflations can be ruled out are restrictive. They show that $\lim_{m\to 0} \phi_m m > 0$ implies $\lim_{m\to 0} \phi(m) = -\infty$; essentially, money must be so necessary that the utility of the representative agent goes to minus infinity if her real balances fall to zero.[17]

When paths originating to the left of m' cannot be ruled out, the model exhibits multiple equilibria. For example, suppose that the nominal stock of money is held constant, with $M_t = M_0$ for all $t > 0$. Then there is a rational expectations equilibrium path for the price level and real money balances starting at any price level P_0 as long as $M_0/P_0 < m'$. Chapter 4 examines an approach called the *fiscal theory of the price level*, which argues that the initial price level may be determined by fiscal policy.

2.2.2 The Interest Elasticity of Money Demand

Returning to (2.12), this characterizes the demand for real money balances as a function of the nominal rate of interest and real consumption. For example, suppose

17. Speculative hyperinflations are shown by Obstfeld and Rogoff to be ruled out if the government holds real resources to back a fraction of the outstanding currency. This ensures a positive value below which the real value of money cannot fall.

that the utility function in consumption and real balances is of the constant elasticity of substitution (CES) form:

$$u(c_t, m_t) = [ac_t^{1-b} + (1-a)m_t^{1-b}]^{\frac{1}{1-b}}, \tag{2.23}$$

with $0 < a < 1$ and $b > 0$, $b \neq 1$. Then

$$\frac{u_m}{u_c} = \left(\frac{1-a}{a}\right)\left(\frac{c_t}{m_t}\right)^b$$

and (2.12) can be written as[18]

$$m_t = \left(\frac{1-a}{a}\right)^{\frac{1}{b}}\left(\frac{i}{1+i}\right)^{-\frac{1}{b}} c_t. \tag{2.24}$$

In terms of the more common specification in log form used to model empirical money demand equations (Goldfeld and Sichel 1990),

$$\log \frac{M_t}{P_t N_t} = \frac{1}{b} \log\left(\frac{1-a}{a}\right) + \log c - \frac{1}{b} \log \frac{i}{1+i}, \tag{2.25}$$

which gives the real demand for money as a negative function of the nominal rate of interest and a positive function of consumption.[19] The consumption (income) elasticity of money demand is equal to 1 in this specification. The elasticity of money demand with respect to the opportunity cost variable $\Upsilon_t = i_t/(1+i_t)$ is $1/b$. For simplicity, this will often be referred to as the *interest elasticity of money demand*.[20]

For $b = 1$, the CES specification becomes $u(c_t, m_t) = c_t^a m_t^{1-a}$. Note from (2.25) that in this case, the consumption (income) elasticity of money demand and the elasticity with respect to the opportunity cost measure Υ_t are both equal to 1.

18. In the limit, as $b \to \infty$, (2.24) implies that $m = c$. This is then equivalent to the cash-in-advance models we will examine in chapter 3.

19. The standard specification of money demand would use income in place of consumption, although see Mankiw and Summers (1986).

20. The elasticity of money demand with respect to the nominal interest rate is

$$-\frac{\partial m_t}{\partial i_t}\frac{i_t}{m_t} = \frac{1}{b}\frac{1}{1+i_t}.$$

Empirical work often estimates money-demand equations in which the log of real money balances is a function of log income and the *level* of the nominal interest rate. The coefficient on the nominal interest rate is then equal to the semielasticity of money demand with respect to the nominal interest rate $(m^{-1}\partial m/\partial i)$, which for (2.25) is $1/bi(1+i)$.

While the parameter b governs the interest elasticity of demand, the steady-state level of money holdings depends on the value of a. From (2.24), the ratio of real money balances to consumption in the steady state will be[21]

$$\frac{m^{ss}}{c^{ss}} = \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{\Pi^{ss}-\beta}{\Pi^{ss}}\right)^{-\frac{1}{b}},$$

where $\Pi^{ss} = 1 + \pi^{ss}$ is the gross rate of inflation. The ratio of m^{ss} to c^{ss} is decreasing in a; an increase in a reduces the weight given to real money balances in the utility function and results in smaller holdings of money (relative to consumption) in the steady state. Increases in inflation also reduce the ratio of money holdings to consumption by increasing the opportunity cost of holding money.

