I. Game Theory: Basic Concepts

1. Simultaneous games

All players move at the same time. Represent with a *game table*. We’ll stick to 2 players, generally “A and B” or “Row and Col”.

Representation of utilities/preferences

In table form, there are three common conventions:

- Just represent row player’s utilities
  (for *strictly competitive* or *zero-sum* games, in which one player’s gain exactly matches the other’s loss)
- Use (Row, Col) format to list both
- Use staggered payoff representation

Example 1: Matching pennies

<table>
<thead>
<tr>
<th></th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td></td>
</tr>
<tr>
<td>Heads</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Heads</td>
<td>-1</td>
</tr>
<tr>
<td>Tails</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Tails</td>
<td></td>
</tr>
<tr>
<td>Heads</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
**Example 2:** Prisoner’s Dilemma.

\[
\begin{array}{c|cc}
 & \text{Confess} & \text{Don’t} \\
\hline
\text{Confess} & (10, 10) & (3, 15) \\
\text{Don’t} & (15, 3) & (5,5) \\
\end{array}
\]

**Example 3:** Chicken

\[
\begin{array}{c|cc}
 & \text{Hold} & \text{Swerve} \\
\hline
\text{Hold} & (-100, -100) & (10, -5) \\
\text{Swerve} & (-5, 10) & (-1, -1) \\
\end{array}
\]
2. **Sequential games**

Players take turns. Represented with a *game tree* consisting of nodes (representing choices by players) and branches (representing different options).

![Game Tree Diagram]

A player’s *information* at some stage in a sequential game is represented by grouping together nodes at that stage into an *information set* which reflect her knowledge of the other player’s past moves. A player has *perfect information* if her information set has just one node; *imperfect* if more than one.

**Key Principle:** Reason backwards from end points of game, assuming each player wants to maximize utility.

**Examples:** Chess, “3-way duel”, bargaining

a) **3-way duel:**

1) Shooting order: Larry, Moe, Curly
2) Each participant takes one shot on his turn
3) Accuracy: Larry (30%), Moe (80%), Curly (100%)

What is Larry’s best strategy?

**Variant:** **4-way duel:**

Shooting Order: A (25%), B(50%), C(75%), D (100%)

What is A’s best strategy?
b) Bargaining:

1) n rounds; 1/n of the total prize disappears each turn the parties fail to reach agreement
2) Players A and B alternate in proposing a division of the prize
3) If both accept a proposal, the bargaining stops; if not, it continues (unless the prize totally disappears).
4) Both players are maximizers, and assign only a slight positive utility to the other player’s getting a larger share
3. **Equivalence of games**

Two games are *equivalent* if they have the same game tree. (Tic-tac-toe and game of 15: error in Resnik, p. 124)

**Example:** Black Jack

<table>
<thead>
<tr>
<th></th>
<th>Stop on 16</th>
<th>Stop on 17</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>A: 45%</td>
<td>A: 40%</td>
</tr>
<tr>
<td><strong>B (dealer)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stop on 16</td>
<td>A: 40%</td>
<td>A: 45%</td>
</tr>
</tbody>
</table>

Equivalent to “Matching Pennies”

4. **Restrictions**

- Two-person games (simplicity)
- Finite games
  - Note: this means a finite set of **basic choices**

**Example with infinite choices:** Greatest integer game.

5. **Equivalence of representations**

The two methods of representation are formally equivalent.
6. **Key Assumptions**

1. Utility maximization: Each player seeks to maximize utility.
2. Full knowledge: Each player knows the full game tree/table.
   
   N.B. Distinction between this and *imperfect information*. E.g., scissors/rocks/paper; poker; bridge.
3. Each player knows that all other players know the full game tree/table.

A *solution* is a joint pair of strategies (for two players) that satisfies these assumptions.
II. Solutions to Games

- Enough to provide solutions for game tables
- A solution is a specification of one strategy for each player such that all players are jointly rational
- In general, a pair (or triple or n-tuple) of strategies is called a profile — for example, (R1, C2)

Key Questions:
1) Does a solution always exist?
2) If there is a solution, must it be unique?
3) How do we determine which strategy(ies) should be part of a solution? What counts as rational?

Criteria of rationality:
- Dominance and admissibility reasoning
- Equilibrium (Nash equilibrium)
a) Dominance Reasoning

*Definition:* S1 dominates S2 iff:

i) S1 is at least as good as S2 regardless of what other players do;

ii) S1 is superior to S2 in at least one case.

*Definitions:*

i) S is a *dominant* strategy for a player iff S dominates all other strategies.

ii) S is a *dominated* strategy for a player iff the player has an alternative strategy, S' which dominates S.

