II. Labour Demand

1. Comparative Statics of the Demand for Labour

1. Overview

2. Downward Sloping Demand Curve

3. Elasticities
   a. Elasticity of Substitution
   b. Cross-Elasticities and Own-Price Elasticities


5. Estimating the Elasticity of Labour Demand
1.1. Overview

- The broadest definition of the demand for labour involves any decision made by an employer regarding its workers, their employment, compensation, training.

- In Marshall’s statement of neoclassical economics (1920), much of the focus in analyzing labour markets was the employer’s decisions about how many workers to employ and how many hours the employees should work.

- The demand for labour is viewed as derived from consumers’ demand for final goods and services.

- The responses of the number of jobs offered and the number of hours that employees are required to work, to external shocks have been the focus of the analysis, that is,
  
  - the comparative statics of employers’ responses to changes in product demand and factor prices.
• By comparison to labour supply, issues related to labour demand occupied a less voluminous part of the field of labour.

• This trend has been partly reversed with
  ▪ An increased theoretical interest for the firm’s internal labour market or personnel economics.
  ▪ The availability of employer-employee linked data-bases, as well as employer based surveys.
  ▪ The growth of theoretical and empirical studies of the dynamic adjustment of employment and hours (job creation and destruction from the firms’ birth and death).
  ▪ Increased government interventions that change the incentives facing employers making decisions about employment and hours, such as
    o minimum wages
    o overtime pay
    o subsidized training
    o family leaves
    o hiring subsidies
  ▪ An increased interest in technological change, especially skill-biased technological change or routine-biased technological change.
• Our ultimate goal will be to show how the parameters describing the employers’ long-run demand for labour can be inferred from data characterizing their employment, wages, product demand, and in some cases, the prices and quantities of other inputs.

• Our study of the static theory of the firm’s labour demand will mostly on issues of substitution among inputs into production: capital vs. labour, low skilled vs. high skilled labour.
  
  o These substitution effects are at the heart of the theory of the skill premia, that has been paramount in explaining the growth in wage inequality

• But first, let’s consider some polar views of the labor-demand curve, that will used in the context of the impacts of aggregate shocks such as minimum wage increases or immigrants influx.
• There are two polar ways of viewing labour demand depending whether the firm faces an inelastic or elastic labour supply:

1. the wage is exogenous (and thus appears fixed) to the individual firm, who faces an infinitely or perfectly elastic supply (horizontal supply curve).
   - A wage shock will result in an adjustment of employment (e.g. a higher minimum wage will result in less employment).
   - In this case (e.g. perfect competition), wage elasticities of labour demand will allow one to infer the effects of exogenous changes in wage rates.

2. the supply of particular type of labor is exogenous (thus appears fixed) to the firm, who faces an inelastic supply curve (vertical supply curve).
   - A supply shock will result in an adjustment of employment (e.g. the black plague lead to an increase in wages.)
   - Again this case (e.g. long run, full-employment) knowing the slope of the demand curve will provides the information needed to infer the effect of the shock.

• In general, neither perfectly elastic nor completely inelastic supply characterizes labour markets. Instead, the supply of labour has a positive slope.
• So without knowing the slopes of both the demand and the supply curves, one cannot infer the actual size of changes in wages and employment to demand or supply shocks.

• However, from the slope of the demand curve, one can find some upper bounds to the impact of shocks. (see diagram)
  ➢ When supply is assumed exogenous, bounds to changes in wages can be inferred.
  ➢ When wage is assumed exogenous, bounds to changes in employment can be inferred.

• This “Static” theory (no adjustment costs of labour – immediate equilibrium ⇒ elasticity of labour supply) should be distinguished from
  o “Dynamic” theory which introduces adjustment costs indicates form and speed of labour adjustments⇒ useful in presence of shocks / more info on hiring and firing strategies
• We will characterize the slope of labour demand by considering different elasticities

• Own-wage Elasticity = percentage change in labour demanded/percentage change in wage

\[ \eta_{ii} = \frac{\%\Delta L_i}{\%\Delta W_i} \]

• The own-wage elasticity is negative and often elasticities are referred to as the absolute value of the above expression.
  - \(|\eta_{ii}| > 1\) Elastic \(\%\Delta L_i > \%\Delta W_i\)
  - \(|\eta_{ii}| = 1\) Unit Elastic \(\%\Delta L_i = \%\Delta W_i\)
  - \(|\eta_{ii}| < 1\) Inelastic \(\%\Delta L_i < \%\Delta w_i\)

• Short-run vs. long-run elasticities
  - Short-run (hold production technology (or K/L ratio) constant) → Scale effect
  - Long-run → Substitution effect (hold output constant) + scale effect
1.2 Downward Sloping Demand Curve

- Though the basic theorems of labour demand require assuming that there are at least two inputs in production, the motivation for the downward sloping labour demand curve can be derived when only one input is assumed.

- Assume that output $Y$ is obtained from a production progress described by a function that transforms labour services ($L$) into output:
  \[ Y = F(L) \quad \text{where} \quad F_L > 0, F_{LL} < 0 \]
  that is, where there are diminishing returns to the single input (in the background, all other inputs are fixed in the short run).

- Assume that the firm is competitive in all markets and attempts to maximize profits
  \[ \pi(L) = pF(L) - W/p \cdot L \quad \text{where} \quad F_L > 0, F_{LL} < 0 \]
  So that the FOC condition is $F_L(L) - w = 0$, where $w = W/p$ is the real wage and the SOC requires $F_{LL} < 0$. 
“Labour will be hired as long as the revenue that an extra unit of labour generates is greater than the cost of that extra unit of labour”
• Differentiation of the FOC with respect to $w$,

$$F_{LL}(L^*) \frac{dL^*}{dw} - 1 = 0 \implies \frac{dL^*}{dw} = \frac{1}{F_{LL}(L^*)} < 0$$

• Thus the more rapidly diminishing are the returns to labour, the steeper the demand curve for labour.

