CHAPTER IX

DIFFERENCE EQUATIONS AND LAG OPERATORS[1]

Linear difference equations underlie much work in macroeconomics and applied economic dynamics. Lag operators provide a powerful tool for solving systems of linear difference equations, and for gaining insights about their structure. Lag operators constitute a language which much of macroeconomics and dynamic econometrics takes for granted. This chapter provides an introduction to the analysis of nonstochastic difference equations via lag operators. While a variety of economic examples occur in this chapter, additional examples continue to occur throughout the remainder of this book.

1. LAG OPERATORS

The backward shift or lag operator is defined by

$$LX_t = X_{t-1}$$
$$L^n X_t = X_{t-n} \qquad \text{for} \quad n = \cdots, -2, -1, 0, 1, 2, \ldots. \tag{1}$$

Multiplying a variable X_t by L^n thus gives the value of X shifted back n periods. Notice that if $n < 0$ in (1), the effect of multiplying X_t by L^n (more precisely, the effect of "operating on X_t with L^n") is to shift X *forward* in time by $-n$ periods.[2] This language is loose. Actually, we are starting out with a *sequence* $\{X_t\}_{t=-\infty}^{\infty}$ which associates a real number X_t with each integer t. We are operating on the

[1] It would be useful for the reader to be familiar with the material on difference equations in Allen (1960) and Baumol (1959).

[2] This chapter aims to teach the reader to manipulate lag operators, while devoting little or no attention to describing their mathematical foundations. The key Riesz-Fischer theorem which justifies these methods is discussed briefly in Chapter XI, pp. 249–253. The reader interested in increasing his proficiency with these techniques is urged to consult Gabel and Roberts (1973, Chapter 4).

sequence $\{X_t\}$ with the operator L^n to obtain the new sequence $\{y_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty}$. Formally, the operator L^n maps one sequence into another sequence.

We shall consider polynomials in the lag operator

$$A(L) = a_0 + a_1 L + a_2 L^2 + \cdots = \sum_{j=0}^{\infty} a_j L^j$$

where the a_j's are constants and $L^0 \equiv 1$. Operating on X_t with $A(L)$ yields a moving sum of X's:

$$A(L)X_t = (a_0 + a_1 L + a_2 L^2 + \cdots)X_t$$
$$= a_0 X_t + a_1 X_{t-1} + a_2 X_{t-2} + \cdots = \sum_{j=0}^{\infty} a_j X_{t-j}.$$

It is generally convenient to work with polynomials $A(L)$ that are "rational," meaning that they can be expressed as the ratio of two (finite order) polynomials in L:

$$A(L) = B(L)/C(L)$$

where

$$B(L) = \sum_{j=0}^{m} b_j L^j, \qquad C(L) = \sum_{j=0}^{n} c_j L^j$$

where the b_j and c_j are constants. Assuming that $A(L)$ is rational amounts to imposing a more economical and restrictive parametrization on the a_j.

To take the simplest example of a rational polynomial in L, consider[3]

$$A(L) = \frac{1}{1 - \lambda L}. \tag{2}$$

For the scalar $|C| < 1$, we know that

$$\frac{1}{1 - C} = 1 + C + C^2 + \cdots. \tag{3}$$

Thus suggests treating λL of (2) exactly like the C of (3) to get

$$\frac{1}{1 - \lambda L} = 1 + \lambda L + \lambda^2 L^2 + \cdots, \tag{4}$$

[3] Actually, we should write $A(L) = I/(I - \lambda L)$ where I is the identity lag operator defined by $I \equiv 1 + 0L + 0L^2 + \cdots$. So I satisfies $Ix_t = x_t$, and thus acts like unity.

an expansion which is sometimes only "useful" so long as $|\lambda| < 1$. To motivate the equality (4), assume that $|\lambda| < 1$ and operate on both sides of (4) with $1 - \lambda L$ to obtain

$$\frac{1-\lambda L}{1-\lambda L} = 1 = (1 + \lambda L + \lambda^2 L^2 + \cdots) - \lambda L(1 + \lambda L + \lambda^2 L^2 + \cdots) = 1.$$