**Rule 1:** If you have a dominant strategy, use it. If your opponent has one, he/she will use it.

(More formally: if there is a dominant strategy, it will be part of the solution profile.)
Case 1: One or both players have a dominant strategy

Example 1: The right wine.

\[
\begin{array}{cc|cc}
& \text{C1 (Beef)} & \text{C2 (Beef/chicken)} \\
\text{R1 (Red)} & \text{Right} & \text{Right} \\
\text{R2 (White)} & \text{Wrong} & \text{Right} \\
\end{array}
\]

Example 2: Newcomb’s Problem (treated as a game)

\[
\begin{array}{cc|cc}
& \text{C1 ($1M$ in box 1)} & \text{C2 (nothing in box 1)} \\
\text{R1 (only box 1)} & $1M$ (3, 4) & $0$ (1, 2) \\
\text{R2 (both)} & $1,001,000$ (4, 1) & $1000$ (2, 3) \\
\end{array}
\]
Example 3:

<table>
<thead>
<tr>
<th>Row</th>
<th>Col</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C1</td>
</tr>
<tr>
<td>R1</td>
<td>8</td>
</tr>
<tr>
<td>R2</td>
<td>4</td>
</tr>
</tbody>
</table>
Case II: Dominated Strategies

What if neither player has a dominant strategy?

Rule 2: Eliminate all dominated strategies for all players.
(More formally: dominated strategies will not be part of a solution profile.)

Ex. 1: The football game. Numbers represent expected yardage gain for offensive player.

<table>
<thead>
<tr>
<th>Defense</th>
<th>Running</th>
<th>Passing</th>
<th>Quarterback</th>
</tr>
</thead>
<tbody>
<tr>
<td>Defense</td>
<td>Defense</td>
<td>Defense</td>
<td>Blitz</td>
</tr>
<tr>
<td>Run</td>
<td>3</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>Pass</td>
<td>9</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>
**Ex. 2:** A zero-sum game.

<table>
<thead>
<tr>
<th></th>
<th>Col</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C1</td>
</tr>
<tr>
<td>Row</td>
<td>R1</td>
</tr>
<tr>
<td></td>
<td>R2</td>
</tr>
<tr>
<td></td>
<td>R3</td>
</tr>
<tr>
<td></td>
<td>R4</td>
</tr>
</tbody>
</table>

*Admissibility reasoning:* Reasoning which selects dominant strategies and eliminates dominated strategies.
b) Equilibrium Reasoning

Example 1:

<table>
<thead>
<tr>
<th>Row</th>
<th>Col</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C1</td>
<td>C2</td>
<td>C3</td>
</tr>
<tr>
<td>R1</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>R2</td>
<td>0</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>R3</td>
<td>9</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

What is the rational choice for Row and Col?

*Definition:* A profile of strategies is called a *(Nash) equilibrium* if no player can do better by unilaterally changing strategy, given the other players’ strategies.

*Rule 3:* Look for an equilibrium set of strategies.

(More formally: A profile is a solution to a game iff it is a Nash equilibrium.)
Example 2: Modified Prisoners’ Dilemma

Length of Sentences

<table>
<thead>
<tr>
<th></th>
<th>Col</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>(10, 10)</td>
<td>(6, 15)</td>
</tr>
<tr>
<td>Row</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Don’t</td>
<td>(15, 6)</td>
<td>(5, 5)</td>
</tr>
</tbody>
</table>
Some Questions

1) Should an equilibrium profile be considered *rational*? Why?

2) Must equilibria exist in every game?

3) Must equilibria be unique if they exist?

4) What can we say if there are either no equilibria or multiple equilibria?

Answers:

3) Not always unique.

2) Need not always exist.

1) Two arguments for rationality of equilibria:

   i. **Stable**: Each player is making her best response to the other; there is *no incentive to change* strategies.

   ii. **Security (in zero-sum game)**: Each player is “maximizing” the minimum payoff he can expect. Each adopts the strategy which minimizes the *damage* he suffers.

**Minimax Test**: A pair of strategies is an equilibrium if and only if its payoff is the *minimum on its row* and the *maximum on its column.*
4a) No equilibria

*Is the criterion of equilibrium necessary* for a pair of strategies to be rational?

4b) Multiple equilibria

*Is the criterion of equilibrium sufficient* for a pair of strategies to be rational?

**Case A:** 2-person, zero-sum game.

**Proposition:** All equilibria have the same value.

**Homework Problem #1:** In a zero-sum game with more than 2 people, is the Proposition valid?

**Case B:** 2-person, non-zero-sum game.