• This will also be true if the firm has some market power over price, that is, faces the inverse demand function $P = P(Y)$, with elasticity $\eta_P^Y \equiv YP'(Y)/P(Y)$ and reap profits:

$$\pi(L) = P(Y)Y - WL$$

• So that the FOC condition is $\pi'(L) = F_L(L)[P(Y) + P'(Y)Y] - W = F_L(L)P(Y)[1 + \eta_P^Y] - W = 0$

• When the $(1 + \eta_P^Y) > 0$, labour demand is defined by

$$F_L(L) = \nu \frac{W}{P} \text{ with } \nu \equiv \frac{1}{1+\eta_P^Y}$$
• That is, the firm maximizes profit when the marginal productivity of labour is equal to the real wage multiplied by a markup $\nu \geq 1$, a measure of market power.

• Differentiation the FOC with respect to $w$, again

$$\frac{dL^*}{dw} = \frac{\nu}{(F_L^2 p' + F_{LL})} < 0$$

• Short-run labour demand is a decreasing function of labour cost

• In the longer run, the firm may contemplate replacing parts of its workforce with machines, depending on technical feasibility.
1.3a. Elasticity of Substitution (of Capital for Labour)

- Many interesting insights from neoclassical production theory come from examining the demand for homogeneous labour in the case of two inputs.

- Assume that production exhibits **constant returns to scale**, as described by a linearly homogeneous function, $\delta Y = F(\delta L, \delta K)$, such that

$$Y = F(L, K) \quad \text{where } F_i > 0, F_{ii} < 0, F_{ij} > 0 \quad i = K, L \quad (1)$$

where $Y$ is output, $L$ is labour and $K$ is capital.

- Assuming that the firm maximizes profits

$$\pi = Y - wL - rK$$

where $w$ is the exogenous wage, $r$ is the **exogenous** price of capital services, and the price of output has been normalized to one, subject the technology (1).
• The FOCs will be: $F_L = w$ and $F_K = r$, yielding the familiar statement that the ratio of the values of the marginal product, $VMP$, the marginal rate of technical substitution equals the factor price ratio for a profit maximizing firm.

$$MRT_{KL} = \frac{VMP_L}{VMP_K} = \frac{1 \cdot F_L}{1 \cdot F_K} = \frac{w}{r}$$  \hspace{1cm} (2)

• Allen (1938) defines the elasticity of substitutions between the services of capital and labour as the effect of a change in relative factor prices on relative inputs of the two factors, holding output constant (i.e. along an isoquant).

• In the two factor linearly homogeneous case, the elasticity of substitution is:

$$\sigma = \left. \frac{d(K/L)}{d(w/r)} \right|_{Ycons} = \left. \frac{d \ln(K/L)}{d \ln(w/r)} \right|_{Ycons}$$  \hspace{1cm} (3)

• Intuitively, this elasticity measures the ease of substituting one input for the other when the firm can only respond to a change in one or both of the input prices by changing the relative use of two factors without changing output.
• Example 1: With the Cobb-Douglas technology, \( Y = AL^\alpha K^{(1-\alpha)} \),

the profit maximizing condition (2) becomes:

\[
\frac{F_L}{F_K} = \frac{K}{L} = \frac{(1-\alpha)w}{\alpha r}
\]

Taking the logarithms gives

\[
\ln\left(\frac{K}{L}\right) = \alpha' + \ln\left(\frac{w}{r}\right), \text{ where } \alpha' \text{ is a constant}
\]

Taking the differential on both sides

\[
d \ln\left(\frac{K}{L}\right) = d \ln\left(\frac{w}{r}\right) \Rightarrow \frac{d \ln(K / L)}{d \ln(w / r)} = 1 = \sigma
\]
• Using the fact that the ratio of input prices will equal the ratio of marginal products from FOC (2), the elasticity of substitution can be written in terms of marginal products

\[
\sigma = \frac{\frac{d \ln(K/L)}{d \ln(F_L/F_k)}|_{\text{Ycons}}}{\frac{d(F_L/F_k)}{d(K/L)}(K/L)|_{\text{Ycons}}} = \frac{d(K/L)}{d(F_L/F_k)}(F_L/F_k)\]

• The elasticity of substitution, thus, compares the movement in the chord from L/K to L’/K’ (denoted by \(\Delta^R\) in the Figure) to the movement in the MRTS from \(F'_K/F'_L\) to \(F_K/F_L\) (represented by \(\Delta^M\)). The elasticity of substitution is thus, intuitively speaking, merely \(\sigma = \frac{\Delta^R}{\Delta^M}\).
• It can be shown (not so easily, see Appendix) that
\[
\sigma = -\frac{F_L F_K (F_L \cdot L + F_K \cdot K)}{L \cdot K (F_K^2 \cdot F_{LL} - 2F_L F_K F_{KL} + F_L^2 \cdot F_{KK})} \Bigg|_{Y_{\text{const}} \tan t}
\] (4)

• In the case of a linear homogeneous production function, Euler’s theorem applied to the marginal product functions yields: \(F_{LL} = -F_{LK} \frac{K}{L}\) and the expression (4) simplifies to
\[
\sigma = \frac{F_L F_K}{Y \cdot F_{LK}} \Bigg|_{Y_{\text{const}} \tan t}
\] (5)

• Equation (5) shows that \(\sigma\) is always non-negative. The value of \(F_{LK}\) depends on the shape of the production function, but is always positive under usual production function assumptions.
• So in the case of the Cobb-Douglas, we get the same as above:

\[
\sigma = \frac{\alpha(1 - \alpha)L^{(2\alpha-1)}K^{(1-2\alpha)}}{(L^\alpha K^{(1-\alpha)}) \cdot \alpha(1 - \alpha)L^{a-1}K^{-a}} = 1
\]

• The elasticity of substitution becomes a property of the curvature of the isoquant and is thus always positive. The larger the value of \( \sigma \), the flatter the constant product curve (isoquant) and the more slowly does the marginal rate of substitution increase as \( K \) is substituted for \( L \).