The reason that we say that (4) is sometimes "useful" only if $|\lambda| < 1$ derives from the following argument. We intend often to multiply $1/(1 - \lambda L)$ by X_t to obtain the infinite moving sum

$$\frac{1}{1-\lambda L} X_t = (1 + \lambda L + \lambda^2 L^2 + \cdots)X_t = \sum_{i=0}^{\infty} \lambda^i X_{t-i}. \tag{5}$$

Consider this sum for a path of X that is constant over time, so that $X_{t-i} = \bar{X}$ for all i and all t. Then the sum (5) becomes

$$\frac{1}{1-\lambda L} X_t = \bar{X} \sum_{i=0}^{\infty} \lambda^i.$$

The sum $\sum_{i=0}^{\infty} \lambda^i$ equals $1/(1 - \lambda)$ if $|\lambda| < 1$. But if $|\lambda| \geq 1$ that sum is unbounded, being $+\infty$ if $\lambda \geq 1$. We shall sometimes (though not always) be applying the polynomial in the lag operator (4) in situations in which it is appropriate to go infinitely far back in time; and we sometimes find it necessary to insist that in such cases the infinite sum in (5) exist where X has been constant through time. This is what leads to the requirement sometimes imposed that $|\lambda| < 1$ in (4). As we shall see, however, in standard analyses of difference equations, which take the starting point of all processes as some point only finitely far back into the past, the requirement that $|\lambda| < 1$ need not be imposed in (4).

It is useful to note that there is an alternative expansion for the "geometric" polynomial $1/(1 - \lambda L)$. For notice that formally

$$\frac{1}{1-\lambda L} = \frac{-(\lambda L)^{-1}}{1-(\lambda L)^{-1}} = \frac{-1}{\lambda L}\left(1 + \frac{1}{\lambda}L^{-1} + \left(\frac{1}{\lambda}\right)^2 L^{-2} + \cdots\right)$$
$$= \frac{-1}{\lambda}L^{-1} - \left(\frac{1}{\lambda}\right)^2 L^{-2} - \left(\frac{1}{\lambda}\right)^3 L^{-3} - \cdots, \tag{6}$$

an expansion which is especially "useful" where $|\lambda| > 1$, i.e., where $|1/\lambda| < 1$. So (6) implies that

$$\frac{1}{1-\lambda L} X_t = -\frac{1}{\lambda} X_{t+1} - \left(\frac{1}{\lambda}\right)^2 X_{t+2} - \cdots = -\sum_{i=1}^{\infty} \left(\frac{1}{\lambda}\right)^i X_{t+i},$$

which shows $(1/(1 - \lambda L))X_t$ to be a geometrically declining weighted sum of *future* values of X. Notice that for this infinite sum to be finite for a constant time path $X_{t+i} = \bar{X}$ for all i and t, the series $-\sum_{i=1}^{\infty} (1/\lambda)^i$ must be convergent, which requires that $|1/\lambda| < 1$.

More generally, let $\{x_t\}_{t=-\infty}^{\infty}$ be any bounded sequence of real numbers, i.e., for some $M > 0$, $|x_t| < M$ for all t. Then applying the operator $1/(1 - \lambda L)$ to the sequence $\{x_t\}_{t=-\infty}^{\infty}$ can be taken to give either the sequence $\{y_t\}_{t=-\infty}^{\infty}$ where

$$y_t = \frac{1}{1-\lambda L} x_t = \sum_{j=0}^{\infty} \lambda^j x_{t-j}$$

or the sequence $\{z_t\}_{t=-\infty}^{\infty}$ where

$$z_t = \frac{-(\lambda L)^{-1}}{1-(\lambda L)^{-1}} x_t = -\sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j x_{t+j}.$$

In general, $\{y_t\}$ is a bounded sequence if $|\lambda| < 1$, while $\{z_t\}$ is a bounded sequence if $|\lambda| > 1$. In many (though not all) contexts, we want application of $(1 - \lambda L)^{-1}$ to map all bounded sequences into bounded sequences, so that we choose the "backward" expansion if $|\lambda| < 1$ and the forward expansion if $|\lambda| > 1$.

