A Simple Model of Monetary Exchange
Based on Nonconvexities and Sunspots*

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Abstract

We construct a simple model of monetary exchange where, as in Lagos and Wright (2004), trades take place in both centralized and decentralized markets. However, in constrast to Lagos and Wright, we allow for general preferences and introduce nonconvexities (indivisibilities) and sunspots. In the centralized market, agents can trade state-contingent commodities. We show that nonconvexities and sunspots make the model tractable by reducing the heterogeneity generated by the randomness of the trading process in the decentralized market. We show that the allocations in the centralized market are determined independently. In particular, the unemployment rate in the centralized market is independent of inflation. While the allocation in the centralized market is efficient, the allocation in the decentralized market is in general inefficient.

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1 Introduction

Everyone should agree that it is desirable to construct models in macro and monetary economics with better microfoundations. Cass and Shell (1980) advocate this position with respect to the overlapping generations model of money described in Samuelson (1958), Shell (1971) and Wallace (1980), for example. Cass and Shell argue "this basic structure has two general features which we believe are indispensable to the development of macroeconomics as an intellectually convincing discipline ... First it is genuinely dynamic ... Second it is fundamentally disaggregative." They also "firmly believe that a satisfactory general theory must, at a minimum, encompass some diversity among households as well as some variety among commodities." We think these words still ring true today, although in the generation since they were written, a new microfounded monetary model based on search theory has been developed that fits this description as well or better than overlapping generations models. A drawback of early search models is that for tractability reasons they had to make extreme assumptions about how much money agents could carry, since otherwise the endogenous distribution of money across agents became difficult to handle. These restrictions not only seem inelegant, they also greatly hindered the use of the models for monetary policy as it is usually formulated.

The subsequent literature developed in several directions. Some people tried to relax these restrictions and see just how far one could get analytically, while others proceeded with numerical methods (e.g. Green and Zhou (1998), Zhou (1999), Camera and Corbae (1999), and Zhu (2003) proceed analytically, while Molico (1999, 2004) uses computational methods).
Still others tried to find assumptions to make the model less complicated, so that analytic results would be relatively easy to derive, and hence the model could be put to theoretical and practical use. There are two main approaches. Shi (1997) assumes the fundamental decision-making units in the model are not individuals, but large households consisting of agents who share their money after each round of decentralized trade, and he appeals to the law of large numbers to conclude that all households start the next round of trade with the same amount of cash. Lagos and Wright (2002) alternatively assume that there are centralized markets in cash and other goods that open after each round of decentralized trade, and show that as long as preferences are quasi-linear, all agents will leave these markets with the same money holdings because, heuristically speaking, quasi-linearity implies no wealth effect in the demand for money (assuming interior solutions).

While these approaches are quite useful, each comes with some baggage. Several conceptual and technical issues with household models are discussed in Lagos and Wright (2002); a clear problem with the alternative centralized market approach is that quasi-linear preferences are obviously special and entail several other implications that may be undesirable for some purposes. In this paper we propose an alternative structure that also generates a degenerate distribution of money holdings in search-based models: we adopt the centralized market structure in Lagos and Wright (2002), but rather than quasi-linearity, we allow general utility functions and assume that some good is indivisible. As is well known, in models with indivisibilities or some other non-convexities, efficient allocations may require randomization – i.e. they may involve lotteries (see e.g. Rogerson 1988, or the recent special issue of JET). Intuitively, the idea is this: since expected utility will be linear in the
probabilities associated with these lotteries, one might conjecture that the
agents in our model may behave like the quasi-linear agents in Lagos and
Wright. While it is going to take a little work to make this precise, this
intuition turns out to be correct.

The way we proceed is to look for competitive equilibria that generate
randomized allocations via sunspots. Following Shell and Wright (1992) our
equilibrium concept is exactly that in Debreu (1959, chapter 7), where state-
contingent commodities are traded in centralized markets, and where it just
so happens that the state is a sunspot variable – i.e. it constitutes extrinsic
uncertainty in the sense that it has no impact on preferences, endowments
or technology. In equilibrium, we prove that agents who enter the market
with different amounts of money will generally buy or sell the indivisible
good with different probabilities, but they will all choose the same bundle
of divisible goods. In particular this implies that they all take the same
amount of money out of the centralized market and hence into next period’s
decentralized market. Hence we get a degenerate distribution of money
in the decentralized market (just like Lagos-Wright). A caveat is that this
works here (just like Lagos-Wright) only if all agents choose interior solutions
for the set of states where they buy or sell the indivisible good. We provide
conditions on fundamentals under which this is true.

The bottom line in terms of monetary theory is that we provide a new en-
vironment where money is valued due to explicit descriptions in the model of
frictions in the trading process, and yet things are analytically very tractable
because the distribution of money is degenerate, and hence as in other sim-
ple models of money, we can reduce the equilibrium conditions to a single
difference equation. At the same time, our results provide another example
of the role of sunspots in economics, although far from being detrimental
to economic activity, as in much of the early work following Cass and Shell
(1980), here sunspots are a good thing. As emphasized in Shell and Wright
(1992), in nonconvex economies sunspots can help us achieve an efficient
randomized allocation as a standard competitive equilibrium in the sense of
Debreu (1959, chapter 7). One thing not noticed in Shell and Wright (1992)
however is that in this equilibrium, assuming an interior solution, agents
will act as if they have quasi-linear preferences. That is, wealth effects van-
ish from the demand for divisible goods, including the demand for money.
From the perspective of trying to build tractable models of money, this is
very nice.

The rest of the paper is organized as follows...

2 A simple model with non-convexities and sunspots

We consider a model with indivisible labor and sunspots based on Shell and
Wright (1993). There is a measure space \((I, \Omega, \alpha)\) of consumers where \(I\) is
the set of agents, \(\Omega\) a \(\sigma\)-algebra of subsets of \(I\) and \(\alpha\) a measure defined on
\(I\), and a finite set, \(K\), of firms indexed by \(k\).\(^1\) Consumers, who can differ in
terms of their endowments and preferences, are divided into a finite number
of different types. There are also \(J + 2\) commodities: \(J\) consumption goods
indexed by \(j \in \{1, \ldots, J\}\), a good in fixed supply called land, and leisure.\(^2\)
The commodity space is \(\mathbb{R}^{J+2}_+\). In the following section, we will reinterpret
land as money by embedding this framework into a model of monetary

\(^1\)For the notion of competitive equilibrium with a continuum of traders, see Aumann

\(^2\)One can view the model as dynamic by reinterpreting the vector \(c\) as having com-
modities indexed by date.
Agent $i$ is endowed with one indivisible unit of labor, some consumption goods $c^i_0 \in \mathbb{R}^J_+$, and some land $m^i_0 \in \mathbb{R}_+$. The functions $c^i_0$ and $m^i_0$ that map $I$ into $\mathbb{R}^J_+$ and $\mathbb{R}$, respectively, are assumed to be $I-$measurable. We denote $M = \int m^i_0 di$ the aggregate stock of land and $C = \int c^i_0 di$ the aggregate endowment in consumption goods. Consumer $i$’s preferences are represented by a von Neumann-Morgenstern utility function $U^i(c, h, m)$ where $c \in \mathbb{R}^J_+$ is consumption, $m \in \mathbb{R}_+$ is land and $h \in \{0, 1\}$ is labor. The utility function is twice continuously differentiable, strictly increasing in each of its arguments, and strictly concave. The consumption set for each agent is

$$X = \{(c, h, m) \in \mathbb{R}^J_+ \times \{0, 1\} \times \mathbb{R}_+\}.$$ 

Note that $X$ is non-convex.