• There are two limiting cases:

1) if \( K \) and \( L \) are perfect substitutes, so that constant product is maintained by increasing \( K \) in proportion as \( L \) is decreased, then the isoquant is a straight line and \( d^2K / dL^2 = 0 \) and \( \sigma \) is infinite.
2) if $K$ and $L$ are entirely incapable of substitution, being needed in a fixed proportion, then an increase in one of the factors from this proportion must leave the product unchanged. The isoquant has a right angle, and $\frac{d^2 K}{dL^2}$ is infinite and $\sigma = 0$. 
1.3 b. Cross-Elasticities and Own-Price Elasticities

- The cross-elasticities of labour demand with respect to the price of capital services or of the demand for capital services with respect the wage rate are found from the comparative statics of cost-minimization (holding output constant):

$$\tilde{\eta}_r^L = \frac{r}{r} \frac{\partial L}{\partial r} \quad \text{and} \quad \tilde{\eta}_w^K = \frac{w}{w} \frac{\partial K}{\partial w}$$

- A firm chooses $L$ and $K$ to minimize: $C = wL + rK$, subject to output takes on a particular value of output: $Y = F(L, K)$

- After solving for $L^*, K^*$ from the first order conditions,

$$\frac{F_L(L^*, K^*)}{F_K(L^*, K^*)} = \frac{w}{r} \quad \text{and} \quad F(L^*, K^*) = Y$$

we can express costs that minimize a certain level of production, subject to $w$, $r$, and $Y$:

$$C^* = C(w, r, Y)$$

- This is the cost function, which has several useful properties that are derived from the assumptions about the production function and the firm’s optimizing behaviour.
Elasticity of Substitution

A very important concept is that of elasticity of substitution between labour and capital (the formula of the elasticity of substitution is derived in Section 1.2.2 of chapter 4 - CZ).

“The elasticity of substitution measures the ease with which labor and capital can be substituted for one another when their relative prices change”

\[
\sigma = \frac{\% \Delta \left( \frac{K}{L} \right)}{\% \Delta \left( \frac{w}{r} \right)} = \frac{\Delta \left( \frac{K}{L} \right)}{\Delta \left( \frac{w}{r} \right)} \left( \frac{K}{L} \right) \left( \frac{w}{r} \right)
\]

Recall that Cost = \( rK + wL \)
So
\( K = \frac{C}{r} + \frac{w}{r} \cdot L \)

\( \Delta (\text{capital/labour}) = (\frac{K_2}{L_2} - \frac{K_1}{L_1}) \)
\( \text{capital/ labour} = \frac{K_2}{L_2} \)
• Among them, $C_w > 0$, $C_r > 0$, $C_{ww} < 0$, $C_{rr} < 0$ and homogeneity of degree 1 in (w,r), so that optimal levels for labor and capital demanded are equal to their respective partial derivatives [Shepard’s Lemma]:

$$L^* = C_w(w, r, Y) \text{ and } K^* = C_r(w, r, Y)$$

(6)

• Differentiation gives us

$$\frac{\partial L^*}{\partial w} = C_{ww} < 0, \quad \frac{\partial L^*}{\partial r} = \frac{\partial K^*}{\partial w} = C_{wr},$$

(7)

• From the following expression for the elasticity of substitution,

$$\sigma = \left. \frac{d(K / L)}{d(w / r)} \frac{(w / r)}{(K / L)} \right|_{Y \text{ constant}}$$

• It can be shown (again, not so simply) using Shepard’s Lemma and the homogeneity of degree 1 of the cost function that

$$\sigma = \frac{C C_{wr}}{C_w C_r}$$

where it represents the elasticity of the ratio $\frac{L^*}{K^*}$ in relation to the relative cost $\frac{w}{r}$. 
• We will then be able to obtain expressions for the cross-elasticities above in terms of $\sigma$. First, we can write $\bar{\eta}_r^L = \frac{r}{L^*} C_w r$ using (7), which leads to

$$\bar{\eta}_r^L = \frac{r C_w C_r}{L^* C} \sigma$$

• Using the labour share $s \equiv w L^*/C$, and the fact $L^* = C_w$ and $K^* = C_r$, leads to

$$\bar{\eta}_r^L = (1 - s) \sigma \quad (8)$$

• The intuition for including $(1 - s)$ here is that if capital’s share is very small, a 1 percent change in its price cannot induce a large percentage change in labour demand.

• We are also, of course, interested in the straightforward response to the change in demand for labor from a change in its wage.

• There also exists a link between the own-price elasticity $\eta_w^L$ and the elasticity of substitution $\sigma$. 
• The conditional demand for labour depending only on \( Y \) and on the ratio \( \frac{w}{r} \), we have \( \frac{\partial L^*}{\partial w} = -\frac{r}{w} \frac{\partial L^*}{\partial r} \) and consequently
\[
\bar{\eta}_w^L = -\bar{\eta}_r^L = -(1 - s)\sigma
\] (9)

• Intuitively, \( \bar{\eta}_w^L \) is smaller in absolute value (less negative) for a given technology \( \sigma \) when labour’s share is greater, because there is relatively less capital toward which to substitute when the wage rises.

• When output requires substantial amounts of labour for production, the constant output labor demand elasticity will be smaller, because the possible change in spending on other factors is small relative to the amount of labour being used.

• Note again that (8) and (9) reflect only substitution along an isoquant.

• More realistic effects should also include scale effects.
**FIGURE 4-12  Substitution and Scale Effects**

A wage cut generates substitution and scale effects. The substitution effect (the move from point $P$ to point $Q$) encourages the firm to use a more labor-intensive method of producing a given level of output. The scale effect (from $Q$ to $R$) encourages the firm to expand, further increasing the firm's employment.
1.4. Hicks-Marshall Rules of Derived Demand

- The scale effect (which is analogous to the income effect in labour supply), depends on the (absolute value) of the elasticity of product demand, and on the share of labour in total costs (which determines the percentage increase in price).