To illustrate how polynomials in the lag operator can be manipulated, consider the difference equation

$$Y_t = \lambda Y_{t-1} + bX_t + a, \qquad t = -\infty, \ldots, 0, 1, 2, \ldots, \tag{7}$$

where X_t is an exogenous variable and Y_t is an endogenous variable and $\lambda \neq 1$. Here, X_t is a sequence of real numbers $t = \cdots, -1, 0, 1, 2, \ldots$. Write the above equation as

$$(1 - \lambda L)Y_t = a + bX_t.$$

Operating on both sides of this equation by $(1 - \lambda L)^{-1}$ gives

$$Y_t = \frac{a}{1-\lambda L} + \frac{b}{1-\lambda L} X_t + c\lambda^t$$
$$= \frac{a}{1-\lambda} + b\sum_{i=0}^{\infty} \lambda^i X_{t-i} + c\lambda^t, \tag{8}$$

since $a/(1 - \lambda L) = a\sum_{i=0}^{\infty} \lambda^i = a/(1 - \lambda)$. Here c is any constant. The reason that we must include the term $c\lambda^t$ in (8) is that for any constant c, $(1 - \lambda L)c\lambda^t = c\lambda^t - c\lambda\lambda^{t-1} = 0$.[4] Therefore, it follows that application of $1 - \lambda L$ to both sides of (8) gives Equation (7) once again. Consequently, (8) is the general "solution" of the difference equation (7) and describes the entire time path of Y associated with a given time path of X. In order to get a "particular solution" we must be able to tie down the constant c. This requires an additional bit of information in the form of a specified value of Y_t at some particular time or some conditions on

[4] Technically, we are free to add to the solution any function of time $f(t)$ for which $(1 - \lambda L)f(t) = 0$. It can be proved that $f(t) = c\lambda^t$ is the only such function.

the path of $\{Y_t\}$ such as boundedness. Notice that for the Y_t defined by (8) to be finite, $\lambda^i X_{t-i}$ must be "small" for large i. More precisely, we require

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \lambda^i X_{t-i} = 0 \qquad \text{for all } t. \tag{9}$$

For the case of X constant for all time, $X_{t-i} = \bar{X}$ all i and t, this condition requires $|\lambda| < 1$. Notice also that the infinite sum $a\sum_{i=0}^{\infty}\lambda^i$ in (8) is finite only if $|\lambda| < 1$, in which case it equals $a/(1-\lambda)$, or if $a = 0$, in which case it equals zero regardless of the value of λ. We tentatively assume that $|\lambda| < 1$.

For analyzing difference equations with arbitrary initial conditions given, it is convenient to rewrite (8) for $t > 0$ as

$$Y_t = a\sum_{i=0}^{t-1}\lambda^i + a\sum_{i=t}^{\infty}\lambda^i + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + b\sum_{i=t}^{\infty}\lambda^i X_{t-i} + c\lambda^t$$

$$= \frac{a(1-\lambda^t)}{1-\lambda} + \frac{a\lambda^t}{1-\lambda} + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + b\lambda^t\sum_{i=0}^{\infty}\lambda^i X_{0-i} + c\lambda^t,$$

$$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + \lambda^t\left\{\frac{a}{1-\lambda} + b\sum_{i=0}^{\infty}\lambda^i X_{0-i} + c\lambda^0\right\}, \qquad t \geq 1. \tag{10}$$

The term in braces equals Y_0, as reference to expression (8) will confirm. So (10) becomes

$$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + \lambda^t Y_0$$

or

$$Y_t = \frac{a}{1-\lambda} + \lambda^t\left(Y_0 - \frac{a}{1-\lambda}\right) + b\sum_{i=0}^{t-1}\lambda^i X_{t-i}, \qquad t \geq 1. \tag{11}$$