Consumption goods are produced by firms using labor only. Firm $k$’s technology is represented by a strictly concave production function $f^k(n^k) \in \mathbb{R}^J_+$ where $n^k \in \mathbb{R}^J_+$ is the vector of labor services used by firm $k$ in the production of the $J$ consumption goods. The $j^{th}$ component of this vector, $n^k_j \in \mathbb{R}_+$, indicates how much labor firm $k$ uses to produce consumption good $j$. The profits of the firm $k$ expressed in an arbitrary unit of account are $\Delta_k = \int f^k(n^k) di$ where $\Delta^i_k$ are the dividends paid by firm $k$ to consumer $i$. We also denote $\Delta^i = \sum_k \Delta^i_k$ the total dividends received by consumer $i$.

The economy is subject to extrinsic uncertainty represented by the probability space $(S, \Sigma, \pi)$ where $S = [0, 1)$ is the set of states, $\Sigma$ the Borel sets on $S$ and $\pi$ is the uniform distribution over $S$. As shown by Garratt, Keister, Qin and Shell (2002) the choice of a uniform distribution for the sunspot variable can be assumed with no loss in generality. The realization of a state
does not affect the fundamentals of the economy (such as preferences, technology or endowments). Markets are complete and agents can trade state-contingent commodities. To take into account the fact that agents’ behavior can depend on the realization of the state, we reformulate the commodity space as the set of $\pi$-measurable functions of the state, $x : S \rightarrow \mathbb{R}^{J+2}$. The consumption set is now the set of such functions such that $x(s) \in X$ for all $s$. Denote $p(s) \in \mathbb{R}^{J+2}$ the price vector of consumption goods in state $s$, $w(s) \in \mathbb{R}_+$ the price of labor in state $s$, and $p_m(s) \in \mathbb{R}_+$ the price of land in state $s$.

The following defines a competitive equilibrium for the economy with extrinsic uncertainty, called a sunspot equilibrium.

**Definition 1** A sunspot equilibrium is a list $\{(c^i(s), h^i(s), m^i(s))_{i \in I}, (n^k(s))_{k \in K}, w(s), p(s), p_m(s)\}$ such that

(i) Given $(w(s), p(s), p_m(s))$, $(c^i(s), h^i(s), m^i(s))$ solves

$$\max_{h^i, c^i, m^i} \int_S U^i[c^i(s), h^i(s), m^i(s)] ds$$

s.t. $\int_S \left\{ p(s) [c^i(s) - c^i_0] + p_m(s) [m^i(s) - m^i_0] - w(s)h^i(s) - \Delta^i(s) \right\} ds = 0.$

(ii) Given $(w(s), p(s))$, $n^k(s)$ solves

$$\max_{n^k} \int_S \left\{ p(s)f^k[n^k(s)] - w(s) \sum_j n^k_j(s) \right\} ds,$$

---

3 More precisely, the commodity space is normed vector space and the price system is represented by a linear functional. We restrict our attention to price systems that have a inner product representation. See Stockey and Lucas (1989, ch.15).
(iii) Markets clear,

$$\sum_{j,k} n_{j}^{k}(s) = \int h^{i}(s)di, \forall s \in S,$$

$$M = \int m^{i}(s)di, \forall s \in S,$$

$$\sum_{k} f^{k}(n^{k}) = \int c^{i}(s)di - C, \forall s \in S.$$  

The requirement (i) in Definition 1 indicates that each consumer maximizes his expected utilities subject to the budget constraint (2) where each commodity is contingent on the realization of the state $s$. The condition (ii) requires that each firm maximizes its expected profits by choosing a vector of labor services in each state. Finally, the conditions in (iii) impose that the markets for labor services, land and consumption goods clear.

Garratt, Keister, Qin and Shell (2002, Theorem 1) showed that every sunspot equilibrium allocation can be supported by prices, when adjusted for probabilities, that are constant across states. Therefore, in the following we restrict ourselves to equilibria where prices are constant across states. We normalize the price of land, $p_{m}$, to one. Denote $S_{e}^{i} = \{ s \in S : h^{i}(s) = 1 \}$ the set of states where agent $i$ supplies his indivisible unit of labor, and $S_{u}^{i} = S \setminus S_{e}^{i}$ (where the subscript $e$ refers to “employed” and the subscript $u$ refers to “unemployed”).

**Lemma 1** Suppose $(w(s), p(s), p_{m}(s)) = (w, p, p_{m})$ for all $s \in S$. (i) $n^{k}(s) = n^{k}$, for all $s \in S$. (ii) $(c^{i}(s), m^{i}(s)) = (c^{i}_{e}, m^{i}_{e})$ for all $s \in S_{e}^{i}$ and $(c^{i}(s), m^{i}(s)) = (c^{i}_{u}, m^{i}_{u})$ for all $s \in S_{u}^{i}$. (iii) If $\partial^{2}U^{i}/\partial c_{j}\partial m = \partial^{2}U^{i}/\partial h\partial m = 0$ then $m^{i}_{e}(s) = m^{i}_{u}(s') = m^{i}$ for all $s \in S$.

**Proof.** (i) Direct from the strict concavity of the profit function. (ii) Direct
from the strict concavity of $U^i(c, h, m)$. (iii) Immediate from the separability of the utility function in terms of land. ■

Lemma 1 simply indicates that when prices are constant across states firms make the same labor choice in all states and consumers make the same choices in terms of consumption and land in all states where they are employed. If the utility function is additively separable in $m$ then consumers make the same choice in terms of land in all states.