- Let $Y^*$ be the level of output that maximizes profit

$$\pi(W, R, Y) = P(Y)Y - C(W, R, Y)$$

- That is, satisfy FOC:

$$P(Y) = \nu C_Y(W, R, Y)$$

where $\nu$ is the mark-up as before.

- And let the unconditional labour demand function $L^* = C_w(w, r, Y^*)$

  Differentiating with respect to $w$, we get

$$\frac{\partial L^*}{\partial w} = C_{ww} + C_{wy} \frac{\partial Y^*}{\partial w}$$

- Multiplying both sides by $\frac{w}{L^*}$ to write this in terms of the elasticities $\eta_w^L$ and $\eta_w^Y$,

$$\eta_w^L = \frac{w}{L^*} C_{ww} + \left( \frac{C_{wy} Y^*}{L^*} \right) \eta_w^Y$$
• But the first term is just the conditional labour demand elasticity, $\bar{\eta}_w^L = \frac{w}{L^*} C_{ww} < 0$, and the second is the elasticity of labour demand with respect to output, $\bar{\eta}_Y^L = \frac{C_{wY} Y^*}{L^*}$, we get

$$\eta_w^L = \bar{\eta}_w^L + \bar{\eta}_Y^L \bar{\eta}_W^Y$$

• The relation show the different effects of a rise in wage on the demand for labour.
  o The first term is the substitution effect, a rise in the cost of labour always lead to a reduced utilization of this factor.
  o The second term is a scale effect, which is always negative ($\bar{\eta}_W^Y$ is always of opposite sign to $C_{wY}$ from SOC) and accentuates the substitution effect.

• Under constant returns to scale and perfect competition, scale effects can be expressed as a function of the labour share of total cost and of the elasticity of product demand, we get (again, not so easily, see Dixit (1976, p.79))

$$\eta_w^L = -(1 - s)\sigma - s\eta_P^Y \quad \text{and} \quad \eta_r^K = -(1 - s)\sigma - s\eta_P^Y \quad (10)$$
• Equation (10) is known as the fundamental law of factor demand: it divides labour (capital) demand elasticity into substitution (holding output constant) and scale effects (holding capital and other inputs constant: short run).

• It states the effect of an increase in the wage of a group of workers on the amount of their labour is negative. This negative response consists of a negative effect at a constant level of output, and a negative scale effect.

• When wages increase, production costs rise and raises product prices. If the elasticity of demand for the product is large then there will be large declines in output following price increases.
  o Greater the decrease in output, greater in the decline in employment in labour.
  o Greater the elasticity of product demand, greater is the elasticity of demand for labour.

• Output with a high share of labour will be affected more. If labour’s share in total costs is only 20% then a 10% increase in the wage rate will raise costs by 2%. But if the initial share is 80% then the same 10% increase in wages raises costs by 8%.
  o The greater the cost of labour in total costs, the higher the elasticity of labour demand.
However if it is easy to substitute other factors for labour then even a small share may result in a large elasticity of demand.

- Alfred Marshall (1920) used four “laws” to summarize the effects of factors that influence the own-wage elasticity. The first three laws can be seen as working through the expressions for $\eta^L_w$ and $\bar{\eta}^L_w$ above.

- The own-wage elasticity of demand for a type of labour, other things being equal, will be higher (that is, the larger the reduction in employment in response to a wage change):

  1. the more easily the other factors can be substituted for labour (proportionality to $\sigma$)

  2. the more elastic the demand for the product being produced is (increasing in absolute value with $\eta^Y_P$)

  3. the greater the share of employing the type of labor in of the total cost of production (proportionality to $s$; holds only when $|\eta^Y_P| > \sigma$).
4. the greater the supply elasticity of other factors of production (that is, usage of the other factors of production can be increased without substantially increasing their prices) (omits the maintained assumption of a constant \( r \).)

Application of Hicks-Marshall laws

- Airline competition & pilots salaries
- Pre-1978/Canada low wage elasticity of demand
  1. Difficult to substitute other inputs
  2. Low price elasticity of demand due to lack of competition amongst airlines
  3. Airline pilots small part of total costs
  4. Not as relevant

- Post-1978/U.S. de-regulation, larger wage elasticity of demand
  - Competition changes 2.
The two-input model can be generalized to $N$ factors of production. The only substantial difficulty comes in generalizing the concept of elasticity of substitution.

While the Allen elasticity of substitution can be defined using only derivatives of the cost function, its magnitude in the multifactor case may depend on particular values of the input prices, its sign however does not.

The neoclassical theory of production thus gives us a useful framework and terminology to classify demand relationships.

Using the partial elasticities of conditional factor demand $\bar{\eta}_{ij} = s_j \sigma_{ij}$, inputs $i$ and $j$ are said to be p-substitutes $\bar{\eta}_{ij} > 0$ and p-complements if $\bar{\eta}_{ij} < 0$.

Two factors are called p-substitutes (or p-complements) if the conditional demand for one of them increases (or falls off) when the cost of the other factor rises.
• For example, if skilled and unskilled labour are p-substitutes, one may infer that a rise in the price of skilled labour, will increase the mix of unskilled labour in production.

• Using the partial elasticities of factor prices \( \varepsilon_{ij} = \frac{\partial \ln w_i}{\partial \ln X_j} = s_j c_{ij} \), where \( c_{ij} = \frac{Yf_{ij}}{f_if_j} \)

inputs \( i \) and \( j \) are said to be q-complements (q-substitutes) if \( \varepsilon_{ij} > 0(<0) \).

• If two factors are also q-complements, an increase in the number of skilled workers will raise the wage of unskilled workers by increasing their relative scarcity.