Now textbooks on difference equations often analyze the special case in which $X_t = 0$ for all $t \geq 0$. Under this special circumstance (11) becomes

$$Y_t = \frac{a}{1-\lambda} + \lambda^t\left(Y_0 - \frac{a}{1-\lambda}\right), \tag{12}$$

which is the solution of the first-order difference equation $Y_t = a + \lambda Y_{t-1}$ subject to the initial condition that Y equals the arbitrary given value Y_0 at time 0. Notice that if $Y_0 = a/(1-\lambda)$, then (12) implies $Y_t = Y_0$ for all $t \geq 0$, which shows $a/(1-\lambda)$ to be a "stationary point" or long-run equilibrium value of Y. Notice also that if, as we are assuming, $|\lambda| < 1$, then (12) implies that

$$\lim_{t\to\infty} Y_t = \frac{a}{1-\lambda},$$

which shows that the system is "stable," tending to approach the stationary point as time passes.

Now consider the first-order system (7) under the assumption that $a = 0$, so that $a\sum_{i=0}^{\infty}\lambda^i$ equals zero regardless of the value of λ. Then the appropriate counterpart to (10) is

$$Y_t = b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + \lambda^t\left\{b\sum_{i=0}^{\infty}\lambda^i X_{0-i} + c\lambda^0\right\}.$$

Assuming that condition (9) is met even where $|\lambda| > 1$ (so that the second term in the equation is finite), the above equation becomes

$$Y_t = b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + \lambda^t Y_0, \qquad t \geq 1.$$

As before we analyze the special case where $X_t = 0$ for all $t > 0$. Then the above equation becomes

$$Y_t = \lambda^t Y_0, \qquad t \geq 1.$$

The stationary point of this solution is zero since if $Y_0 = 0$, Y will remain equal to zero forever, regardless of the value of λ. However, if $|\lambda| > 1$, the system will diverge farther and farther from this stationary point if either $Y_0 > 0$, or $Y_0 < 0$. If $\lambda > 1$, Y_t will tend toward $+\infty$ as $t \to \infty$ provided $Y_0 > 0$; Y_t will tend toward $-\infty$ as $t \to \infty$ if $Y_0 < 0$. If $\lambda < -1$, Y_t will display explosive oscillations of periodicity two time periods.

We can also solve the difference equation (7) by applying the "forward inverse" of $1 - \lambda L$ to get the general solution

$$Y_t = \frac{-\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}\,a + b\left(\frac{-\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}\right)X_t + d\lambda^t,$$

$$Y_t = \frac{a}{1-\lambda} - b\sum_{i=0}^{\infty}\left(\frac{1}{\lambda}\right)^{i+1} X_{t+i+1} + d\lambda^t, \tag{8'}$$

where d is a constant to be determined from some side condition on the path of Y_t, such as an initial condition or terminal condition. If $a = 0$, then for *any* value of $\lambda \neq 1$, in general (8) and (8′) *both* represent solutions of the difference equation (7). They are simply alternative representations of the solution in the sense that for any given initial condition or other side condition, one can generally find values of d and c which guarantee that both (8) and (8′) satisfy (7). The equivalence of the solutions (8) and (8′) will hold whenever $(b/(1-\lambda L))X_t$ and $(b\lambda^{-1}L^{-1}/(1-\lambda^{-1}L^{-1}))X_t$ are both finite for all t.

It often happens, however, that either $(b/(1-\lambda L))X_t$ or

$$\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}X_t$$

fails to be finite, i.e., the infinite sum fails to converge. In this case, one or the other of the representations (8) or (8′) breaks down, i.e., fails to give a Y_t sequence that is finite for all finite t. For example, if the sequence $\{X_t\}$ is bounded, this is sufficient to imply that $\{(b/(1-\lambda L))X_t\}$ is a bounded sequence if $|\lambda| < 1$, but not sufficient to imply that

$$\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}X_t$$

is a convergent sum for all t. Similarly, if $|\lambda| > 1$, boundedness of the sequence $\{X_t\}$ is sufficient to imply that

$$\left\{\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}X_t\right\}$$

is a bounded sequence, but fails to guarantee finiteness of $(b/(1-\lambda L))X_t$. In instances where one of $(b/(1-\lambda L))X_t$ or

$$\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}X_t$$

is always finite and the other is not we shall take as our solution to (7) either (8) where the backward sum in X_t is finite, or (8′) where the forward sum in X_t is finite. This procedure assures us that we shall find the unique solution of (7) that is finite for all finite t, provided that such a solution exists. Such a solution is guaranteed to exist where $\{X_t\}$ is a bounded sequence.