Agents only care about the measure of states in which they are employed, $\ell^i = \int_{S^i_e} ds$, and not about which particular state is in $S^i_e$. The dual of the sets $(S^i_e)_{i \in I}$ and $(S^i_u)_{i \in I}$ are the sets $I_e(s) = \{ i \in I : h^i(s) = 1 \}$ and $I_u(s) = I \setminus I_e(s)$ that denote the set of agents who work and do not work, respectively, in a given state $s$. The sets $I_e(s)$ and $S^i_e$ are related as follows: $I_e(s) = \{ i : s \in S^i_e \}$. Definition 1 can now be reformulated as follows:

**Definition 2** A sunspot equilibrium with constant prices across states is a list $\{ (c^i_e, c^i_u, \ell^i, m^i_e, m^i_u)_{i \in I}, I_e(s), (n^k)_{k \in \{1, \ldots, K\}}, (w, p) \}$ such that:

(i) Given $(w, p)$, $(c^i_e, c^i_u, \ell^i, m^i_e, m^i_u)$ solves

\[
W^i(m^i_0, c^i_0) = \max_{\ell^i, c^i_e, c^i_u, m^i_e, m^i_u} \left\{ \ell^i U^i(c^i_e, 1, m^i_e) + (1 - \ell^i) U^i(c^i_u, 0, m^i_u) \right\} \tag{7}
\]

s.t. $\ell^i (pc^i_e + m^i_e) + (1 - \ell^i) (pc^i_u + m^i_u) = pc^i_0 + w\ell^i + m^i_0 + \Delta^i. \tag{8}$

(ii) Given $(w, p)$, $n^k$ solves

\[
\max_{n^k} \left[ p f^k(n^k) - w \sum_j n^k_j \right]. \tag{9}
\]
(iii) Markets clear, i.e.,

$$\sum_{j,k} n^k_j = \int_{I_e(s)} di, \quad \forall s \in S,$$

(10)

$$M = \int_{I_e(s)} m_e^i di + \int_{I_u(s)} m_u^i di, \quad \forall s \in S,$$

(11)

$$\sum_k f^k(n^k) = \int_{I_e(s)} c_e^i di + \int_{I_u(s)} c_u^i di - C, \quad \forall s \in S.$$  

(12)

(iv) The employment allocation rule and the consumers’ choices are consistent, i.e., \( \ell^i = \int 1_{\{i \in I_e(s)\}} ds. \)

In order to establish that a sunspot equilibrium with constant prices exists one would have to show first that there exists a list \( \{(e^i_c, c^u, \ell^i, m^i_e, m^i_u)_{i \in I}, (n^k)_{k \in \{1, ..., K\}}, (w, p)\} \) that satisfies the requirements (i) and (ii) in Definition 1 and the following market clearing conditions (derived from (10)-(12) by integrating across states)

$$\sum_{j,k} n^k_j = \int I \ell^i di,$$

(13)

$$M = \int I [\ell^i m^i_e + (1 - \ell^i)m^i_u] di,$$

(14)

$$\sum_k f^k(n^k) = \int I [\ell^i c^i_e + (1 - \ell^i)c^i_u] di - C.$$  

(15)

Second, one can construct an employment allocation rule \( I_e(s) \) that satisfies the requirements in Definition 2. With no loss we assume that \( I \) can be written as \( \bigcup_\tau I_\tau \) where each \( I_\tau \) is an interval of consumers of type \( \tau \). Denote \( \alpha_i \) the measure of the set \( I_i \) that contains consumer \( i \). Then, the employment allocation rule obeys

$$S^i_e = \left[ \frac{i}{\alpha_i} \frac{i}{\alpha_i} + \ell^i \right] \mod \ell^i.$$
It can easily be checked that the measure of the sets $I_e(s) = \{ i \in I : S_i^c \ni s \}$ are invariant across states. This allocation rule is illustrated in Figure 1 where we assume that there are two types of consumers, 1 and 2: Consumers of type 1 wish to work $\ell^1$ and consumers of type 2 wish to work $\ell^2$. The measure of consumers of type 1 is $\alpha$. The grey areas represent their $(i, s)$ such that $s \in S_i^c$, or equivalently, $i \in I^c(s)$.

![Figure 1: Employment allocation rule.](image)

In the following we focus on allocations where $\ell^i$ is interior for all consumers $i$. We will focus on related equilibria in the following section on monetary economies.

**Proposition 1** Consider two consumers $i$ and $i'$ with the same preferences
but different endowments. In any sunspot equilibrium with constant prices across states, 
\((c^i_e, c^i_u, m^i_e, m^i_u) = (c^{i'}_e, c^{i'}_u, m^{i'}_e, m^{i'}_u)\) whenever \(\ell^i\) and \(\ell^{i'}\) are interior (i.e., \(\ell^i, \ell^{i'} \in (0, 1)\)).

**Proof.** Denote \(\lambda^i\) the Lagrange multiplier associated with (8). Assuming \(\ell^i \in (0, 1)\), the first-order conditions for \((\ell^i, c^i_e, c^i_u, m^i_e, m^i_u)\) are

\[
U^i \left( c^i_e, 1, m^i_e \right) - U^i \left( c^i_u, 0, m^i_u \right) + \lambda^i \left( w - pc^i_e - m^i_e + pc^i_u + m^i_u \right) = 0,
\]

\[
U^i_j(\ell^i, c^i_u, 1, m^i_e) = U^i_j(\ell^i, c^i_u, 0, m^i_u) = \lambda^i p_j, \quad \forall j = 1, ..., J,
\]

\[
U^i_m(\ell^i, c^i_e, 1, m^i_e) = U^i_m(\ell^i, c^i_u, 0, m^i_u) = \lambda^i,
\]

where \(U^i_j\) is the partial derivative of \(U^i\) with respect to \(c_j\) and \(U^i_m\) is the partial derivative of \(U^i\) with respect to \(m\). It is checked in Appendix 1 that the second-order conditions for a maximum are satisfied. Equations (16)-(18) determine \((c^i_e, c^i_u, m^i_e, m^i_u, \lambda^i)\) independently of agents’ initial endowments \((m^i_0, c^i_0)\). ■

Proposition 1 states that consumers make the same choices in terms of consumption and land whenever their choice of labor is interior even though they may have different initial endowments. In other words, the presence of indivisible labor and sunspots eliminates wealth effects on consumption and land. In the absence of such non-convexities, elimination of wealth effects requires a special class of utility functions, namely quasi-linear utility functions.

**Proposition 2** If \(\ell^i\) is interior then \(W^i(m, c) = \lambda^i(m + pc) + \tilde{W}\) where \(\tilde{W} \in \mathbb{R}\) and \(\partial \lambda^i / \partial m^i_0 = \partial \lambda^i / \partial c^i_0 = 0\).

**Proof.** From (7)-(8), \(W^i\) satisfies

\[
W^i(m^i_0, c^i_0) = \ell^i U^i(\ell^i, c^i_e, 1, m^i_e) + (1 - \ell^i) U^i(\ell^i, c^i_u, 0, m^i_u)
\]
\[
+ \lambda^i \left\{ p_c^i + w \ell^i + m_0^i + \Delta^i - \ell^i (p_{c_e}^i + m_e^i) - (1 - \ell^i) (p_{c_u}^i + m_u^i) \right\}
\]

(19)

Using (16), i.e.,

\[
U^i(c_e^i, 1, m_e^i) - U^i(c_u^i, 0, m_u^i) + \lambda^i \left( w - p_{c_e}^i - m_e^i + p_{c_u}^i + m_u^i \right) = 0,
\]

(19) can be simplified to

\[
W^i(m_0^i, c_0^i) = \lambda^i \left( p_{c_0}^i + m_0^i \right) + \bar{W},
\]

where \( \bar{W} = U^i(c_u^i, 0, m_u^i) + \lambda^i \left( \Delta^i - p_{c_u} - m_u \right) \). From Proposition 1, \( \lambda^i \) is independent of agent \( i \)'s wealth, \( m_0^i \) and \( c_0^i \).