• The hypothesis of skill-biased technological change can be formulated as reflecting a complementarity between capital and skilled labour, and a substitution between capital and unskilled labour
  
  o e.g. robots replacing assembly workers but requiring engineers for design and programming.
Summary  The following parametric changes result in an increase in short-run supply:

decrease in the price of an input when only one input is variable;
decrease in the price of a normal input (two or more inputs variable);
increase in the price of an inferior input (two or more inputs variable);
increase in a fixed input that raises the marginal product of a single variable input;
decrease in a fixed input that lowers the marginal product of a single variable input;
neutral technological change;
biased technological change that raises the marginal product of the variable input,
5. Estimating the Elasticity of Labour Demand

- Empirically (see Table 8.2), in developed economies in the late twentieth century, the labour-demand elasticity was estimated to be in the range 0.15-0.5, which puts a limit on the likely effects of wage subsidies to change the relative labor intensity of production.

- The “game” of estimating these elasticities is to propose a production function that ameliorates the estimation process. For example, forget using Cobb-Douglas: the elasticity of substitution is fixed at one.

- As another example, another production function is the Constant Elasticity of Substitution function (CES), which, as you might guess from the name, the elasticity of labor demand does not depend on current production, or costs. The CES function is:

  \[ Y^\rho = \alpha L^\rho + (1 - \alpha)K^\rho \quad \Leftrightarrow \quad Y = [\alpha L^\rho + (1 - \alpha)K^\rho]^{1/\rho} \]

- The marginal products are \( \frac{\partial y}{\partial L} = \alpha \left( \frac{y}{L} \right)^{1-\rho} \) and \( \frac{\partial y}{\partial K} = (1 - \alpha) \left( \frac{y}{K} \right)^{1-\rho} \)
So that
\[
\frac{F_k}{F_L} = \frac{1 - \alpha}{\alpha} \left( \frac{L}{K} \right)^{1-\rho} = \frac{r}{w}
\]

taking the logarithm and differentiating with respect to \(\ln(w/r)\) gives
\[
\frac{\partial \ln(L/K)}{\partial \ln(w/r)} = \sigma = \frac{1}{1 - \rho}
\]

We could try to estimate this equation, adding an error term.
\[
\ln(L/K) = \beta_0 + \beta_1 \ln(w/r) + \varepsilon
\]

where an estimate of the constant-output elasticity of labour demand would be \(\hat{\beta}_1\).

Unfortunately, this specification seems grossly unrealistic: the elasticity does not depend on the current level of production, or the current relative use of each factor.

Note, if the price of capital is constant, we are in effect estimating a regression equation similar to one seen before for labour supply.
• We need some way of determining whether the wage fluctuations are due to exogenous changes in labor supply, or exogenous changes in labor demand. We also need to assure no omitted variables bias.

• The credibility of these estimates depends crucially on the research design of the analysis.

• Perhaps the most popular method of estimating the elasticities of labor demand is to use the **translog cost function** (introduced by Erwin Diewert), which is often interpreted as a second-order approximation to an unknown functional form, using the cost shares of many inputs (see Berman, Bound and Griliches, 1994).
4 The Translog Cost Function

4.1 The Model

The cost function has as its arguments the level of output and input prices. In particular, the translog cost function could be considered as a second-order Taylor’s series approximation in logarithms to an arbitrary cost function (see Christensen et al., 1973). The more general specification of the translog cost function imposes no prior restriction on the production structure, that is, it does not impose, ex ante, neutrality, homotheticity, homogeneity, constant returns to scale, or unitary elasticities of substitution; in fact, it allows to test these alternative production configurations.

The translog cost function can be written as

\[ \ln C = \alpha_0 + \sum_{i=1}^{i,N} \alpha_i \ln P_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln P_i \ln P_j + \alpha_y \ln Y + \frac{1}{2} \gamma_{yy} (\ln Y)^2 + \sum_{i=1}^{N} \gamma_{iy} \ln P_i \ln Y, \]  

where \( i, j = 1, \ldots, N \) index the \( N \) different inputs considered and \( \gamma_{ij} = \gamma_{ji} \), \( C \) is total cost, \( Y \) is output and the \( P_i \)'s are the prices of the factor inputs. For a cost function to be well behaved it must be homogeneous of degree one in prices, implying that, for a fixed level of output, total cost must increase proportionally when all prices increase proportionally. Thus, the following restrictions on equation (2) apply

\[ \sum_{i=1}^{i,N} \alpha_i = 1, \]  

\[ \sum_{i=1}^{i,N} \gamma_{iy} = 0, \]  

\[ \sum_{i=1}^{i,N} \gamma_{ij} = \sum_{j=1}^{j,N} \gamma_{ij} = \sum_{i=1}^{i,N} \sum_{j=1}^{j,N} \gamma_{ij} = 0. \]  

As mentioned, a number of additional parameter restrictions can be imposed on the translog cost function, which implicitly represent the underlying technology. Homotheticity means that the cost function can be written as a separable function in output and factor prices.\(^\text{14}\) For homotheticity, it is necessary and sufficient that

\[ \gamma_{iy} = 0, \forall i. \]  

The cost function is homogeneous in output if the elasticity of cost with respect to output is constant, this occurs with the following restrictions

\[ \gamma_{iy} = 0, \quad \gamma_{yy} = 0, \]  

in this case the degree of homogeneity equals \( \frac{1}{\alpha_y} \). There are constant returns to scale (CRS) of the dual production function when, in addition to equation (7)

\[ \alpha_y = 1. \]  

\(^{14}\)With nonhomothetic cost functions their ratios of cost-minimising inputs demands are allowed to depend on the level of output, by contrast, with homothetic functions relative input demands are independent of the level of output.
Ultimately, the translog function becomes a constant returns to scale Cobb-Douglas function if, besides to the previous restrictions, each of the

\[ \gamma_{ij} = 0, \quad \forall i. \]  

(9)

Actually, the elasticities of substitution can all be restricted to unity by the elimination of the second-order terms in the prices from the translog cost function and can be applied to the translog, homothetic, homogeneous and/or the CRS models.