Now if we desired to impose that the $\{Y_t\}$ sequence given by (8) or (8′) is bounded—as we are free to do if no other side condition has been imposed—then it is evident that we must set $c = 0$ in (8) or $d = 0$ in (8′). This is necessary since if $\lambda > 1$ and $c > 0$,

$$\lim_{t\to\infty} c\lambda^t = \infty;$$

while if $\lambda < 1$ and $c > 0$,

$$\lim_{t\to-\infty} c\lambda^t = \infty.$$

It follows that Y_t will be bounded for all t only if c or d is zero. For the solution in the forward direction to be finite for all finite t, we clearly require a condition analogous to (9)

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\left(\frac{1}{\lambda}\right)^i X_{t+i} = 0. \tag{9′}$$

The principle of solving "stable roots" ($|\lambda| < 1$) backward and "unstable roots" ($|\lambda| > 1$) forward was encountered in Chapter I. It is a device designed to ensure that the solution of the differential (or difference) equation maps

bounded functions (or sequences) as driving processes into bounded functions (or sequences). Below, we shall see that a formal justification for this procedure is sometimes available in the context of difference equations that emerge from optimum problems.

2. SECOND-ORDER DIFFERENCE EQUATIONS

Consider the second-order difference equation

$$Y_t = t_1 Y_{t-1} + t_2 Y_{t-2} + a + bX_t, \tag{13}$$

where $\{X_t\}$ is again an exogenous sequence of real numbers for $t = \cdots, -1, 0, 1, \ldots$. Using lag operators, (13) can be written as

$$(1 - t_1 L - t_2 L^2)Y_t = a + bX_t.$$

A solution to this difference equation is given by

$$Y_t = \frac{a}{1-t_1L-t_2L^2} + \frac{b}{1-t_1L-t_2L^2}X_t \tag{14}$$

where we have temporarily ignored the terms analogous to $c\lambda^t$ which appeared in the general solution of the first-order equation. By long division it is easy to verify that

$$\frac{b}{1-t_1L-t_2L^2} = \sum_{i=0}^{\infty} w_i L^i \tag{15}$$

where $w_0 = b$, $w_1 = bt_1$, and

$$w_j = t_1 w_{j-1} + t_2 w_{j-2} \qquad \text{for} \quad j \geq 2.$$

That is,

$$\begin{array}{r|l} & 1 + t_1L + (t_2 + t_1^2)L^2 + (t_1(t_2 + t_1^2) + t_1t_2)L^3 + \cdots \\ \hline 1 - t_1L - t_2L^2 & 1 \\ & \underline{1 - t_1L - t_2L^2} \\ & t_1L + t_2L^2 \\ & \underline{t_1L - t_1^2L^2 \qquad - t_1t_2L^3} \\ & (t_2 + t_1^2)L^2 + t_1t_2L^3 \\ & \underline{(t_2 + t_1^2)L^2 - t_1(t_2 + t_1^2)L^3 - t_2(t_2 + t_1^2)L^4} \\ & \cdots \end{array}$$

Notice that the weights in (15) follow a geometric pattern if $t_2 = 0$, as we would expect, since then (13) collapses to a first-order equation.

It is convenient to write the polynomial $1 - t_1L - t_2L^2$ in an alternative way, given by the *factorization*

$$\begin{aligned} 1 - t_1L - t_2L^2 &= (1 - \lambda_1 L)(1 - \lambda_2 L) \\ &= 1 - (\lambda_1 + \lambda_2)L + \lambda_1\lambda_2 L^2, \end{aligned} \tag{16}$$