This establishes that consumer’s value function \( W \) is linear in wealth.

In the following we show how the equilibrium allocation when the initial distribution of endowments is non-degenerate relates to the allocation when it is degenerate. We focus on the case where all consumers have the same preferences, \( U^i = U \). Denote \( (\hat{c}_e, \hat{c}_u, \hat{\ell}, \hat{m}_e, \hat{m}_u, \hat{p}, \hat{w}) \) an equilibrium when the initial distribution of endowments is degenerate, i.e., \( m_0^i = M \), \( c_0^i = C \) and \( \Delta^i = \Delta \) for all \( i \in I \). Furthermore, denote \( \hat{\lambda} = U_m(\hat{c}_u, 0, \hat{m}_u) \) the Lagrange multiplier associated with (8). We assume that fundamentals (preferences, technology...) are such that \( \hat{\ell} \in (0, 1) \). Consider next the same economy except that the distribution of endowments is non-degenerate. Let \( F_0 \) be the distribution of agents’ wealth, \( m_0^i + \hat{p}c_0^i + \Delta^i \), where the endowments in terms of consumption goods are valued according to the price vector \( \hat{p} \). It satisfies \( F_0(\omega) = \int_{[0, \omega]} \left( m_0^i + \hat{p}c_0^i + \Delta^i \right) di \). In the following we assume \( U(\hat{c}_u, 0, \hat{m}_u) \neq U(\hat{c}_e, 1, \hat{m}_e) \).

\[\text{If } U(c_e, 1, m_e) = U(c_u, 0, m_u) \text{ then, from (16), } w = p_{c_e} + m_e - p_{c_u} - m_u \text{ and, from (8), } p_{c_u} + m_u = p_{c_0} + m_0 + \Delta. \text{ When unemployed agents consume their initial endowments and when employed they consume their initial endowments plus the wage. Such consumption patterns occur for preferences such that consumption and leisure are perfect substitutes,}\]
Proposition 3 Consider an economy where consumers have the same preferences. There exists $\omega$ and $\bar{\omega}$ such that if $\text{supp}(F_0) \subseteq [\omega, \bar{\omega}]$ then an equilibrium exists with the following properties: $(c'_i, c'_u, m'_e, m'_u) = (\hat{c}_e, \hat{c}_u, \hat{m}_e, \hat{m}_u)$ for all $i \in I$ and $\int I \ell^i di = \hat{\ell}$.

Proof. We construct an equilibrium where $(p, w) = (\hat{p}, \hat{w})$. From Lemma 1, if $\ell^i$ is interior for all $i \in I$ then $(c'_i, c'_u, m'_e, m'_u) = (\hat{c}_e, \hat{c}_u, \hat{m}_e, \hat{m}_u)$. Integrating (8),

$$(p\hat{c}_e + \hat{m}_e) \int I \ell^i di + (1 - \int I \ell^i di)(p\hat{c}_u + \hat{m}_u) = pC + w \int I \ell^i di + M + \sum_j \Delta_j.$$ 

Since the previous equation holds when $F^0$ is degenerate or non-degenerate, assuming all $\ell^i$ are interior, $\int I \ell^i di = \hat{\ell}$. Since aggregate quantities are the same than those of the economy with degenerate distribution of endowments, $(\hat{p}, \hat{w})$ are market-clearing prices. The interval $[\omega, \bar{\omega}]$ corresponds to the range of values for the endowments $\omega^i = m^i_0 + pc^i_0 + \Delta^i$ such that $\ell^i \in (0, 1)$.

From (8) and (16), and after some simplification,

$$\ell^i = \frac{\hat{\lambda}(M + pC + \Delta - \omega^i)}{U(c_u, 0, m_e) - U(c_e, 1, m_e)} + \hat{\ell}. \quad (20)$$

If $U(c_u, 0, m_e) - U(c_e, 1, m_e) > 0$ then $\ell^i \in (0, 1)$ requires $\omega^i \in (\omega, \bar{\omega})$ where

$$\omega = M + \Delta + pC + [U(c_u, 0, m_e) - U(c_e, 1, m_e)] \hat{\ell}/\hat{\lambda}, \quad (21)$$

$$\bar{\omega} = M + \Delta + pC - [U(c_u, 0, m_e) - U(c_e, 1, m_e)](1 - \hat{\ell})/\hat{\lambda}. \quad (22)$$

If $U(c_u, 0, m_e) - U(c_e, 1, m_e) < 0$ then $\omega$ and $\bar{\omega}$ are given by

$$\omega = M + \Delta + pC - [U(c_u, 0, m_e) - U(c_e, 1, m_e)](1 - \hat{\ell})/\hat{\lambda}, \quad (23)$$

$$\bar{\omega} = M + \Delta + pC + [U(c_u, 0, m_e) - U(c_e, 1, m_e)] \hat{\ell}/\hat{\lambda}. \quad (24)$$

i.e., $U(c, \ell) = \phi(kc + \ell)$ where $\phi$ is concave. Note that this class of utility functions are not jointly strictly concave and therefore do not satisfy our assumptions. See Cooper (1987, p.16).
Proposition 3 shows that economies with different distributions of endowments have the same allocations for \((c_e, c_u, m_e, m_u)\) whenever the distribution of endowments is not too disperse. In particular, they are characterized by the same ex-post distribution of land with at most two mass points.

In the following, we consider two examples to illustrate Proposition 3. In these examples, \(c_0 = 0\) and land is additively separable in the utility function.

**Example 1** Assume \(U(c, h, m) = A \log c + L(h) + V(m)\) where \(L(.)\) is decreasing and \(V(m)\) is strictly increasing and strictly concave. Normalize \(L(1)\) to 0 and denote \(L(0) = L\) with \(L > A\). There is no endowment in consumption goods and the production function is linear, \(f(n) = \alpha n\). Then, \(m_e = m_u = M, c_e = c_u = \alpha A/L\) and \(n = A/L\). From (21) and (22), \(\bar{\omega}\) and \(\bar{\omega}\) satisfy

\[
\bar{\omega} = M + A/V'(M),
\]

\[
\bar{\omega} = M - (L - A)/V'(M).
\]

As in the specification studied in Rogerson (1988), consumption and leisure are additively separable in the previous example so that the reduced form utility function, \(A \log c - Lh\), coincides with a quasi-linear specification. We provide next an example where the reduced form utility function is not quasi-linear.