Direct estimation of equation (2) can be carried out. However, gains in efficiency can be obtained if the optimal cost-minimising input demand equations, cost-share equations, are estimated jointly with equation (2). More specifically, with a set of cost-share equations directly related with the translog cost function as implied by duality theory. Following Shepard's Lemma, the derived demand for an input is obtained by partially differentiating the cost function with respect to input prices \( \frac{\partial C}{\partial P_i} = Z_i \). Thus, partially differentiating equation (2) and using Shepard’s Lemma, such cost-share equations can be obtained

\[ \frac{\partial \ln C}{\partial \ln P_i} = \frac{P_i}{C} \frac{\partial C}{\partial P_i} = \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln P_j + \gamma_{iy} \ln Y, \]

(10)

where \( \sum_{i=1}^{N} P_i X_i = C \). If \( S_i \equiv \frac{P_i X_i}{C} \), then \( \sum_{i=1}^{N} S_i = 1 \).

The necessary restrictions given by equations (3), (4), and (5) are imposed to the constraint \( \sum_{i=1}^{N} S_i = 1 \) as well, which also implies that only \( N - 1 \) of the share equations in (10) are linearly independent.

Once the coefficients are estimated, one can construct Allen partial elasticities of substitution between two factors \( i \) and \( j \) (Uzawa, 1962). These elasticities are crucial to describe the pattern and degree of substitutability and complementarity amongst the factors of production. Basically, they measure the percentage change in factor proportions due to a one-percent change in their relative prices. For the translog this implies

\[ \sigma_{ij} = \frac{\gamma_{ij}}{S_i S_j} + 1 \quad \text{for } i \neq j. \]

(11)

In addition, one can compute own- and cross-price elasticities of factor demand (ceteris paribus, how the demand for input \( i \) responds with respect to changes in its own price or to changes in the price of input \( j \)) as \( \eta_{ij} = S_i \sigma_{ij} \), which can be calculated as

\[ \eta_{ii} = \frac{\gamma_{ii}}{S_i} + S_i - 1, \]

\[ \eta_{ij} = \frac{\gamma_{ij}}{S_i} + S_j \quad \text{for } i \neq j. \]

(12)

Based on Hanoch (1975), economies of scale must be evaluated along the expansion path, that is, where factor prices are constant and costs are minimised at every level of output; whereas returns to scale are usually defined along an arbitrary input-mix ray. In fact, if the production function is homothetic, both returns to scale and economies of scale will be the same.

Economies of scale are defined in terms of the relative increase in output resulting from a proportional increase in all inputs. This is expressed as one minus the elasticity of total cost with respect to output

\[ \Psi = 1 - \frac{\partial \ln C}{\partial \ln Y}. \]