Empirical Evidence on the Interest Elasticity of Money Demand The empirical literature on money demand is vast. See, for example, the references in Judd and Scadding (1982), Laidler (1985), or Goldfeld and Sichel (1990) for earlier surveys. More relevant for our discussion is Holman (1998), who directly estimates the parameters of the utility function under various alternative specifications of its functional form, including (2.23), using annual U.S. data from 1889 to 1991.[22] She obtains estimates of b of around 0.1 and a of around 0.95. This value of b implies an elasticity of money demand equal to 10. However, in shorter samples, the data fail to reject $b = 1$, the case of Cobb-Douglas preferences, indicating that the interest elasticity of money demand is estimated very imprecisely.

Chari, Kehoe, and McGrattan (2000) estimate (2.25) using quarterly U.S. data and the $M1$ definition of money. They obtain an estimate for a of around 0.94 and an estimate of the interest elasticity of money demand of 0.39, implying a value of b on the order of $1/.39 \approx 2.6$. Hoffman, Rasche, and Tieslau (1995) conduct a cross-country study of money demand and find a value of around 0.5 for the United States and Canada money demand interest elasticity, with somewhat higher values for the United Kingdom and lower values for Japan and Germany. An elasticity of 0.5 implies a value of 2 for b. Ireland (2001a) estimates the interest elasticity as part of a general equilibrium model and obtains a value of 0.19 for the pre-1979 period and 0.12 for the post-1979 period. These translate into values for b of 5.26 and 8.33, respectively.

Most empirical estimates of the interest elasticity of money demand employ aggregate time-series data. At the household level, many U.S. households hold

21. This makes use of the fact that $1 + i = R\Pi = \Pi/\beta$ in the steady state.

22. Holman (1998) considers a variety of specifications for the utility function, including Cobb-Douglas ($b = 1$) and nested CES functions of the form we will use later in section 2.5.

no interest-earning assets, so the normal substitution between money and interest-earning assets as the nominal interest rate changes is absent. As nominal interest rates rise, more households find it worthwhile to hold interest-earning assets. Changes in the nominal interest rate then affect both the extensive margin (the decision whether to hold interest-earning assets) and the intensive margin (the decision on how to allocate wealth between money and interest-earning assets). Mulligan and Sala-i-Martin (2000) focus on these two margins and use cross-sectional evidence on household holdings of financial assets to estimate money demand interest elasticity. They find that the elasticity increases with the level of nominal interest rates and is low at low nominal rates of interest.

2.2.3 Limitations

Before moving on to use this framework to analyze the welfare cost of inflation, we need to consider the limitations of the money in the utility approach. In the MIU model, there is a clearly defined reason for individuals to hold money—it provides direct utility. However, this essentially "solves" the problem of generating a positive demand for money by assumption; it doesn't address the reasons why money, particularly money in the form of unbacked pieces of paper, might yield utility. The money in the utility function approach has to be thought of as a shortcut for a fully specified model of the transactions technology faced by households that gives rise to a positive demand for a medium of exchange.[23]

Shortcuts are often extremely useful. But one problem with such a shortcut is that it does not provide any real understanding of, or possible restrictions on, such quantities as u_m or u_{cm} that play a role in determining equilibrium and the outcome of comparative static exercises. One possible story that can generate money in the utility function is a shopping-time story, and we will return to this idea in chapter 3.

2.3 The Welfare Cost of Inflation

Because money holdings yield direct utility and higher inflation reduces real money balances, inflation generates a welfare loss. This raises two questions: 1) how large is the welfare cost of inflation? and 2) is there an optimal rate of inflation that maximizes the steady-state welfare of the representative household? While chapter 4 (and chapter 11) will provide a more detailed discussion of the optimal rate of inflation,

23. For a general equilibrium analysis of asset prices in an MIU framework, see LeRoy (1984a, 1984b).

we can illustrate here some important results on both of these questions, taking them in reverse order.

The second question—the optimal rate of inflation—was originally addressed by Bailey (1956) and M. Friedman (1969). Their basic intuition was the following. The private opportunity cost of holding money depends on the nominal rate of interest (see 2.12). The social marginal cost of producing money—that is, running the printing presses—is essentially zero. The wedge that arises between the private marginal cost and the social marginal cost when the nominal rate of interest is positive generates an inefficiency. This inefficiency would be eliminated if the private opportunity cost were also equal to zero, and this will be the case if the nominal rate of interest equals zero. But $i = 0$ requires that $\pi = -r/(1+r) \approx -r$. So the optimal rate of inflation is a rate of *deflation* approximately equal to the real return on capital.[24]