**Example 2** Assume \(U(c, h, m) = c^a(1 - h + b)^{1-a} + V(m)\) with \(a \in (0,1)\) and \(a(1 + b) \in (0,1)\). There is no endowment in consumption goods and the production is linear, \(f(n) = \alpha n\). Then, \(c_u = a(1 + b)\alpha/(1 - a), c_e =\)
$\alpha/(1 - a)$ and $n = a(1 + b)$. From (21) and (22), $\bar{\omega}$ and $\bar{\omega}$ satisfy

$$\bar{\omega} = M + \left( \frac{\alpha}{1 - a} \right)^a a(1 + b)/V'(M),$$

$$\bar{\omega} = M - \left( \frac{\alpha}{1 - a} \right)^a (1 - a - ab)/V'(M).$$

This analysis shows that one round of trading in this competitive market can eliminate the heterogeneity in terms of endowments assuming agents’ initial endowments are not too far away from the average. All agents have the same consumption conditional on being employed or unemployed. The only choice variable that adjusts to make up for the differences in endowments is the measure of states in which agents are employed.

3 A model of monetary exchange

The previous section has shown how in the presence of indivisible labor the ex-ante heterogeneity in terms of endowments was endogenously washed out. This formalization can provide a useful device to make some models with heterogeneity tractable. For instance, search-theoretic models of monetary exchange assume that agents are subject to idiosyncratic risks regarding their trading opportunities so that the distribution of money balances in equilibrium is non-degenerate. This non-degenerate distribution is what makes the model very difficult to analyze. To reduce this complexity, Lagos and Wright (2003) assumed that agents access competitive markets periodically, and were endowed with a quasi-linear utility function in order to eliminate wealth effects. As a consequence, all agents start each round of decentralized meetings with the same money balances. We show here an alternative way to obtain a degenerate distribution of money balances based on the formalization described in the previous section. In contrast to Lagos
and Wright’s device, we do not need the quasi-linear specification for the utility function but we do need indivisible labor and extrinsic uncertainty. Also, we will use the full strength of the competitive paradigm by allowing agents to trade state-contingent commodities.\footnote{Faig (2004) analyzes a somewhat related model where the nonconvexity comes from agents’ choices of being buyers or sellers and agents can use lotteries.}

We now consider an intertemporal economy where there is an essential role for money. Time is discrete and indexed by $t$. Each period of time is divided into two subperiods called day and night. During the day, there is a centralized market similar to the one described in the previous section where agents trade state-contingent commodities and indivisible labor. We assume that agents can only trade contemporaneous commodities. In particular, agents cannot trade commodities contingent on their trading histories since trading histories are private information. With no loss, we assume that there is a single consumption good and a single firm in the centralized market, and agents have no endowment in terms of this good ($c^i_0 = 0$). The utility function over these goods is $U(c, h)$.

At night, there is a decentralized market where agents trade anonymously. Each agent receives an idiosyncratic shock that determines whether he is a buyer or a seller in this market, or none of them. With probability $\sigma \leq 1/2$ an agent receives a preference shock that makes him willing to consume. With the same probability $\sigma$ an agent receives a technological shock that makes him able to produce. Not that an agent who wants to consume cannot produce while an agent who can produce does not want to consume. Therefore, there is a double coincidence of wants problem in the decentralized market that generates an essential role for money. With
We will consider different trading mechanisms at night. We will first assume that agents trade in a standard competitive market where they take the price of night goods as given. We will also consider the case where agents are matched bilaterally. The idiosyncratic shocks can then be interpreted as an outcome of the randomness of the matching process. Terms of trade in bilateral matches are then determined by a bargaining protocol.

Agents’ utility function in the decentralized market is \( u(q^b) - c(q^s) \) where \( q^b \) is the amount bought and \( q^s \) the amount sold. We assume that \( u(q^b) - c(q^s) \) is strictly concave. Furthermore, we denote \( q^* \) the quantity that solves \( u'(q) = c'(q) \). The discount factor between the day and night is denoted \( \beta^d \) and the discount factor between the night and the following day is \( \beta^n \). We denote \( \beta = \beta^d \beta^n \).

Fiat money is perfectly divisible, durable and intrinsically useless. The stock of money in the economy in period \( t \) is \( M_t \). The supply of money is growing at the gross growth rate \( \gamma > \beta \), i.e., \( M_{t+1} = \gamma M_t \). Money is injected, or withdrawn, through lump-sum transfers in the centralized market. (If \( \gamma < 1 \) money is withdrawn through lump-sum taxes at the end of the centralized market while if \( \gamma > 1 \) money is injected through lump-sum transfers at the beginning of the centralized market).

---

\(^6\)Suppose that the set of night commodities is \( J^p \) where \( |J^p| \geq 3 \) and that the set of agents \( I \) is divided evenly among \( |J^p| \) types. An agent of type \( j \) produces good \( j \in J^p \) and consumes good \( j + 1 \) (modulo \( |J^p| \)). As a consequence, there are no double coincidence of wants meetings. The probability for an agent to meet someone who produces the good he likes is \( \sigma = 1/|J^p| \). Symmetrically, the probability to meet someone who likes the good that one produces is also \( \sigma \).
3.1 Centralized market

As in the previous section, we focus on equilibria with constant prices across states. The problem of agent $i$ in the centralized market is

$$W_t(m_i) = \max_{h_i^t, c_i^t, \hat{m}_i^t} \int_S \left\{ U(c_i^t(s), h_i^t(s)) + \beta^d V_i(\hat{m}_i^t(s)) \right\} ds,$$  
(25)

s.t. \[ \int_S \{ p_t c_i^t(s) + \hat{m}_i^t(s) \} ds = \int_S \{ w_t h_i^t(s) + m_i^t + \Delta_i^t(s) + T_t \} ds. \]  
(26)

where $V_i(m)$ is the value function of an agent with $m$ units of money in the decentralized sector in period $t$ and where $T_t$ is the lump-sum transfer (or tax) at time $t$. The problem (25)-(26) is formally equivalent to (1)-(2).

Since we focus on equilibria with constant prices across states, we have the following Lemma.

**Lemma 2** The solution to (25)-(26) is such that: $c_i^t(s) = c_{e,t}^i$ for all $s \in S_{e,t}^i = \{ s \in S : h_i^t(s) = 1 \}$; $c_i^t(s) = c_{u,t}^i$ for all $s \in S_{u,t}^i = \{ s \in S : h_i^t(s) = 0 \}$; $\hat{m}_i^t(s) = \hat{m}_i^t$ for all $s \in S$.

