Phil 321
Assignment 3: Solutions (Total marks: 20)

1. a) R1, C3 **(1 mark)**
b) R3, C2 -- by admissibility reasoning. **(1 mark)**

BONUS (up to +2 marks)
c) No pure equilibrium. Use q_1, q_2 $1 - q_1 - q_2$ for the three columns and p, 1-p for the two rows. Then:

EU(Row) $= 2pq_1 - 4pq_2 + p - q_1 + 3q_2 + 5$

Write in two ways:

$= p(2q_1 - 4q_2 + 1) - q_1 + 3q_2 + 5$
$= q_1(2p - 1) + q_2(-4p + 3) + p + 5$

Row wants to maximize; Col wants to minimize this.

- If $2q_1 - 4q_2 + 1 = 0$, then Row will have no incentive to change strategy. Further, this condition (call it *) is actually necessary for an equilibrium.

- Given that * is satisfied, the minimum value for EU(Row) is achieved if $q_1 = 0$, $q_2 = ¼$. For any other choice (consistent with *), Col has an incentive to switch to this one.

- Finally, the second way of writing EU(Row) tells us that $p = ¾$ is required for an equilibrium. For $p < ¾$, there is an incentive for Col to set $q_2 = 0$, and for $p > ¾$, there is an incentive to set $q_2 = 1$.

Hence: $p = ¾$, $q_1 = 0$, $q_2 = ¼$ is the only equilibrium.

d) No 'pure' equilibrium. Using the formula for 2 by 2 zero-sum games: p = ½ and q = 1/6. **(1 mark)**

2. a) If A places two stones, s/he wins no matter what B does. **(1 mark)**
b) If n = 3k, B wins: this is clear if n = 3, and if n = 6, 9, 12, etc. B just **(2 marks)** puts down 2 if A puts down 1, and 1 if A puts down 2, until the game ends.

If n = 3k + 1 or n = 3k + 2, then A wins. For A's first move is to put down 2 if n = 3k+2, or 1 if n=3k+1, and from then on A can then use the strategy just described for B. (Rigorous proof would be by mathematical induction on k.)

3. a) Row 1 is dominated by Row 2. Col 2 is then dominant in the sub-game. Then R2, C2 is the solution. **(1 mark)**

b) This is "Chicken". Two pure equilibria: (R1, C2) and (R2, C1). **(3 marks)**

There is also a mixed equilibrium. Letting p and q be as usual, we get

EU(Row) = -3pq + p - q + 2. Usual technique of factoring out q fails; factoring out p yields q = 1/3.

EU(Col) = -3pq + q - p + 2. Factoring out q yields p = 1/3.

So p = 1/3, q = 1/3 is an equilibrium.

In the absence of communication, this is the most reasonable solution. Each player has an EU of 4/3. With communication, the players might try to co-ordinate by flipping a coin between the two pure equilibria, so that each has an EU of 2.

4. Game table (using EMV for each outcome): **(3 marks)**

		XYZ: Bid $225k	XYZ: Bid $260k
ABC	Bid $225k	(12.5k, 2.5k)	(25k, 0)
ABC	Bid $260k	(0, 5k)	(30k, 20k)

We use EMV instead of utility here (or assume utility is linear in EMV).

Two pure equilibria: (R1, C1) and (R2, C2).

One mixed equilibrium. Use p for ABC, q for XYZ as usual. Get:

EMV(ABC) = -5p + 17.5pq - 30q + 30

Usual technique of factoring out q fails; factoring out p yields q=2/7 .

EMV(XYZ) = 17.5pq - 20p - 15q + 20

Usual technique fails; factoring out q yields p = 6/7 .

So the equilibrium mixed strategies are p = 6/7 for ABC, and q = 2/7 for XYZ.

With communication, the players will opt for the solution (R2, C2). I'd say this would even be the solution without communication, because it is optimal. But one could argue that the mixed strategy, with a guaranteed EMV, would be adopted.

5. No, the result fails in 3-person games. Example: **(2 marks)**

(1, 1, -2)	(2, -1, -1)
(0, 0, 0)	(1, -1, 0)

(1, 2, -3)	(1, -1, 0)
(3, -4, 1)	(2, -3, 1)

Here, both (A1, B1, C1) and (A2, B2, C2) are equilibria, yet payoffs differ.

6. "Always cheat" does the trick: gets A up to 30.95%. **(5 marks)**

However, my instructions for doing this problem were misleading. I stated that you find the proportion p′(A) at the end of one year using v(A) / [v(A) + v(B) + v(C) + v(D)]. That's correct, but only for the first year, because the starting proportions are all equal. In general, the correct formula for getting to the next year, assuming p(A), p(B), p(C) and p(D) are the previous proportions, is:

$$p'(A) = p(A)\, v(A) / [p(A)\, v(A) + p(B)\, v(B) + p(C)\, v(C) + p(D)\, v(D)].$$

The denominator is just the 'average payoff' in the population, and the ratio v(A) over this denominator is the relative payoff to group A. In the special case of the first year, all the proportions are equal and they cancel out.

Although we did examples in class using the correct formula, I have been very lenient in grading the question because it was quite natural for you to carry over the wrong formula for the second year.

a) Cheat. (Table shows payoffs for Row player only.)

b)

	A	B	C	D	EU
A	12	12	24	16	16
B	12	12	24	16	16
C	6	6	18	14	11
D	10	10	20	16	14

c) The expected utility (or expected payoff) for each group is calculated using the initial proportion of 0.25 for each group. So for instance,

$$EU(A) = 0.25(12) + 0.25(12) + 0.25(24) + 0.25(16) = 16.$$

The average earned utility is

$$U = 0.25(16) + 0.25(16) + 0.25(11) + 0.25(14) = 14\tfrac{1}{4}$$

For the final proportions p′ in the population at the end of year one, compute the relative fitness EU / U for each strategy and multiply by the initial proportion of 0.25. For instance,

$$p'(A) = (16 / 14.25)(0.25) = 0.281.$$

The full set of answers is as follows:

$p'(A) = 0.281$
$p'(B) = 0.281$
$p'(C) = 0.193$
$p'(D) = 0.246.$

d) First, EU is computed from the above table, weighted by the new proportions. For instance,

$$EU(A) = 0.281(12) + 0.281(12) + 0.193(24) + 0.246(16) = 15.3$$

The full set of answers:

$EU(A) = EU(B) = 15.3$
$EU(C) = 10.3$
$EU(D) = 13.4$

The average earned utility is now

$$U = 0.281(15.3) + 0.281(15.3) + 0.193(10.3) + 0.246(13.4) = 13.9$$

For the final proportions p'' in the population at the end of year two, compute the new relative fitness EU / U for each strategy and multiply by the proportion p'. For instance,

$$p''(A) = (15.3 / 13.9)(0.281) = 0.309$$

The full set of answers is as follows:

$p''(A) = 0.309$
$p''(B) = 0.309$
$p''(C) = 0.143$
$p''(D) = 0.237$

The proportion for A (and B) at the end of two years is thus about 30.9%, which exceeds the required 30%.