(13)
<table>
<thead>
<tr>
<th>Theoretical forms</th>
<th>Estimating forms and demand elasticities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Cobb–Douglas</td>
<td></td>
</tr>
<tr>
<td>(a) Cost</td>
<td>ln ( C/Y = \sum a_i \ln w_i )</td>
</tr>
<tr>
<td>( C = Y^a \prod w_i^{\alpha_i}; \quad \alpha_i = 1 ) if CRS</td>
<td></td>
</tr>
<tr>
<td>Production</td>
<td>ln ( Y = \sum \beta_i \ln X_i ); \eta_{ii} = \left[ 1 - \beta_i \right] )</td>
</tr>
<tr>
<td>( Y = \prod X_i^{\beta_i}; \quad \beta_i = 1 ) if CRS</td>
<td></td>
</tr>
<tr>
<td>2. CES</td>
<td></td>
</tr>
<tr>
<td>(a) Cost</td>
<td>ln ( X_i = a_0 + \sigma \ln w_i + \sigma \ln Y; \eta_{ii} = \sigma )</td>
</tr>
<tr>
<td>( C = Y^a \left[ \sum \alpha_i w_i^{\alpha_i(1-\sigma)} \right]^{1/(1-\sigma)}, \quad \alpha_i = 1 ) if CRS</td>
<td>Little use</td>
</tr>
<tr>
<td>(b) Production</td>
<td></td>
</tr>
<tr>
<td>( Y = \left[ \sum \beta_i X_i \right]^{b/\sigma}, \quad b = 1 ) if CRS</td>
<td></td>
</tr>
<tr>
<td>3. Generalized Leontief</td>
<td></td>
</tr>
<tr>
<td>(a) Cost</td>
<td></td>
</tr>
<tr>
<td>( C = Y \sum a_{ij} w_i^{0.5} w_j^{0.5}, \quad a_{ij} = a_{ji} )</td>
<td></td>
</tr>
<tr>
<td>( X_i = a_{ii} + \sum_j a_{ij} \left( w_j / w_i \right)^{0.5}, \quad i = 1, \ldots, N; \eta_{ij} = \frac{s_j a_{ij}}{2 [X_i X_j s_i s_j]^{0.5}} )</td>
<td></td>
</tr>
<tr>
<td>(b) Production</td>
<td></td>
</tr>
<tr>
<td>( Y = \sum b_{ij} X_i^{0.5} X_j^{0.5}, \quad b_{ij} = b_{ji} )</td>
<td></td>
</tr>
<tr>
<td>( w_i = b_{ii} + \sum_j b_{ij} \left( X_j / X_i \right)^{0.5}, \quad i = 1, \ldots, N )</td>
<td></td>
</tr>
<tr>
<td>( e_{ij} = \frac{s_j b_{ij}}{2 [w_i w_j s_i s_j]^{0.5}} )</td>
<td></td>
</tr>
<tr>
<td>( e_{ii} = \frac{b_{ii} - w_i}{2 w_i} )</td>
<td></td>
</tr>
<tr>
<td>4. Translog</td>
<td></td>
</tr>
<tr>
<td>(a) Cost</td>
<td></td>
</tr>
<tr>
<td>( \ln C/Y = a_0 + \sum a_i \ln w_i + 0.5 \sum b_{ij} \ln w_j \ln w_i )</td>
<td></td>
</tr>
<tr>
<td>( b_{ij} = b_{ji} )</td>
<td></td>
</tr>
<tr>
<td>( s_i = a_i + \sum_j b_{ij} \ln w_j, \quad i = 1, \ldots, N )</td>
<td></td>
</tr>
<tr>
<td>( \eta_{ij} = [b_{ij} + s_i s_j]/s_i )</td>
<td></td>
</tr>
<tr>
<td>( \eta_{ii} = [b_{ii} + s_i^2 - s_i]/s_i )</td>
<td></td>
</tr>
<tr>
<td>(b) Production</td>
<td></td>
</tr>
<tr>
<td>( \ln Y = a_0 + \sum \alpha_i \ln X_i + 0.5 \sum \beta_{ij} \ln X_j \ln X_i )</td>
<td></td>
</tr>
<tr>
<td>( \beta_{ij} = \beta_{ji} )</td>
<td></td>
</tr>
<tr>
<td>( s_i = \alpha_i + \sum_j \beta_{ij} \ln X_j, \quad i = 1, \ldots, N )</td>
<td></td>
</tr>
<tr>
<td>( \epsilon_{ij} = [\beta_{ij} + s_i s_j]/s_i )</td>
<td></td>
</tr>
<tr>
<td>( \epsilon_{ii} = [\beta_{ii} + s_i^2 - s_i]/s_i )</td>
<td></td>
</tr>
<tr>
<td>Author and source</td>
<td>Data and industry coverage</td>
</tr>
<tr>
<td>------------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td><strong>I. Labor demand studies</strong></td>
<td></td>
</tr>
<tr>
<td>A. Marginal productivity condition on labor (estimates of ( \eta_{LL}/[1-s] ))</td>
<td>( \eta_{LL} )</td>
</tr>
<tr>
<td>Dhrymes (1969)</td>
<td>Private hours, quarterly, 1948–60</td>
</tr>
<tr>
<td>Drazen et al. (1984)</td>
<td>Manufacturing hours, quarterly, 10 OECD countries, mostly 1961–80</td>
</tr>
<tr>
<td>Hamermesh (1983)</td>
<td>Private nonfarm, quarterly, based on labor cost, 1955–78</td>
</tr>
<tr>
<td>Liu and Hwa (1974)</td>
<td>Private hours, monthly, 1961–71</td>
</tr>
<tr>
<td>Lucas and Rapping (1970)</td>
<td>Production hours, annual, 1930–65</td>
</tr>
<tr>
<td>Rosen and Quandt (1978)</td>
<td>Private production hours, annual, 1930–73</td>
</tr>
<tr>
<td><strong>B. Labor demand with price of capital</strong></td>
<td></td>
</tr>
<tr>
<td>Chow and Moore (1972)</td>
<td>Private hours, quarterly, 1948:IV–1967</td>
</tr>
<tr>
<td>Clark and Freeman (1980)</td>
<td>Manufacturing quarterly, 1950–76: Employment</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Ndiri (1968)</td>
<td>Manufacturing quarterly, 1947–64: Employment</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Nickell (1981)</td>
<td>Manufacturing quarterly, 1958–74, United Kingdom (materials prices)</td>
</tr>
<tr>
<td>Tinsley (1971)</td>
<td>Private nonfarm, quarterly, 1954–65: Employment</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>C. Interrelated factor demand</strong></td>
<td></td>
</tr>
<tr>
<td>Coen and Hickman (1970)</td>
<td>Private hours, annual, 1924–40, 1949–65</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Schott (1978)</td>
<td>British industry, annual, 1948–70: Employment</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>II. Production and cost function studies</strong></td>
<td></td>
</tr>
<tr>
<td>A. CES production functions</td>
<td></td>
</tr>
<tr>
<td>Brown and deCani (1963)</td>
<td>Private nonfarm hours, annual, 1933–58</td>
</tr>
<tr>
<td>David and van de Klundert (1965)</td>
<td>Private hours, annual, 1899–1960</td>
</tr>
<tr>
<td>McKinnon (1963)</td>
<td>2-digit SIC manufacturing, annual, 1947–58</td>
</tr>
<tr>
<td>B. Translog cost functions</td>
<td></td>
</tr>
<tr>
<td>Berndt and Khaled (1979)</td>
<td>Manufacturing, annual, 1947–71; capital, labor, energy and materials: Homogeneous, neutral technology change</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Magnus (1979)</td>
<td>Enterprise sector, annual, 1950–76, Netherlands; capital, labor and energy</td>
</tr>
<tr>
<td>Morrison and Berndt (1981)</td>
<td>Manufacturing, annual, 1952–71; capital, labor energy and materials</td>
</tr>
<tr>
<td>Pindyck (1979)</td>
<td>10 OECD countries, annual, 1963–73; capital, labor and energy</td>
</tr>
</tbody>
</table>

<sup>a</sup>Simple average of country estimates.

<sup>b</sup>Estimates calculated at the sample end-point.
### Table 8.4
Studies of substitution of production and nonproduction workers.