**Proof.** Direct from Lemma 1. 

From Lemma 2, the maximization problem (25)-(26) can be simplified to

$$W_t(m_i) = \max_{\ell_i^t, c_{e,t}^i, c_{u,t}^i, \hat{m}_i^t} \left\{ \ell_i^t U (c_{e,t}^i, 1) + (1 - \ell_i^t) U (c_{u,t}^i, 0) + \beta^d V_i (\hat{m}_i^t) \right\}$$

$$+ \lambda_i^t \left[ w_t \ell_i^t + m_i^t + \Delta_i^t + T_t - \ell_i^t p_t c_{e,t}^i - (1 - \ell_i^t) p_t c_{u,t}^i - \hat{m}_i^t \right],$$  
(27)

where $\lambda_i^t$ is the Lagrange multiplier associated with (26). From Proposition 2 the value function $W_t(m)$ is linear in $m$ with $W_t'(m) = \lambda_i^t$. Furthermore, $\lambda_i^t$ is independent of $i$’s initial money balances and is the same across agents. Let $n_t$ be employment in the centralized market, i.e., $n_t = \int \ell_i^t di$. 

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Lemma 3 Assuming $\ell_t^i$ is interior for all $i \in I$, the allocation $(c_{e,t}, c_{u,t}, n_t)$ satisfies

$$U(c_{e,t}, 1) - U(c_{u,t}, 0) + U(c_{u,t}, 0) [f'(n_t) - c_{e,t} + c_{u,t}] = 0,$$  \hspace{1cm} (28)

$$U(c_{e,t}, 1) = U(c_{u,t}, 0),$$  \hspace{1cm} (29)

$$f(n_t) = n_t c_{e,t} + (1 - n_t) c_{u,t}.$$  \hspace{1cm} (30)

Proof. Direct from (12), (16) and (17). \qed

From Proposition 1, $\hat{m}_t^i = M_t$ for all $i \in I$ assuming $m_t^i$ is in some relevant range. So, the distribution of money balances at the beginning of the decentralized market is degenerate. From (17) the choice of money balances satisfies

$$-\lambda_t + \beta^d V_t'(\hat{m}_t) \leq 0, \text{ if } \hat{m}_t = 0.$$  \hspace{1cm} (31)

$$= 0 \text{ if } \hat{m}_t > 0.$$  \hspace{1cm} (32)

3.2 Pricing

Let us turn to the allocation in the decentralized market. We will make different assumptions regarding the trading mechanism at night. Assume first that agents trade in a standard competitive market. The price of night goods is $\tilde{p}_t$. The problem of an agent $i$ holding $m_t^i$ units of money who wants to consume is

$$\max_{q_t^b} \left[ u(q_t^b) + \beta^n W_{t+1}(m_t^i - \tilde{p}_t q_t^b) \right] \text{ s.t. } \tilde{p}_t q_t^b \leq m_t^i.$$  \hspace{1cm} (33)

The problem of an agent who can produce is

$$\max_{q_t^s} \left[ -c(q_t^s) + \beta^n W_{t+1}(m_t^i + \tilde{p}_t q_t^s) \right].$$  \hspace{1cm} (34)

Since there is the same measure $\sigma$ of buyers and sellers, the clearing of the night market requires $q_t^b = q_t^s$. 

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Lemma 4  (i) The solution to the buyer’s problem (33) is

\[ u'(q^h_t) = \beta^n \lambda_{t+1} \tilde{p}_t, \]

if \( m^i_t \geq m^*_i \) where \( m^*_i \) satisfies \( u'(m^*_i / \tilde{p}_t) = \beta^n \lambda_{t+1} \tilde{p}_t \) and \( q^h_t = m^i_t / \tilde{p}_t \) otherwise. (ii) The solution to the seller’s problem (34) is

\[ c'(q^s_t) = \beta^n \lambda_{t+1} \tilde{p}_t. \]

Proof. From the linearity of \( W_{t+1}(m) \), \( W_{t+1}(m^i_t - \tilde{p}_t q^h_t) = W_{t+1}(m^i_t - \lambda_{t+1} \tilde{p}_t q^h_t) \) and \( W_{t+1}(m^i_t + \tilde{p}_t q^s_t) = W_{t+1}(m^i_t) + \lambda_{t+1} \tilde{p}_t q^s_t. \] □

Let us turn to a different trading mechanism. We now follow the search-theoretic literature since Shi (1995) and Trejos and Wright (1995) and assume that terms of trade are determined according to the generalized Nash bargaining solution. Assume that agents are matched bilaterally and randomly. Consider a meeting between a buyer \( i \) holding \( m^i_t \) units of money and a seller \( i' \) holding \( m^{i'}_t \) units of money in period \( t \). The terms of trade \( (q_t, d_t) \), where \( q_t \) is the production of the seller and \( d_t \) the amount of money he receives from the buyer, satisfy

\[
\max_{q_t, d_t \leq m^i_t} \left[ u(q_t) + \beta^n W_{t+1}(m^i_t - d_t) - \beta^n W_{t+1}(m^i_t) \right]^\theta \times \left[ -c(q_t) + \beta^n W_{t+1}(m^{i'}_t + d_t) - \beta^n W_{t+1}(m^{i'}_t) \right]^{1-\theta}, \quad (35)
\]

where \( \theta \in (0,1] \) is the buyer’s bargaining power.

Lemma 5  The solution to the bargaining problem in (35) is

\[ (q_t, d_t) = (q^*, m^*_i), \]

if \( m^i_t \geq m^*_i = g(q^*) / \beta^n \lambda_{t+1}, \) and

\[ (g(q_t), d_t) = (\beta^n \lambda_{t+1} m^i_t, m^i_t), \]
otherwise, where
\[
g(q) = \frac{\theta u'(q)c(q) + (1 - \theta)c'(q)u(q)}{\theta u'(q) + (1 - \theta)c'(q)}.
\] (36)

**Proof.** From the linearity of \(W_{t+1}(m)\), (35) can be simplified to
\[
\max_{q_t,d_t \leq m_i^\ell} [u(q_t) - \beta^n \lambda_{t+1} d_t] \theta [-c(q_t) + \beta^n \lambda_{t+1} d_t]^{1-\theta}.
\] (37)
The rest is straightforward. ■

Note that the terms of trade \((q_t, d_t)\) do not depend directly of the money balances \(m_{i'}\) of the seller \(i'\) in the match. They do depend, however, on the marginal value of money \(\lambda_{t+1}^{i'}\) of the seller, but \(\lambda_{t+1}^{i'} = \lambda_{t+1}^i = \lambda_{t+1}\).

### 3.3 Decentralized market

The utility of agent \(i\) in the decentralized market obeys the following Bellman equation
\[
V_t(m^i) = \sigma \{ u[q_t(m^i)] + \beta^n W_{t+1}[m^i - d_t(m^i)] \}
+ \sigma \mathbb{E}_{(q_t,d_t)} \left[ -c(q_t) + \beta^n W_{t+1}(m^i + d_t) \right] + (1 - 2\sigma) \beta^n W_{t+1}(m^i),
\] (38)
where \(\mathbb{E}_{(q_t,d_t)}\) captures the fact that the trade is random in the case of random matching. According to (38), an agent is a buyer at night with probability \(\sigma\). In this event, he consumes \(q_t\) and spends \(d_t\) dollars where \((q_t, d_t)\) depends on his money balances. With probability \(\sigma\) an agent is a seller at night. In this event he produces \(q_t\) and receives \(d_t\) where \((q_t, d_t)\) are independent of his money balances. Using the linearity of \(W_t(m)\), the Bellman equation (38) can then be rewritten as
\[
V_t(m^i) = \sigma \{ u[q_t(m^i)] - \beta^n \lambda_{t+1} d_t(m^i) \} + \beta^n \lambda_{t+1} m^i
+ \sigma \mathbb{E}_{(q_t,d_t)} \left[ -c(q_t) + \beta^n \lambda_{t+1} d_t \right] + \beta^n W_{t+1}(0).
\] (39)
3.4 Equilibrium

Substituting $V_t(m)$ by its expression given by (39) into (27) the choice of money balances satisfies

$$\max_{\hat{m}_t} \left\{ - (\lambda_t - \beta \lambda_{t+1}) \hat{m}_t + \beta^d \sigma \{ u[q_t(\hat{m}_t)] - \beta^n \lambda_{t+1} d_t(\hat{m}_t) \} \right\}. \quad (40)$$

**Lemma 6** In any equilibrium, $\lambda_t \geq \lambda_{t+1} \beta$ and $\hat{m}_t \leq m_t^*$.  

**Proof.** If $\lambda_t < \lambda_{t+1} \beta$ then the agent’s problem (40) has no solution. Therefore, $\lambda_t \geq \lambda_{t+1} \beta$. Consider first the Walrasian pricing case. The term $u(\hat{m}_t/\hat{p}_t) - \beta^n \lambda_{t+1} \hat{m}_t$ is maximized for $\hat{m}_t = m_t^*$. Consequently, if $\lambda_t \geq \lambda_{t+1} \beta$ then $\hat{m}_t \leq m_t^*$. Consider next the bargaining case. For all $\theta < 1$, it is easy to check that $u[q_t(m)] - g[q_t(m)]$ is maximized for some $m < m_t^*$. Therefore, the solution $\hat{m}_t$ to (40) is smaller than $m_t^*$. If $\theta = 1$ then $u[q_t(m)] - g[q_t(m)] = u[q_t(m)] - c[q_t(m)]$ that is maximized for $m = m_t^*$. Consequently, if $\lambda_t > \lambda_{t+1} \beta$ then $\hat{m}_t < m_t^*$. If $\lambda_t = \lambda_{t+1} \beta$ we choose the solution by taking the limit when $\lambda_t - \beta \lambda_{t+1}$ approaches 0. 

In equilibrium, $m_t^i = M_t \leq m_t^*$. The first-order condition from (40) yields

$$\lambda_t = \beta^d \sigma u'(q_t) q_t'(M_t) + (1 - \sigma) \beta \lambda_{t+1}. \quad (41)$$

Under Walrasian pricing, $q_t'(m) = 1/\hat{p}_t = \beta^n \lambda_{t+1}/c'(q_t)$ and $\beta^n \lambda_{t+1} M_t = c'(q_t)q_t$. Therefore (41) becomes

$$\frac{\gamma}{\beta} c'(q_{t-1}) q_{t-1} = \left\{ \sigma \frac{u'(q_t)}{c'(q_t)} + 1 - \sigma \right\} c'(q_t)q_t. \quad (42)$$

Under bargaining $q_t'(m) = \beta^n \lambda_{t+1}/g'(q_t)$ and $\beta^n \lambda_{t+1} M_t = g(q_t)$. Therefore (41) gives

$$\frac{\gamma}{\beta} g(q_{t-1}) = \left\{ \sigma \frac{u'(q_t)}{g'(q_t)} + 1 - \sigma \right\} g(q_t). \quad (43)$$

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We now have to impose conditions to guarantee agents’ choice of labor is interior so that \( W_t(m) \) is linear for the values of \( m \) in the support of \( F_0 \). Suppose that at the end of the centralized round of trading in period \( t-1 \) the distribution of money balances is degenerate, i.e., all agents hold \( M_{t-1} \). In the decentralized market in period \( t-1 \), each agent faces one of the following three idiosyncratic events: He is a buyer in which case he spends all his money balances; He is the seller in which case he receives all the money balances of his partner; He is not matched in which case he carries his money balances to the following period. At the end of the round of decentralized trade, the support of the distribution of money balances is \( \{0, M_{t-1}, 2M_{t-1}\} \) with \( \Pr[2M_{t-1}]=\Pr[0]=\sigma \) and \( \Pr[M_{t-1}]=1-\sigma \). In the centralized market of following period \( t \) agents receive a lump-sum transfer equal to \( T_t = M_t - M_{t-1} = (\gamma - 1)M_{t-1} \). The support of the distribution \( F_0 \) at time \( t \) is then \( \{M_t - M_{t-1}, M_t, M_t + M_{t-1}\} \). Note that we allow \( M_t - M_{t-1} \) to be negative: in this case the agent’s wealth is negative and equal to his tax liabilities. Denote \( z(q_t) = \beta^n \lambda_{t+1} M_t \) the relationship between real balances and the quantities traded at night. Under Walrasian pricing \( z(q_t) = c'(q_t) q_t \) and under bargaining \( z(q_t) = g(q_t) \).

**Lemma 7** Assume that the support of the distribution of money balances at the beginning of period \( t \) is \( \{M_t - M_{t-1}, M_t, M_t + M_{t-1}\} \). Then \( \tilde{m}_i^t = M_t \) for all \( i \) if

\[
z(q_{t-1}) \leq \beta^n |U(c_{u,t}, 0) - U(c_{e,t}, 1)| \min(1 - n_t, n_t).
\]

**Proof.** From Proposition 3 we need to impose conditions so that \( \{M_t - M_{t-1}, M_t, M_t + M_{t-1}\} \subseteq [\underline{w}, \bar{w}] \) where \( \underline{w} \) and \( \bar{w} \) are defined in the proof of Proposition 3. Assume \( U(c_{u,t}, 0) - U(c_{e,t}, 1) > 0 \). Conditions (21) and (22)
require
\[ \lambda_t M_{t-1} \leq [U(c_{u,t}, 0) - U(c_{e,t}, 1)] \min (1 - n_t, n_t). \]  

(45)

Since \( \beta^n \lambda_{t+1} M = z(q_t) \), (45) can be rewritten as
\[ z(q_{t-1}) \leq \beta^n [U(c_{u,t}, 0) - U(c_{e,t}, 1)] \min (1 - n_t, n_t). \]  

(46)

Similarly, if \( U(c_{u,t}, 0) - U(c_{e,t}, 1) < 0 \) then (23) and (24) yield
\[ z(q_{t-1}) \leq -\beta^n [U(c_{u,t}, 0) - U(c_{e,t}, 1)] \min (1 - n_t, n_t). \]  

(47)

Definition 3 A steady-state monetary equilibrium is a sequence \((q_t, c_{e,t}, c_{u,t}, n_t)\) that satisfies (28), (29), (30), (42) or (43), and (44).

There is a dichotomy between the centralized market and the decentralized one. The allocation in the centralized sector, \((c_{e,t}, c_{u,t}, n_t)\), is determined by (28)-(30) while the allocation in the search sector is determined by the difference equation (43). Note also that the allocation in the centralized market is identical to the one that would prevail if money was not valued. A consequence of this dichotomy is that the inflation rate does not affect aggregate production and consumption in the centralized sector. In particular, the fraction of agents who are unemployed, \(n_t\), is independent of the money growth rate, \(\gamma\). So, the model predicts a vertical long-run Phillips curve. Even though inflation does not affect aggregate employment, it does affect how employment is distributed across agents. Assuming leisure is a normal good, the fraction of rich agents who are unemployed falls and the fraction of poor agents who are unemployed increases as the inflation rate increases. In other words, the Phillips curve is downward-sloping for rich
agents, upward-sloping for poor agents, and vertical for agents holding the average money holdings.

4 Conclusion
Appendix 1: Second-order conditions for the consumer’s problem

We check the second-order sufficient conditions for a strict local maximum to the consumer’s problem. We assume here that there is only one consumption good and that money (land) enters additively in the utility function. The problem is as follows:

$$\max_{c_e, c_u, \hat{m}} \{ \ell U(c_e, 1) + (1 - \ell)U(c_u, 0) + V(\hat{m}) \}$$

s.t. $\ell pc_e + (1 - \ell)pc_u + \hat{m} = \ell w + m$

The Lagrangian associated with this problem is

$$\mathcal{L}(c_e, c_u, \hat{m}, \ell, \lambda) = \ell U(c_e, 1) + (1 - \ell)U(c_u, 0) + V(\hat{m}) + \lambda[\ell w + m + \Delta + T - \ell pc_e - (1 - \ell)pc_u - \hat{m}]$$

The first derivatives of the Lagrangian are

$$\frac{\partial \mathcal{L}}{\partial c_e} = \ell U_e' - \lambda \ell p,$$

$$\frac{\partial \mathcal{L}}{\partial c_u} = (1 - \ell)U_u' - \lambda(1 - \ell)p,$$

$$\frac{\partial \mathcal{L}}{\partial \hat{m}} = V'(\hat{m}) - \lambda,$$

$$\frac{\partial \mathcal{L}}{\partial \ell} = U_e - U_u + \lambda[w - pc_e + pc_u],$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \ell w + m + \Delta + T - \ell pc_e - (1 - \ell)pc_u - \hat{m},$$

where $U_e = U(c_e, 1), U'_e = \partial U(c_e, 1)/\partial c_e$ and so on. The bordered Hessian evaluated at a point where the first-order conditions are satisfied is

$$B = \begin{pmatrix}
0 & -\ell p & -(1 - \ell)p & -1 & w - pc_e + pc_u \\
-\ell p & \ell U_e'' & 0 & 0 & 0 \\
-(1 - \ell)p & 0 & (1 - \ell)U_u'' & 0 & 0 \\
-1 & 0 & 0 & V''(\hat{m}) & 0 \\
w - pc_e + pc_u & 0 & 0 & 0 & 0
\end{pmatrix}$$
For a maximum we need to check that the last three leading principal minors of $B$, denoted $|B_3|$, $|B_4|$ and $|B_5|$, alternate in sign with $|B_5| > 0$.

\[
|B_5| = -(w - pe + pu)^2 V''(\hat{m}) \ell U''_e (1 - \ell) U''_u > 0,
\]

\[
|B_4| = -V''(\hat{m}) \left\{ (\ell p)^2 (1 - \ell) U''_u + [(1 - \ell)p]^2 \ell U''_e \right\} - \ell (1 - \ell) U''_e U''_u < 0,
\]

\[
|B_3| = - (\ell p)^2 (1 - \ell) U''_u - [(1 - \ell)p]^2 \ell U''_e > 0.
\]

Therefore any point that satisfies the first-order conditions is a strict local maximum.
Appendix 2: Efficient allocation

We consider a social planner who maximizes the sum of the utilities of all agents subject to some feasibility constraints as well as the search frictions in the decentralized market. Denote \( \tilde{I} \subseteq I \times I \) the set of bilateral matches in the decentralized market where \((i, j) \in \tilde{I}\) if \(i\) and \(j\) are matched and \(i\) is the buyer in the match. The planner’s problem can be written recursively as follows

\[
W^* = \max_{c^i, h^i, q^i} \int \int I U[c^i(s), h^i(s)]dids + \beta^d \left[ \int I [u(q^i) - c(q^i)] d(i, j) \right] + \beta W^*
\]

(48)

s.t. \( \int c^i(s)di \leq f \left[ \int h^i(s)di \right], \forall s \) \hspace{1cm} (49)

\( q^i \leq q^j, \forall (i, j) \in \tilde{I} \)

(50)

Denote \( I^e(s) \) the set of agents who are working in state \( s \) and let \( \ell(s) \) be the measure of this set.

**Proposition 4** The efficient allocation is such that

\[ q^i = q^j = q^*, \forall (i, j) \in \tilde{I}, \]

(51)

\( c^i(s) = c^e \) for all \( i \in I^e(s) \), \( c^i(s) = c^u \) for all \( i \in I \setminus I^e(s) \), \( \ell(s) = \ell \) for all \( s \).

Assuming an interior solution, \((c^e, c^u, \ell)\) satisfies

\[
U_c(c_e, 1) = U_c(c_u, 0),
\]

(52)

\[
U(c_e, 1) - U(c_u, 0) = U_c(c_u, 0) [c_e - c_u - f'(\ell)],
\]

(53)

\[
\ell c^e + (1 - \ell)c^u = f(\ell).
\]

(54)
**Proof.** From (48) and (50), \( q_i = q^j = \arg \max \{u(q) - c(q)\} \) for all \((i,j) \in \tilde{I}\).

From the strict concavity of \( U(c,h) \), \( c^i(s) = c^e(s) \) for all \( i \in I^e(s) \) and \( c^i(s) = c^u(s) \) for all \( i \in I \setminus I^e(s) \). We can therefore rewrite the problem in (48) as

\[
W^* = \max_{c^e,c^u} \ell U(c^e,1) + (1 - \ell)U(c^u,0) + \beta \sigma \left[ u(q^*) - c(q^*) \right] + \beta W^* \tag{55}
\]

\[
\text{s.t. } \ell c^e + (1 - \ell)c^u \leq f(\ell) . \tag{56}
\]

The first-order conditions to this problem are (52) and (53). □
References