In the steady state, real money balances are directly related to the inflation rate, so the optimal rate of inflation is also frequently discussed under the heading of the *optimal quantity of money*. With utility depending directly on m, one can think of the government choosing its policy instrument θ (and therefore π) to achieve the steady-state optimal value of m. Steady-state utility will be maximized when $u(c^{ss}, m^{ss})$ is maximized subject to the constraint that $c^{ss} = f(k^{ss}) - \delta k^{ss}$. But since c^{ss} is independent of θ, the first order condition for the optimal θ is just $u_m(\partial m/\partial \theta) = 0$, or $u_m = 0$, and from (2.12), this occurs when $i = 0$.[25]

The major criticism of this result is that of Phelps (1973), who pointed out that money growth generates revenue for the government—the inflation tax. The implicit assumption so far has been that variations in money growth are engineered via lump-sum transfers. Any effects on government revenue can be offset by a suitable adjustment in these lump-sum transfers (taxes). But if governments only have distortionary taxes available for financing expenditures, then reducing inflation tax revenues to achieve the Friedman rule for optimal inflation requires that the lost revenue be replaced through increases in other distortionary taxes. To minimize the total distortions associated with raising a given amount of revenue, it may be optimal to rely to some degree on the inflation tax. Reducing the nominal rate of interest to zero would increase the inefficiencies generated by the higher level of other taxes needed to replace lost inflation tax revenues. Recent work has reexamined these results (see

24. Since $(1+i) = (1+r)(1+\pi)$, $i = 0$ implies $\pi = -r/(1+r) \approx -r$.

25. Note that the earlier assumption that the marginal utility of money goes to zero at some finite level of real balances ensures that $u_m = 0$ has a solution with $m < \infty$. While we focus here on the steady state, a more appropriate perspective for addressing the optimal inflation question would not restrict attention solely to the steady state. The more general case is considered in chapter 4.

Chari, Christiano, and Kehoe 1991, 1996; Correia and Teles 1996, 1999; Mulligan and Sala-i-Martin 1997). The revenue implications of inflation and optimal inflation are major themes of chapter 4.

Now let's return to the first question posed previously—what is the welfare cost of inflation? Beginning with Bailey (1956), this welfare cost has been calculated from the area under the money demand curve (showing money demand as a function of the nominal rate of interest), since this provides a measure of the consumer surplus lost as a result of having a positive nominal rate of interest. Figure 2.2 is based on the money demand function given by (2.24) with $a = 0.9$ and $b = 2.56$. At a nominal interest rate of i^*, the deadweight loss is measured by the shaded area under the money demand curve.

Because nominal interest rates reflect *expected* inflation, calculating the area under the money demand curve will provide a measure of the costs of anticipated inflation and is therefore appropriate for evaluating the costs of alternative constant rates of inflation. There are other costs of inflation associated with tax distortions and with variability in the rate of inflation; these are discussed in the survey on the costs of inflation by Driffill, Mizon, and Ulph (1990).

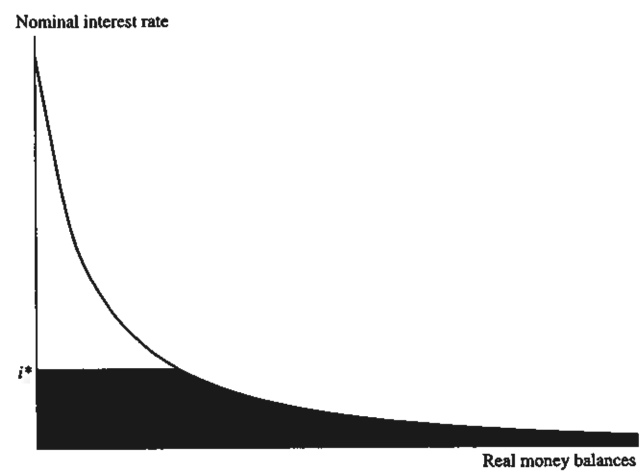

Figure 2.2
The Welfare Costs of Inflation

Lucas (1994) provides estimates of the welfare costs of inflation by starting with the following specification of the instantaneous utility function:

$$u(c,m) = \frac{1}{1-\sigma}\left\{\left[c\varphi\left(\frac{m}{c}\right)\right]^{1-\sigma} - 1\right\}. \tag{2.26}$$

With this utility function, (2.12) becomes

$$\frac{u_m}{u_c} = \frac{\varphi'(x)}{\varphi(x) - x\varphi'(x)} = \frac{i}{1+i} = \Upsilon, \tag{2.27}$$

where $x \equiv m/c$.[26] Normalizing so that steady-state consumption equals 1, $u(1,m)$ will be maximized when $\Upsilon = 0$, implying that the optimal x is defined by $\varphi'(m^*) = 0$. Lucas proposes to measure the costs of inflation by the percentage increase in steady-state consumption necessary to make the household indifferent between a nominal interest rate of i and a nominal rate of 0. If this cost is denoted $w(\Upsilon)$, it is defined by

$$u(1 + w(\Upsilon), m(\Upsilon)) \equiv u(1, m^*), \tag{2.28}$$

where $m(\Upsilon)$ denotes the solution of (2.27) for real money balances evaluated at steady-state consumption $c = 1$.

Suppose, following Lucas, that $\varphi(m) = [1 + Bm^{-1}]^{-1}$ where B is a positive constant. Solving (2.27), one obtains $m(i) = B^{.5}\Upsilon^{-.5}$.[27] Note that $\varphi' = 0$ requires that $m^* = \infty$. But $\varphi(\infty) = 1$ and $u(1,\infty) = 0$, so $w(\Upsilon)$ is the solution to $u(1 + w(\Upsilon), B^{.5}\Upsilon^{-.5}) = u(1,\infty) = 0$. Using the definition of the utility function, we obtain $1 + w(\Upsilon) = 1 + \sqrt{B\Upsilon}$, or

$$w(\Upsilon) = \sqrt{B\Upsilon}. \tag{2.29}$$

Based on U.S. annual data from 1900 to 1985, Lucas reports an estimate of .0018 for B. Hence, the welfare loss arising from a nominal interest rate of 10% would be $\sqrt{(.0018)(.1/1.1)} = .013$, or just over 1% of aggregate consumption.

Because U.S. government bond yields were around 10% in 1979 and 1980, we can use 1980 aggregate personal consumption expenditures of \$2447.1 trillion to get a rough estimate of the dollar welfare loss (although consumption expenditures includes purchases of durables). In this example, 1.3% of \$2447.1 trillion is about \$32 billion. Since this is the annual cost in terms of steady-state consumption, we need the present discounted value of \$32 billion. Using a real rate of return of 2%, this amounts to $\$32(1.02)/.02 = \$1.632$ trillion; at 4%, the cost would be \$832 billion.

26. In the framework Lucas employs, the relevant expression is $u_m/u_c = i$; problem 1 at the end of the chapter provides an example of the timing assumptions Lucas employed.

27. Lucas starts with the assumption that money demand is equal to $m = Ai^{-.5}$ for A equal to a constant. He then derives $\varphi(m)$ as the utility function necessary to generate such a demand function, where $B = A^2$.

An annual welfare cost of \$32 billion seems a small number, especially when compared to the estimated costs of reducing inflation. For example, Ball (1993) reports a "sacrifice ratio" of 2.4% of output per 1% of inflation reduction for the United States. Because inflation was reduced from about 10% to about 3% in the early 1980s, Ball's estimate would put the cost of this disinflation at approximately 17% of GDP (2.4% times an inflation reduction of 7%). Based on 1980 GDP of \$3776.3 trillion (1987 prices), this would be \$642 billion. This looks large when compared to the \$32 billion annual welfare cost, but the trade-off starts looking more worthwhile if the costs of reducing inflation are compared to the present discounted value of the annual welfare cost. (See also Feldstein 1979.)

Gillman (1995) provides a useful survey of different estimates of the welfare cost of inflation. The estimates differ widely. One important reason for these differences arises from the choice of the base inflation rate. Some estimates compare the area under the money demand curve between an inflation rate of zero and, say, 10%. This is incorrect in that a zero rate of inflation still results in a positive nominal rate (equal to the real rate of return) and therefore a positive opportunity cost associated with holding money. Gillman concludes, based on the empirical estimates he surveys, that a reasonable value of the welfare cost of inflation for the United States is in the range of 0.85% to 3% of real GNP per percentage rise in the nominal interest rate above zero, a loss in 2002 dollars of \$88 to \$310 billion per year.[28]

The area under the demand curve is a partial equilibrium measure of the welfare costs of inflation. Gomme (1993) and Dotsey and Ireland (1996) examine the effects of inflation in general equilibrium frameworks that allow for the supply of labor and the average rate of economic growth to be affected. Gomme finds that, even though inflation reduces the supply of labor and economic growth, the welfare costs are small because of the increased consumption of leisure that households enjoy. Dostey and Ireland find much larger welfare costs of inflation in a model that generates an interest elasticity of money demand that matches estimates for the United States. (See also De Gregorio 1993 and Imrohoroğlu and Prescott 1991.)

The Sidrauski model provides a convenient framework for calculating the steady-state welfare costs of inflation, both because the lower level of real money holdings that result at higher rates of inflation has a direct effect on welfare when money enters the utility function and because the superneutrality property of the model means that the other argument in the utility function, real consumption, is invariant across different rates of inflation. This latter property simplifies the calculation since

28. These estimates apply to the United States, which has experienced relatively low rates of inflation. They may not be relevant for high-inflation countries.

it is not necessary to account for both variations in money holdings and variations in consumption when making the welfare cost calculation. Of critical importance, however, is the ability of the MIU function approach to allow the costs of inflation to be calculated based on a model of money demand that is consistent with optimizing behavior on the part of economic agents.

2.4 Extensions

2.4.1 Interest on Money

If the welfare costs of inflation are related to the positive private opportunity costs of holding money, an alternative to deflation as a means of eliminating these costs would be the payment of explicit interest on money. There are obvious technical difficulties in paying interest on cash, but ignoring these, assume that the government pays a nominal interest rate of i^m on money balances. Assume further that these interest payments are financed by lump-sum taxes s. The household's budget constraint, (2.4), now becomes (setting $n = 0$)

$$f(k_{t-1}) - s_t + \tau_t + (1-\delta)k_{t-1} + (1+r_{t-1})b_{t-1} + \frac{1+i_t^m}{1+\pi_t}m_{t-1} = c_t + k_t + m_t + b_t \tag{2.30}$$

and the first order condition (2.8) becomes

$$-u_c(c_t, m_t) + u_m(c_t, m_t) + \frac{\beta(1+i_t^m)V_\omega(\omega_{t+1})}{(1+\pi_{t+1})} = 0, \tag{2.31}$$

while (2.12) is now

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i_t - i_t^m}{1+i_t}.$$

The opportunity cost of money is related to the interest rate gap $i - i_m$, which represents the difference between the nominal return on bonds and the nominal return on money. If $\theta = 0$ so that the rate of inflation in the steady state is also zero, the optimal quantity of money, the quantity such that $u_m = 0$, can be achieved if $i_m = r$.

The assumption that the interest payments are financed by the revenue from lump-sum taxes is critical for this result. One of the problems at the end of this chapter considers what happens if the government simply finances the interest payments on money by printing more money.

2.4.2 Nonsuperneutrality

Calculations of the steady-state welfare costs of inflation in the Sidrauski model are greatly simplified by the fact that the model exhibits superneutrality. But how robust is the result that money is superneutral? The empirical evidence of Barro (1995) suggests that inflation has a negative effect on growth, a finding inconsistent with superneutrality.[29] One channel through which inflation can have real effects in the steady state is introduced if households have a labor supply choice. That is, suppose utility depends on consumption, real money holdings, and leisure:

$$u = u(c, m, l). \tag{2.32}$$

The economy's production function becomes

$$y = f(k, 1-l), \tag{2.33}$$

where the total supply of time is normalized to equal 1 so that labor supply is just $1 - l$. The additional first order condition implied by the optimal choice of leisure is

$$\frac{u_l(c, m, l)}{u_c(c, m, l)} = f_n(k, 1-l). \tag{2.34}$$

Now, both steady-state labor supply and consumption may be affected by variations in the rate of inflation. Specifically, an increase in the rate of inflation reduces holdings of real money balances. If this affects the marginal utility of leisure, then (2.34) implies that the supply of labor will be affected, leading to a change in the steady-state per capita stock of capital, output, and consumption. But why would changes in money holdings affect the marginal utility of leisure? Because money has simply been assumed to yield utility, with no explanation for the reason, it is difficult to answer this question. In chapter 3 we will examine a model in which money helps to reduce the time spent in carrying out the transactions necessary to purchase consumption goods; in this case, a rise in inflation would lead to more time spent engaged in transactions, and this would raise the marginal utility of leisure. But one might expect that this channel is unlikely to be important empirically, so superneutrality may remain a reasonable first approximation to the effects of inflation on steady-state real magnitudes.

29. Of course, the empirical relationship may not be causal; both growth and inflation may be reacting to common factors. As noted in chapter 1, McCandless and Weber (1995) find no relationship between inflation and average real growth.

Equation (2.34) suggests that if u_l/u_c were independent of m, then superneutrality would hold. This is the case since the steady-state values of k, c, and l could then be found from

$$\frac{u_l}{u_c} = f_n(k^{ss}, 1 - l^{ss}),$$

$$f_k(k^{ss}, 1 - l^{ss}) = \frac{1}{\beta} - 1 + \delta,$$

and

$$c^{ss} = f(k^{ss}, 1 - l^{ss}) + \delta k^{ss}.$$

If u_l/u_c does not depend on m, these three equations determine the steady-state values of consumption, capital, and labor independently of inflation. So superneutrality reemerges when the utility function takes the general form $u(c, m, l) = v(c, l)g(m)$. While variations in inflation will affect the agent's holdings of money, the consumption-leisure choice will not be directly affected. As McCallum (1990a) notes, Cobb-Douglas specifications, which are quite commonly used, satisfy this condition. So with a Cobb-Douglas utility function, the ratio of the marginal utility of leisure to the marginal utility of consumption will be independent of the level of real money balances, and superneutrality will hold.

Another channel through which inflation can affect the steady-state stock of capital occurs if money is entered directly into the production function (Fischer 1974). Since steady states with different rates of inflation will have different equilibrium levels of real money balances, they will also then have different marginal products of capital if the capital-labor ratios are the same. With the steady-state marginal product of capital determined by $1/\beta - 1 + \delta$ (see 2.18), the two steady states can have the same marginal product of capital only if their capital-labor ratios differ. If $\partial MPK/\partial m > 0$ (so that money and capital are complements), higher inflation, by leading to lower real money balances, also leads to a lower steady-state capital stock.[30] This is the opposite of the *Tobin effect*; Tobin (1965) argued that higher inflation would induce a portfolio substitution toward capital that would increase the steady-state capital-labor ratio (see also Stein 1969, Fischer 1972). For higher inflation to be associated with a higher steady-state capital-labor ratio requires that $\partial MPK/\partial m < 0$ (that is, higher money balances reduce the marginal product of capital; money and capital are substitutes in production).

This discussion actually has, by ignoring taxes, excluded what is probably the most important reason that superneutrality may fail in actual economies. Taxes generally are not indexed to inflation and are levied on nominal capital gains instead of real capital gains. Effective tax rates will depend on the inflation rate, generating real effects on capital accumulation and consumption as inflation varies. (See, for example, Feldstein 1978, Summers 1981, and Feldstein 1998). We will return to this issue in chapter 4.

30. That is, in the steady state, $f_k(k^{ss}, m^{ss}) = \beta^{-1} - 1 + \delta$, where $f(k, m)$ is the production function and f_i denotes the partial with respect to the ith argument. It follows that $dk^{ss}/dm^{ss} = -f_{km}/f_{kk}$, so with $f_{kk} \leq 0$, $\text{sign}(dk^{ss}/dm^{ss}) = \text{sign}(f_{km})$.

2.5 Dynamics in an MIU Model

The analysis of the MIU function approach has, up to this point, focused on steady-state properties. We are also interested in understanding the implications of the model for the dynamic process the economy follows as it adjusts in response to exogenous disturbances. Even the basic Sidrauski model can exhibit nonsuperneutralities during the transition to the steady state. For example, Fischer (1979a) has shown that, for the constant relative risk aversion class of utility functions, the rate of capital accumulation is positively related to the rate of money growth except for the case of log separable utility.[31]

In addition, theoretical and empirical work in macroeconomics and monetary economics are closely tied, and it is important to reflect on how the theoretical models can help us understand actual observations on inflationary experiences. One way to do this is to use a theoretical model to generate artificial data by simulating the model economy; comparing the simulated data with actual data generated by real economies provides a means of validating the model. This approach has been popularized by the real-business-cycle literature (see Cooley 1995). Since we can vary the parameters of our theoretical models in ways that we cannot vary the characteristics of real economies, simulation methods allow us to answer a variety of "what if" questions. For example, how does the dynamic response to a temporary change in the growth rate of the money supply depend on the degree of intertemporal substitution characterizing individual preferences?

31. Superneutrality holds during the transition if $u(c, m) = \ln(c) + b\ln(m)$. The general class of utility functions Fischer considers is of the form $u(c, m) = \frac{1}{1-\Phi}(c^a m^b)^{1-\Phi}$; log utility obtains when $\Phi = 1$. See also Asako (1983), who shows that faster money growth can lead to slower capital accumulation under certain conditions if c and m are perfect complements. These effects of inflation on capital accumulation apply during the transition from one steady-state equilibrium to another; they differ therefore from the Tobin effect of inflation on the steady-state capital-labor ratio.