<table>
<thead>
<tr>
<th>Study</th>
<th>Data and method</th>
<th>$\sigma_{BK}$</th>
<th>$\sigma_{WK}$</th>
<th>$\sigma_{BW}$</th>
<th>$\eta_{BB}$</th>
<th>$\eta_{WW}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I. Capital excluded</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A. Cost functions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Freeman and Medoff (1982)</td>
<td>Manufacturing plants, 1968, 1970, and 1972; detailed industry dummy variables; CES Union</td>
<td></td>
<td></td>
<td>0.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Nonunion</td>
<td></td>
<td></td>
<td>0.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B. Production functions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dougherty (1972)</td>
<td>States, Census of Population, 1960; CES</td>
<td></td>
<td></td>
<td>4.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>II. Capital Included</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A. Cost functions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Berndt and White (1978)</td>
<td>Manufacturing, 1947–71; translog, 1971 elasticities</td>
<td>0.91</td>
<td>1.09</td>
<td>3.70</td>
<td>-1.23</td>
<td>-0.72</td>
</tr>
<tr>
<td>Clark and Freeman (1977)</td>
<td>Manufacturing, 1950–76; translog, mean elasticities</td>
<td></td>
<td>-1.98</td>
<td>0.91</td>
<td>-0.58</td>
<td>-0.22</td>
</tr>
<tr>
<td>Dennis and Smith (1978)</td>
<td>2-digit manufacturing, 1952–73; translog, mean elasticitiesa</td>
<td>0.14</td>
<td>0.38</td>
<td>-0.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Denny and Fuss (1977)</td>
<td>Manufacturing, 1929–68; translog, 1968 elasticities</td>
<td>1.50</td>
<td>-0.91</td>
<td>2.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Freeman and Medoff (1982)</td>
<td>Pooled states and 2-digit manufacturing industries, 1972; translog, Union</td>
<td>0.94</td>
<td>0.53</td>
<td>-0.02</td>
<td>-0.24</td>
<td>-0.12</td>
</tr>
<tr>
<td></td>
<td>Nonunion</td>
<td>0.90</td>
<td>1.02</td>
<td>0.76</td>
<td>-0.43</td>
<td>-0.61</td>
</tr>
<tr>
<td>Grant (1979)</td>
<td>SMSAs, Census of Population, 1970; translog, Professionals and managers</td>
<td>0.47</td>
<td>0.08</td>
<td>0.52</td>
<td>-0.32</td>
<td>-0.18</td>
</tr>
<tr>
<td></td>
<td>Sales and clericals</td>
<td></td>
<td>0.46</td>
<td>0.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kesselman et al. (1977)</td>
<td>Manufacturing, 1962–71; translog, 1971 elasticities</td>
<td>1.28</td>
<td>-0.48</td>
<td>0.49</td>
<td>-0.34</td>
<td>-0.19</td>
</tr>
<tr>
<td>Woodbury (1978)</td>
<td>Manufacturing, 1929–71; translog, 1971 elasticities</td>
<td></td>
<td></td>
<td>0.70</td>
<td>-0.52</td>
<td></td>
</tr>
</tbody>
</table>
### Table 8.4 continued

<table>
<thead>
<tr>
<th>Study</th>
<th>Data and method</th>
<th>$\sigma_{BK}$</th>
<th>$\sigma_{WK}$</th>
<th>$\sigma_{BW}$</th>
<th>$\eta_{BB}$</th>
<th>$\eta_{WW}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B. Production functions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Berndt and Christensen (1974b)</td>
<td>Manufacturing, 1929–68; translog 1968 elasticities</td>
<td>2.92</td>
<td>-1.94</td>
<td>5.51</td>
<td>-2.10</td>
<td>-2.59</td>
</tr>
<tr>
<td>Chiswick (1978)</td>
<td>States, Census of Population, 1910 and 1920 manufacturing; CES professionals vs. others</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Denny and Fuss (1977)</td>
<td>Manufacturing, 1929–68; translog 1968 elasticities</td>
<td>2.86</td>
<td>-1.88</td>
<td>4.76</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hansen et al. (1975)</td>
<td>3- and 4-digit industries, Census of Manufactures, 1967, translog.*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>highest quartile of plants</td>
<td>6.0</td>
<td>-1.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>lowest quartile of plants</td>
<td>2.0</td>
<td>-1.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Estimates are medians of parameters for individual industries.

*Ranked by value added per manhour. Estimates are medians of parameters for individual industries.

---

### Table 8.3

Industry studies of labor demand.

<table>
<thead>
<tr>
<th>Author and source</th>
<th>Data and industry coverage</th>
<th>$\eta_{LL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ashenfelter and Ehrenberg (1975)</td>
<td>State and local government activities, states, 1958–69</td>
<td>0.67*</td>
</tr>
<tr>
<td>Field and Grebenstein (1980)</td>
<td>2-digit SIC manufacturing, annual, 1947–58</td>
<td>0.29*</td>
</tr>
<tr>
<td>Freeman (1975)</td>
<td>U.S., university faculty, 1920–70</td>
<td>0.26</td>
</tr>
<tr>
<td>Hopcroft and Symons (1983)</td>
<td>U.K. road haulage, 1953–80, capital stock held constant</td>
<td>0.49</td>
</tr>
<tr>
<td>Lovell (1973)</td>
<td>2-digit SIC manufacturing, states, 1958</td>
<td>0.37*</td>
</tr>
<tr>
<td>McKinnon (1963)</td>
<td>2-digit SIC manufacturing, annual, 1947–58</td>
<td>0.29*</td>
</tr>
<tr>
<td>Waud (1968)</td>
<td>2-digit SIC manufacturing, quarterly, 1954–64</td>
<td>1.03*</td>
</tr>
</tbody>
</table>

*Weighted average of estimates, using employment weights.
Take-Away on Estimates of Aggregate Demand Elasticities

- Cahuc & Zylberberg quoting Hamermesh (1993)
  Unconditional $\eta^L_w = -1.0$
  Conditional $\bar{\eta}^L_w = -0.3 \ [−0.15 \text{ to } −0.75]\$

- Given that as shown earlier,
  \[ \bar{\eta}^L_w = -(1 - s)\sigma \]
  where \( s \) is share of labour in total cost and \( \sigma \) is the elasticity of substitution of labour and capital.

- Given conditional elasticity of -0.3 and known share of labour of 0.7 in most advanced countries, then a reasonable elasticity of substitution between capital and labour is \( \sigma \approx 1.0 \), very close to the Cobb-Douglas with \( \alpha = 0.7 \).
Basic readings:
