Rational Choice on Arbitrary Domains: A Comprehensive Treatment

WALTER BOSSERT

Département de Sciences Economiques and CIREQ
Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal QC H3C 3J7
Canada

walter.bossert@umontreal.ca

KOTARO SUZUMURA

Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo 186-8603
Japan

suzumura@ier.hit-u.ac.jp

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Abstract

The rationalizability of a choice function on arbitrary domains by means of a transitive relation has been analyzed thoroughly in the literature. Moreover, characterizations of various versions of consistent rationalizability have appeared in recent contributions. However, not much seems to be known when the coherence property of quasi-transitivity or that of P-acyclicity is imposed on a rationalization. The purpose of this paper is to fill this significant gap. We provide characterizations of all forms of rationalizability involving quasi-transitive or P-acyclical rationalizations on arbitrary domains. Furthermore, we examine the consequences of imposing domain-closedness restrictions on the logical relationships among different versions of rationalizability. *Journal of Economic Literature* Classification No.: D11.

Keywords: Rational Choice, Quasi-Transitivity, P-Acyclicity.

1 Introduction

The question whether observed (individual or social) choice behaviour can be generated by some notion of optimization is one of the most fundamental issues in the analysis of economic decisions. The basic question to be addressed is the following. Given some observed (or, at least, observable) choices from feasible sets, does there exist a preference relation (possibly with some additional properties) such that, for each choice situation under consideration, the set of chosen objects is given by some set of 'best' elements according to this relation? There are two basic forms of rationalizability, namely, *greatest-element rationalizability* and *maximal-element rationalizability.* Greatest-element rationalizability requires the existence of a relation such that, for any feasible set in the domain of a choice function, the set of chosen elements coincides with those elements of the feasible set that are at least as good as all feasible alternatives. Maximal-element rationalizability, on the other hand, demands the existence of a relation such that the set of chosen objects consists of all undominated elements in the feasible set, that is, all elements such that there exists no feasible alternative that is strictly preferred.

In addition to the basic type of rationalizability (in terms of greatest elements or in terms of maximal elements), we can require the rationalizing relation to be endowed with certain fundamental properties. Two standard requirements are *reflexivity* and *completeness,* which we refer to as *richness* properties because they require a relation to contain at least certain pairs of alternatives. Furthermore, it is customary to impose *coherence* properties such as *transitivity, consistency, quasi-transitivity* or *P-acyclicity;* see the following section for formal definitions.

Revealed preference theory has its origins in the theory of consumer demand, where the choices to be analyzed are those of a competitive consumer from budget sets. This area of research has been developed in contributions such as those of Samuelson (1938; 1947, Chapter V; 1948; 1950) and Houthakker (1950).

Uzawa (1957) and Arrow (1959) considered alternative choice situations by introducing the general concept of a choice function defined on the domain of all subsets of a universal set of alternatives. In this setting, Sen (1971), Schwartz (1976), Bandyopadhyay and Sengupta (1991), to name but a few, characterized notions of rational choice under various coherence conditions imposed on rationalizing relations. Most notably, the theory of rational choice on such rich domains was greatly simplified by the equivalence results between several revealed preference axioms, for example, the weak axiom of revealed preference and the strong axiom of revealed preference, whose subtle difference had been regarded as lying at the heart of the integrability problem for a competitive consumer.

Clearly, the above-described assumptions regarding the class of possible choice situations to be considered restrict the applicability of the results obtained. Thus, it is of great interest to examine what the logic of rational choice—and nothing else—entails in general, irrespective of the domain of a choice function. A crucial step along this line was taken by Richter (1966; 1971), Hansson (1968) and Suzumura (1976; 1977; 1983, Chapter 2) who assumed the domain of a choice function to be an arbitrary family of non-empty subsets of an arbitrary non-empty universal set of alternatives without any further structural assumptions.

While the theory of rational choice on arbitrary domains is well-developed if a rationalizing relation is assumed to be transitive, much less is known when weaker coherence properties are imposed. The case of consistent rationalizability has been addressed recently in Bossert, Sprumont and Suzumura (2005a). Furthermore, some versions of maximal-element rationalizability (and, thus, those versions of greatest-element rationalizability that are equivalent to them) have been characterized in Bossert, Sprumont and Suzumura (2002) but a comprehensive treatment of rationalizability on arbitrary domains in the presence of quasi-transitivity or P-acyclicity is still missing. An analysis of some conditions that are necessary and others that are sufficient for some forms of quasi-transitive or P-acyclical rationalizability can be found in Suzumura (1983) and Bossert, Sprumont and Suzumura (2005b) but full characterizations of most of these concepts have not yet been provided.

The purpose of this paper is to fill the above-mentioned gap in the literature, thereby providing characterizations of all relevant notions of rationalizability on arbitrary domains. Thus, the results of this paper provide systematic answers to some important open questions in the literature on rational choice and revealed preference. It may be worth pointing out that answering these open questions have more substantial relevance than simply filling in logical lacunae in the literature. Recollect that assuming the rationalizing weak preference relation to be transitive entices strong empirical criticism. There are many experimental studies that suggest that the imperfect discriminatory power of human beings leads to nontransitive indifference and Armstrong (1948, p.3) went as far as to assert: "That indifference is not transitive is indisputable, and the world in which it were transitive is indeed unthinkable." Thus, to liberate the theory of rationalizable choice functions from the unsustainable assumption of transitive indifference seems to be an important step for the sake of making the theoretical edifice more of empirical relevance than otherwise.

Our basic definitions and some preliminary observations are collected in Section 2. Section 3 introduces our different notions of rationalizability and examines their logical relationships on arbitrary domains. Section 4 is devoted to characterizations of all versions of rationalizability involving quasi-transitivity or P-acyclicity, as well as a version that does not impose any coherence condition. In Section 5, we show that the domain assumption of closedness under union leads to an additional implication regarding the different definitions of rationalizability. Section 6 concludes.

2 Preliminaries

We consider a non-empty (but otherwise arbitrary) universal set of alternatives X, and we let $R \subseteq X \times X$ be a (binary) relation on X. The *asymmetric factor* $P(R)$ of R is defined by $(x,y) \in P(R)$ if and only if $(x,y) \in R$ and $(y,x) \notin R$ for all $x,y \in X$. The *symmetric factor* $I(R)$ of R is defined by $(x,y) \in I(R)$ if and only if $(x,y) \in R$ and $(y,x) \in R$ for all $x,y \in X$. The *non-comparable factor* $N(R)$ of R is defined by $(x,y) \in N(R)$ if and only if $(x,y) \notin R$ and $(y,x) \notin R$ for all $x,y \in X$. If R is interpreted as a *weak preference relation,* that is, $(x,y) \in R$ means that x is considered at least as good as y, then $P(R)$, $I(R)$ and $N(R)$ can be interpreted as the *strict preference relation,* the *indifference relation* and the *non-comparability relation* corresponding to R, respectively. The *diagonal relation* on X is given by $R_d = \{(x,x) \mid x \in X\}$.

Let $\mathbb{N}$ denote the set of positive integers. The following properties of a binary relation R are of importance in this paper.

Reflexivity. For all $x \in X$,

$$(x, x) \in R.$$

Completeness. For all $x, y \in X$ such that $x \neq y$,

$$(x, y) \in R \text{ or } (y, x) \in R.$$

Transitivity. For all $x, y, z \in X$,

$$[(x, y) \in R \text{ and } (y, z) \in R] \;\Rightarrow\; (x, z) \in R.$$

Quasi-transitivity. For all $x, y, z \in X$,

$$[(x, y) \in P(R) \text{ and } (y, z) \in P(R)] \;\Rightarrow\; (x, z) \in P(R).$$

Consistency. For all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$,

$$(x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\} \;\Rightarrow\; (x^K, x^0) \notin P(R).$$

P-acyclicity. For all $K \in \mathbb{N} \setminus \{1\}$ and for all $x^0, \ldots, x^K \in X$,

$$(x^{k-1}, x^k) \in P(R) \text{ for all } k \in \{1, \ldots, K\} \;\Rightarrow\; (x^K, x^0) \notin P(R).$$

A reflexive and transitive relation is called a *quasi-ordering* and a complete quasi-ordering is called an *ordering.*

We refer to reflexivity and completeness as *richness* conditions. This term is motivated by the observation that the properties in this group require that, at least, some pairs must belong to the relation under consideration. In the case of reflexivity, all pairs of the form (x, x) are required to be in the relation, whereas completeness demands that, for any two distinct alternatives x and y, at least one of (x, y) and (y, x) must be in R. Clearly, the reflexivity requirement is equivalent to the set inclusion $R_d \subseteq R$.

Transitivity, quasi-transitivity, consistency and P-acyclicity are *coherence* properties. They require that if certain pairs belong to R, then certain other pairs must belong to R as well (as is the case for transitivity and for quasi-transitivity) or certain other pairs cannot belong to R (which applies to the cases of consistency and of P-acyclicity). Quasi-transitivity and consistency are independent. A transitive relation is quasi-transitive, and a quasi-transitive relation is P-acyclical. Moreover, a transitive relation is consistent, and a consistent relation is P-acyclical. The reverse implications are not true in general. However, the distinction between transitivity and consistency disappears for a reflexive and complete relation; see Suzumura (1983, p.244). Thus, if a relation R on X is reflexive, complete and consistent, then R is transitive, hence an ordering.

Transitivity is *the* classical coherence requirement on preference relations and its significance in theories of individual and collective choice is obvious. Quasi-transitivity was introduced by Sen (1969; 1970, Chapter 1*), and it has been employed in numerous approaches to the theory of individual and social choice, including issues related to rationalizability. P-acyclicity has the important property that it, together with reflexivity and completeness, is not only sufficient for the existence of undominated choices from any arbitrary finite subset of a universal set, but it is also necessary for the existence of such choices from all possible finite subsets of the universal set; see Sen (1970, Chapter 1*).

Violations of transitivity are quite likely to be observed in practical choice situations. For instance, Luce's (1956) well-known coffee-sugar example provides a plausible argument against assuming that indifference is always transitive: the inability of a decision maker to perceive 'small' differences in alternatives is bound to lead to intransitivities. As this example illustrates, transitivity frequently is too strong an assumption to impose in the context of individual choice. In collective choice problems, it is even more evident that the plausibility of transitivity can be questioned. The concept of consistency, which is due to Suzumura (1976b), is of particular interest in this context. To underline its importance, note that this property is exactly what is required to prevent the problem of a 'money pump.' If consistency is violated, there exists a preference cycle with at least one strict preference. In this case, an agent with such preferences is willing to trade (where 'willingness to trade' is assumed to require that the alternative acquired in the trade is at least as good as that relinquished) an alternative x^0 for another alternative x^1, x^1 for an alternative x^2 and so on until we reach an alternative x^K such that the agent *strictly prefers* getting back to x^0 to retaining possession of x^K. Thus, at the end of a chain of exchanges, the agent is willing to pay a positive amount in order to get back to the alternative it had in its possession in the first place—a classical example of a money pump.

There is yet another reason for the importance of the concept of consistency. As shown by Suzumura (1976b; 1983, Chapter 1), consistency is necessary and sufficient for the existence of an ordering which subsumes all the pairwise information contained in the binary relation. This result is a generalization of Szpilrajn's (1930) classical result on extending quasi-orderings to orderings.

Let $\mathcal{X}$ be the set of all non-empty subsets of X. We now introduce the concepts of greatestness and maximality with respect to a relation. Suppose R is a relation on X and $S \in \mathcal{X}$. The set $G(S, R)$ of all *R-greatest elements* of S is defined by

$$G(S, R) = \{x \in S \mid (x, y) \in R \text{ for all } y \in S\} \tag{1}$$

and the set $M(S, R)$ of all *R-maximal elements* of S is defined by

$$M(S, R) = \{x \in S \mid (y, x) \notin P(R) \text{ for all } y \in S\}.$$

As is straightforward to verify, $G(S, R) \subseteq M(S, R)$ for all relations R on X and for all $S \in \mathcal{X}$. Furthermore, if R is reflexive and complete, then $G(S, R) = M(S, R)$; for relations R that are not reflexive or not complete, the set inclusion can be strict.

A *choice function* is a mapping that assigns, to each feasible set in its domain, a subset of this feasible set. This subset is interpreted as the set of chosen alternatives. The domain of the choice function depends on the choice situation to be analyzed, but it will always be a set of subsets of X, that is, a subset of $\mathcal{X}$. We assume this subset of $\mathcal{X}$ to be non-empty to avoid degenerate situations. Thus, letting $\Sigma \subseteq \mathcal{X}$ be a non-empty domain, a choice function defined on that domain is a mapping $C\colon \Sigma \to \mathcal{X}$ such that, for all $S \in \Sigma$, $C(S) \subseteq S$. The *image of* Σ *under* C is given by $C(\Sigma) = \cup_{S\in\Sigma} C(S)$.

The *direct revealed preference relation* R_C of a choice function C with domain Σ is defined as

$$R_C = \{(x, y) \in X \times X \mid \exists S \in \Sigma \text{ such that } x \in C(S) \text{ and } y \in S\}.$$

The *axiom of choice* is employed in some of the results discussed here. It is defined as follows.

Axiom of choice. Suppose that $\mathcal{T}$ is a collection of non-empty sets. Then there exists a function $\varphi\colon \mathcal{T} \to \cup_{T\in\mathcal{T}} T$ such that $\varphi(T) \in T$ for all $T \in \mathcal{T}$.

We conclude this section with the statement of a fundamental result regarding the existence of *ordering extensions.*. A relation R' is an *extension* of a relation R if and only if we have (i) $R \subseteq R'$ and (ii) $P(R) \subseteq P(R')$. Conversely, R is said to be a *subrelation* of R' if and only if R' is an extension of R.

The following classical theorem, which is a variant of the basic theorem due to Szpilrajn (1930), specifies a sufficiency condition for the existence of an extension that is an ordering, to be called an *ordering extension.* This convenient variant of Szpilrajn's theorem was stated by Arrow (1951, p.64) without a proof, whereas Hansson (1968) provided a full proof thereof on the basis of Szpilrajn's original theorem. The proof makes use of the axiom of choice; see Szpilrajn (1930). Arrow (1951) and Suzumura (1976b; 1983, Chapter 1; 2004) provide generalizations of this result.

Theorem 1 *Any quasi-ordering R on X has an ordering extension.*

3 Definitions of Rationalizability

There are two basic forms of rationalizability properties that are commonly considered in the literature. The first is *greatest-element rationalizability* which requires the existence of a relation such that, for any feasible set, every chosen alternative is at least as good as every alternative in the set. Thus, this notion of rationalizability is based on the view that chosen alternatives should weakly dominate all feasible alternatives. *Maximal-element rationalizability,* on the other hand, demands the existence of a relation such that, for each feasible set, there exists no alternative in this set that is strictly preferred to any one of the chosen alternatives. Hence, this version of rationalizability does not require chosen alternatives to weakly dominate all elements of the feasible set but, instead, demands that they are not strictly dominated by any other feasible alternative.

In addition to one or the other of these two concepts of rationalizability, we have a choice regarding the properties that we require a rationalizing relation to possess. We consider the standard richness requirements of *reflexivity* and *completeness* and, in addition, the coherence properties of *transitivity, quasi-transitivity, consistency* and *P-acyclicity.* By combining each version of rationalizability with one or both (or none) of the richness conditions and with one (or none) of the coherence properties, various definitions of rationalizability are obtained. Some of these definitions are equivalent, others are independent, and some are implied by others. To get an understanding of what each of these definitions entails, we summarize all logical relationships between them in this section.

A choice function C is *greatest-element rationalizable, G-rationalizable* for short, if there exists a relation R on X, to be called a *G-rationalization* of C, such that $C(S) = G(S, R)$ for all $S \in \Sigma$. Analogously, a choice function C is *maximal-element rationalizable, M-rationalizable* for short, if there exists a relation R on X, to be called an *M-rationalization* of C, such that $C(S) = M(S, R)$ for all $S \in \Sigma$. If a rationalization R is required to be reflexive and complete, the notion of greatest-element rationalizability coincides with that of maximal-element rationalizability because, in this case, $G(S, R) = M(S, R)$ for all $S \in \mathcal{X}$. Without these properties, however, this is not necessarily the case. Greatest-element rationalizability is based on the idea of chosen alternatives weakly dominating all alternatives in the feasible set under consideration, whereas maximal-element rationalizability requires chosen elements not to be strictly dominated by any other feasible alternative.

The following theorem presents a fundamental relationship between the direct revealed preference relation and a G-rationalization of a choice function. This observation, which is due to Samuelson (1938; 1948), states that any G-rationalization of a G-rationalizable choice function must respect the direct revealed preference relation of this choice function. This result follows immediately from combining the definitions of the direct revealed preference relation R_C and of G-rationalizability.

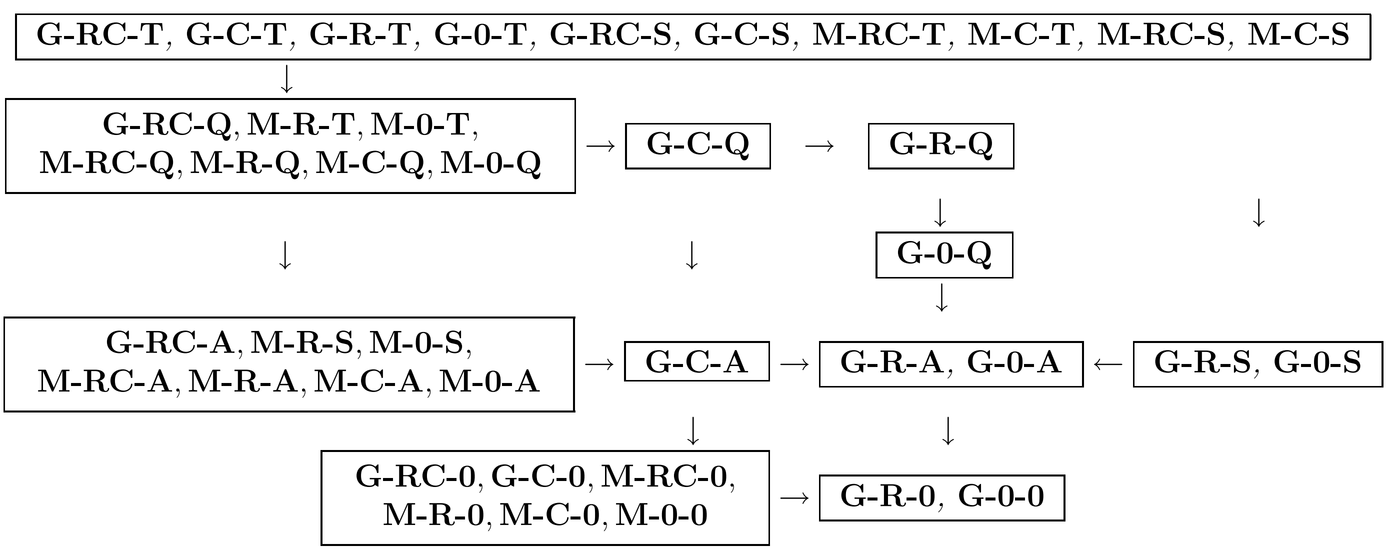
Theorem 2 *Suppose $C: \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$ and R is a relation on X. If R is a G-rationalization of C, then $R_C \subseteq R$.*

An analogous set inclusion is not valid for M-rationalizability. An M-rationalization does not necessarily have to respect the direct revealed preference relation because chosen alternatives merely have to be undominated within the feasible set from which they are chosen. The only restriction that is imposed is that the union of an M-rationalization and its non-comparable factor must respect the revealed preference relation. This is a straightforward observation because a strict preference of any feasible alternative over a chosen one immediately contradicts the definition of M-rationalizability.

Depending on the additional properties that we might want to impose on a rationalization (if any), different notions of rationalizability can be defined. For simplicity of presentation, we use the following convention when formulating a rationalizability axiom. We distinguish three groups of properties of a relation, namely, *rationalization* properties, *richness* properties and *coherence* properties. The first group consists of the two rationalizability properties of G-rationalizability and M-rationalizability, the second of the two requirements of reflexivity and completeness and, finally, the third of the axioms of transitivity, quasi-transitivity, consistency and P-acyclicity. Greatest-element rationalizability is abbreviated by **G**, **M** is short for maximal-element rationalizability, **R** stands for reflexivity and **C** is completeness. Transitivity, quasi-transitivity, consistency and P-acyclicity are denoted by **T**, **Q**, **S** and **A**, respectively. We identify the property or properties to be satisfied within each of the three groups and separate the groups with hyphens. If none of the properties within a group is required, this is denoted by using the symbol **0**. Either greatest-element rationalizability or maximal-element rationalizability may be required. In addition to imposing one of the two richness properties only, reflexivity and completeness may be required simultaneously and we may require rationalizability properties without either of the two. We only consider notions of rationalizability involving at most one of the coherence properties at a time. As is the case for the richness properties, imposing none of the coherence properties is a possibility. Formally, a rationalizability property is identified by an expression of the form α-β-γ, where $\alpha \in \{\mathbf{G}, \mathbf{M}\}$, $\beta \in \{\mathbf{RC}, \mathbf{R}, \mathbf{C}, \mathbf{0}\}$ and $\gamma \in \{\mathbf{T}, \mathbf{Q}, \mathbf{S}, \mathbf{A}, \mathbf{0}\}$. For example, greatest-element rationalizability by a reflexive, complete and transitive relation is denoted by **G-RC-T**, maximal-element rationalizability by a complete relation is **M-C-0**, greatest-element rationalizability by a reflexive and consistent relation is **G-R-S** and maximal-element rationalizability without any further properties of a rationalizing relation is **M-0-0**. Clearly, according to this classification, there are $2 \cdot 4 \cdot 5 = 40$ versions of rationalizability.

We now provide a full description of the logical relationships between these different notions of rationalizability. This result synthesizes contributions due to Bossert, Sprumont and Suzumura (2002; 2005a; 2005b) and, therefore, we do not provide a proof; see the original papers for details. For convenience, a diagrammatic representation is employed. All axioms that are depicted within the same box are equivalent, and an arrow pointing from one box b to another box b' indicates that the axioms in b imply those in b', and the converse implication is not true. In addition, of course, all implications resulting from chains of arrows depicted in the diagram are valid.

Theorem 3 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. Then*



Although the equivalences established in the above theorem reduce the number of distinct notions of rationalizability from a possible forty to eleven, there remains a relatively rich set of possible definitions. In particular, note that none of the coherence properties of transitivity, quasi-transitivity, consistency and P-acyclicity is redundant: eliminating any one of them reduces the number of distinct definitions, and so does the elimination of the versions not involving any coherence property.

Furthermore, it is worth pointing out an important and remarkable difference between G-rationalizability by a transitive or a consistent relation on the one hand and G-rationalizability by a quasi-transitive or a P-acyclical relation on the other. In the case of G-rationalizability with transitivity or consistency, the reflexivity requirement is redundant in all cases. That is, irrespective of whether or not completeness is imposed as a richness condition, any version of G-rationalizability with transitivity or consistency and without reflexivity is equivalent to the definition that is obtained if reflexivity is added. This observation applies to the case where no coherence property is imposed as well. In contrast, G-rationalizability by a complete and quasi-transitive relation is not equivalent to G-rationalizability by a reflexive, complete and quasi-transitive relation, and the same is true for the relationship between G-rationalizability by a complete and P-acyclical relation and G-rationalizability by a reflexive, complete and P-acyclical relation. In addition, while G-rationalizability by a P-acyclical relation and G-rationalizability by a reflexive and P-acyclical relation are equivalent, there is yet another discrepancy in the quasi-transitive case: G-rationalizability by a quasi-transitive relation is not the same as G-rationalizability by a reflexive and quasi-transitive relation.

In the case of M-rationalizability, only four distinct notions of rationalizability exist although, in principle, there are twenty definitions, as in the case of G-rationalizability. This means that there is a dramatic reduction of possible definitions due to the equivalences established in the theorem. Note that, within the set of definitions of M-rationalizability, there is a substantial degree of redundancy. In particular, it is possible to generate all possible versions of M-rationalizability with merely two coherence properties: any of the combinations of transitivity and consistency, transitivity and P-acyclicity, quasi-transitivity and consistency is sufficient to obtain all four notions of M-rationalizability (provided, of course, that the option of not imposing any coherence property is retained). Moreover, reflexivity is redundant in all forms of M-rationalizability, irrespective of the coherence property imposed (if any), including quasi-transitivity and P-acyclicity: any version of M-rationalizability without reflexivity is equivalent to the version obtained by adding this richness property.

There is an interesting feature that distinguishes the notions of transitive or consistent M-rationalizability from those involving quasi-transitivity, P-acyclicity or none of the coherence properties. All four M-rationalizability properties involving quasi-transitivity are equivalent, and so are all four notions involving P-acyclicity as well as the four versions without any coherence property. In contrast, there are two distinct notions of transitive M-rationalizability and two distinct notions of consistent M-rationalizability.

As is apparent from Theorem 3, M-rationalizability does not add any new versions of rationalizability, provided that all definitions of G-rationalizability with all of the four coherence properties are present. Therefore, the characterization results stated in the following section can be proved without explicitly having to work with M-rationalizability.

4 Characterizations

This section constitutes the main contribution of this paper. We present necessary and sufficient conditions for all notions of G-rationalizability involving quasi-transitivity or P-acyclicity as the required coherence property, in addition to the version with completeness but without any coherence property. We focus on these eight versions because the remaining three variants have already been characterized by Richter (1966), Hansson (1968) and Suzumura (1977) in the case of **G-RC-T** and its equivalents, by Richter (1971) in the case of **G-R-0** and **G-0-0** and by Bossert, Sprumont and Suzumura (2005a) in the case of **G-R-S** and **G-0-S**. Of the eight versions stated here, the three that are equivalent to one of the notions of M-rationalizability have been characterized by Bossert, Sprumont and Suzumura (2002) and, therefore, we do not provide proofs. We include their statements here, however, because they make use of necessary and sufficient conditions that are in the same spirit as those employed in our new results and, therefore, permit us to provide a comprehensive and exhaustive treatment of quasi-transitive and P-acyclical rationalizability.

The results of this section are extremely general because they apply to any arbitrary domain, and they serve to close an important gap in the literature: they provide characterizations of all notions of rationalizability that have not been axiomatized before.

The notion of rationalizability considered in this section involve rather complex formulations of necessary and sufficient conditions. The reason is that there is no such thing as a *smallest* quasi-transitive relation or a *smallest* P-acyclical relation for a given arbitrary relation. In contrast, any relation R has a well-defined *transitive closure* and a well-defined *consistent closure* which are, respectively, the smallest transitive relation containing R and the smallest consistent relation containing R. Intuitively, when moving from R to its transitive or consistent closure, pairs are added that are *necessarily* in any transitive or consistent relation containing R. As soon as there exist alternatives $x^0, \ldots, x^K$ connecting two alternatives x and y via a chain of weak preferences, transitivity demands that the pair (x, y) is included in any transitive relation that contains R. Analogously, a chain of that nature implies that if, in addition, the pair (y, x) is in R, (x, y) must be added if the resulting relation is to be consistent. In contrast, there are no necessary additions to a relation in order to transform it into a quasi-transitive relation by augmenting it. For instance, suppose we have $(x, y) \in P(R)$, $(y, z) \in P(R)$ and $(z, x) \in P(R)$. In order to define a quasi-transitive relation that contains R, at least two of the three strict preferences must be converted into indifferences but *any two* will do. Thus, there is no unique smallest quasi-transitive relation containing R. Similarly, if we have a P-cycle, a P-acyclical relation containing R merely has to have the property that at least *one* of the pairs along the cycle, representing a strict preference, must be converted into an indifference. But, without further information, there is nothing that forces this indifference on a specific pair along the cycle. As a consequence, there is, in general, no smallest P-acyclical relation containing an arbitrary relation R. The same difficulty arises when G-rationalizability involving completeness without any coherence conditions is considered. For that reason, we introduce some further concepts in order to be able to formulate necessary and sufficient conditions for the definitions of rationalizability considered in this section.

Let $C \colon \Sigma \to \mathcal{X}$ be a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$, and define

$$\mathcal{A}_C = \{(S, y) \mid S \in \Sigma \text{ and } y \in S \setminus C(S)\}$$

and

$$\mathcal{F}_C = \{f \colon \mathcal{A}_C \to X \mid f(S, y) \in S \text{ for all } (S, y) \in \mathcal{A}_C\}.$$

The set $\mathcal{A}_C$ consists of all pairs of a feasible set and an element that belongs to the set but is not chosen by C. If $C(S) = S$ for all $S \in \Sigma$, the set $\mathcal{A}_C$ is empty; in all other cases, $\mathcal{A}_C \neq \emptyset$. The functions in $\mathcal{F}_C$ have an intuitive interpretation. They assign a feasible element to each pair of a feasible set S and an alternative y that is in S, but not chosen from S. Within the framework of G-rationalizability, the intended interpretation is that $f(S, y)$ is an alternative in S that can be used to prevent y from being chosen in the sense that y is not at least as good as $f(S, y)$ according to a G-rationalization. Clearly, the existence of such an alternative for each (S, y) in $\mathcal{A}_C$ is a necessary condition for G-rationalizability.

We now introduce two categories of crucial properties which prove instrumental in our subsequent axiomatizations. The first category consists of properties which impose restrictions on the relationship between a choice function C and a function $f \in \mathcal{F}_C$.

Self exclusion (SE). For all $(S, x) \in \mathcal{A}_C$, for all $T \in \Sigma$ and for all $x \in T$,

$$f(S, x) = x \Rightarrow x \notin C(T).$$

Direct exclusion (DRE). For all $(S, y) \in \mathcal{A}_C$, for all $T \in \Sigma$ and for all $x \in T$,

$$f(S, y) = x \Rightarrow y \notin C(T).$$

Revelation exclusion (RE). For all $K \in \mathbb{N}$, for all $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$\left[f(S^k, x^k) = x^{k-1} \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\}\right] \Rightarrow x^K \notin C(S^0).$$

Distinctness exclusion (DSE). For all $K \in \mathbb{N}$, for all $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$\left[f(S^k, x^k) = x^{k-1} \text{ and } x^{k-1} \neq x^k \text{ for all } k \in \{1, \ldots, K\}\right] \Rightarrow x^K \notin C(S^0).$$

Indirect exclusion (IE). For all $K \in \mathbb{N}$, for all $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, for all $S^0 \in \Sigma$ and for all $x^0 \in S^0$,

$$f(S^k, x^k) = x^{k-1} \text{ for all } k \in \{1, \ldots, K\} \Rightarrow x^K \notin C(S^0).$$

The second category consists of properties which avoid incoherent behavior of a function $f \in \mathcal{F}_C$ itself.

Self irreversibility (SI). For all $(S, x) \in \mathcal{A}_C$,

$$f(S, x) \neq x.$$

Direct irreversibility (DRI). For all $(S, y), (T, x) \in \mathcal{A}_C$,

$$f(S, y) = x \Rightarrow f(T, x) \neq y.$$

Revelation irreversibility (RI). For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$,

$$\left[f(S^k, x^k) = x^{k-1} \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } (x^K, x^0) \in R_C\right] \Rightarrow f(S^0, x^0) \neq x^K.$$

Distinctness irreversibility (DSI). For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$,

$$\left[f(S^k, x^k) = x^{k-1} \text{ and } x^{k-1} \neq x^k \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K \neq x^0\right] \Rightarrow f(S^0, x^0) \neq x^K.$$

Indirect irreversibility (II). For all $K \in \mathbb{N}$ and for all $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$,

$$f(S^k, x^k) = x^{k-1} \text{ for all } k \in \{1, \ldots, K\} \Rightarrow f(S^0, x^0) \neq x^K.$$

These properties enable us to introduce several axioms which we employ in our characterizations of various concepts of rationalizability. To begin with, the axiom that turns out to be necessary and sufficient for **G-RC-Q** (and, thus, for all rationalizability properties equivalent to it) postulates the existence of a function $f \in \mathcal{F}_C$ satisfying indirect exclusion and indirect irreversibility, provided that the set $\mathcal{A}_C$ is non-empty. The following characterization result is due to Bossert, Sprumont and Suzumura (2002); see that paper for a proof.

Theorem 4 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-RC-Q**, **M-R-T**, **M-0-T**, **M-RC-Q**, **M-R-Q**, **M-C-Q**, **M-0-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **IE** *and* **II**.

The next three results establish characterizations of the three versions of greatest-element rationalizability by a quasi-transitive relation, none of which coincides with any other notion of rationalizability. All three of them are new. We begin with **G-C-Q**.

Theorem 5 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-C-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **SE**, **DSE** *and* **DSI**.

Proof. We first prove that **G-C-Q** implies the existence of $f \in \mathcal{F}_C$ which satisfies **SE**, **DSE** and **DSI** whenever $\mathcal{A}_C \neq \emptyset$. Let R be a complete and quasi-transitive G-rationalization of C. Consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To show that f satisfies **SE**, suppose $(S, x) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, x) = x$. By the definition of f, we have $(x, x) \notin R$. Because R is a G-rationalization of C, it follows that $x \notin C(T)$.

To prove that f satisfies **DSE**, let $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ be such that $f(S^k, x^k) = x^{k-1}$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$ and the completeness of R implies $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and, thus, $(x^K, x^0) \notin R$. Because R is a G-rationalization of C, we obtain $x^K \notin C(S^0)$.

The final property of the function f that remains to be established is **DSI**. Suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. By definition, $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$ and, because R is complete, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and hence $(x^0, x^K) \in R$. By the definition of f, this implies $f(S^0, x^0) \neq x^K$.

Now suppose that, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying **SE**, **DSE** and **DSI**. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a complete and quasi-transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **SE**, **DSE** and **DSI**. Define

$$\begin{aligned} R \;=\;& \{(x, y) \in X \times X \mid \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ & \quad y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } x^{k-1} \neq x^k \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x\} \\ \setminus\;& \{(x, x) \in R_d \mid \exists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = x\}. \end{aligned}$$

To prove that R is complete, suppose, by way of contradiction, that there exist $x, y \in X$ such that $x \neq y$, $(x, y) \notin R$ and $(y, x) \notin R$. By definition, this implies that there exist $K, L \in \mathbb{N}$, $x^0, y^0 \in X$ and $(S^1, x^1), \ldots, (S^K, x^K), (T^1, y^1), \ldots, (T^L, y^L) \in \mathcal{A}_C$ such that $y = x^0$, $x^{k-1} = f(S^k, x^k)$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$, $x^K = x$, $x = y^0$, $y^{\ell-1} = f(T^\ell, y^\ell)$ and $y^{\ell-1} \neq y^\ell$ for all $\ell \in \{1, \ldots, L\}$ and $y^L = y$. Letting $M = K + L - 1$, $(U^0, z^0) = (T^L, y^L)$, $(U^m, z^m) = (S^m, x^m)$ for all $m \in \{1, \ldots, K\}$ and $(U^m, z^m) = (T^{m-K}, y^{m-K})$ for all $m \in \{K+1, \ldots, K+L\} \setminus \{K+L\}$, it follows that $z^{m-1} = f(U^m, z^m)$ and $z^{m-1} \neq z^m$ for all $m \in \{1, \ldots, M\}$ and $z^M = f(U^0, z^0)$, contradicting the property **DSI**.

Next, we show that R is quasi-transitive. Suppose $x, y, z \in X$ are such that $(x, y) \in P(R)$ and $(y, z) \in P(R)$. This implies that $x \neq y$ and $y \neq z$ and, by the definition of R, there exist $K, L \in \mathbb{N}$, $x^0, y^0 \in X$ and $(S^1, x^1), \ldots, (S^K, x^K), (T^1, y^1), \ldots, (T^L, y^L) \in \mathcal{A}_C$ such that $x = x^0$, $x^{k-1} = f(S^k, x^k)$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$, $x^K = y$, $y = y^0$, $y^{\ell-1} = f(T^\ell, y^\ell)$ and $y^{\ell-1} \neq y^\ell$ for all $\ell \in \{1, \ldots, L\}$ and $y^L = z$. Letting $M = K + L$, $z^0 = x^0$, $(U^m, z^m) = (S^m, x^m)$ for all $m \in \{1, \ldots, K\}$ and $(U^m, z^m) = (T^{m-K}, y^{m-K})$ for all $m \in \{K+1, \ldots, K+L\}$, it follows that $(z, x) \notin R$. Because R is complete, we obtain $(x, z) \in P(R)$.

Finally, we prove that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$ and, by way of contradiction, that there exists $y \in S$ such that $(x, y) \notin R$. If $x = y$ and there exists $S \in \Sigma$ such that $(S, x) \in \mathcal{A}_C$ and $f(S, x) = x$, we obtain a contradiction to the property **SE**. If there exist $K \in \mathbb{N}$, $x^0 \in X$ and $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$ such that $y = x^0$, $x^{k-1} = f(S^k, x^k)$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$ and $x^K = x$, letting $S^0 = S$ yields a contradiction to the property **DSE**. Therefore, $x \in G(S, R)$. Thus, $C(S) \subseteq G(S, R)$.

Now suppose $x \notin C(S)$. Let $y = f(S, x)$. By definition, this implies $(x, y) \notin R$ and hence $x \notin G(S, R)$. Thus, $G(S, R) \subseteq C(S)$. ■

The definition of the relation R in the above proof is based on the following intuition. Recall that f is intended to identify, for each feasible set S and for each element y of S that is not chosen by C, an alternative x in S such that y is not at least as good as x. Because reflexivity is not required, the observation that $(y, x) \notin R$ is not equivalent to $(x, y) \in P(R)$—it is possible that $x = y$, in which case we must have non-comparability instead. As a consequence, a strict preference is to be declared not in all situations where an alternative x is the alternative assigned to y by f for some feasible set S, but only when $x \neq y$, in which case a strict preference of x over y is indeed required because of the completeness assumption imposed on a rationalization. In addition, whenever an element x is not chosen in a set S because, according to f, x is not at least as good as itself, the pair (x, x) cannot be in R. Because R is also required to be quasi-transitive, chains of strict preference have to be respected as well. In order to arrive at a complete and quasi-transitive G-rationalization of C, we define R to be composed of all pairs $(x, y) \in X \times X$ such that y does not have to be strictly preferred to x according to the above-described criterion. The properties of f ensure that R is indeed a complete and quasi-transitive G-rationalization of C.

Let us now turn to a characterization of **G-R-Q**.

Theorem 6 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-R-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE**, **RE**, **SI** *and* **RI**.

Proof. Suppose C satisfies **G-R-Q** and let R be a reflexive and quasi-transitive G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$ and consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that f satisfies **DRE**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To show that f satisfies **RE**, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Thus, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and the quasi-transitivity of R implies $(x^0, x^K) \in P(R)$. Therefore, $(x^K, x^0) \notin R$ and, because R is a G-rationalization of C, we obtain $x^K \notin C(S^0)$.

To prove that **SI** is satisfied, suppose there exists $(S, x) \in \mathcal{A}_C$ such that $f(S, x) = x$. By the definition of f, this implies $(x, x) \notin R$, contradicting the reflexivity of R.

To establish the property **RI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in R_C$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Therefore, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and thus $(x^0, x^K) \in R$ which, by the definition of f, implies $f(S^0, x^0) \neq x^K$.

Suppose that, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying **DRE**, **RE**, **SI** and **RI**. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a reflexive and quasi-transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE**, **RE**, **SI** and **RI**. Define

$$\begin{aligned}
R \;=\; & R_C \cup R_d \\
\cup\; & \{(x, y) \in X \times X \mid (y, x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y \\
& \quad \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\
& \quad y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x\} \\
\cup\; & \{(x, y) \in X \times X \mid \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\
& \quad x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\}.
\end{aligned}$$

Clearly, R is reflexive because $R_d \subseteq R$.

To prove that R is quasi-transitive, suppose $x, y, z \in X$ are such that $(x, y) \in P(R)$ and $(y, z) \in P(R)$. This implies that $x \neq y$ and $y \neq z$ and, thus, $(x, y) \notin R_d$ and $(y, z) \notin R_d$. Therefore, $(x, y) \in R$ implies

$$(x, y) \in R_C \tag{2}$$

or

$$\begin{array}{c}(y, x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y \\ \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x\end{array} \tag{3}$$

or

$$\begin{array}{c}\exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y.\end{array} \tag{4}$$

Analogously, $(y, x) \notin R$ implies

$$(y, x) \notin R_C \tag{5}$$

and

$$\begin{array}{c}(x, y) \notin R_C \text{ or } \exists S \in \Sigma \text{ such that } (S, y) \in \mathcal{A}_C \text{ and } f(S, y) = x \\ \text{or } \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\end{array} \tag{6}$$

and

$$\begin{array}{c}\nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x.\end{array} \tag{7}$$

Because (5) must be true, (3) must be false. Therefore, it follows that (2) or (4) is true and that (6) is true. Because (2) and $(x, y) \notin R_C$ are incompatible, it follows that we must have

$$(2) \text{ and } \exists S \in \Sigma \text{ such that } (S, y) \in \mathcal{A}_C \text{ and } f(S, y) = x \tag{8}$$

or

$$(2) \text{ and } (4) \tag{9}$$

or (4). Clearly, (8) implies (4) and (9) implies (4) trivially. Thus, (4) follows in all possible cases. Analogously, $(y, z) \in P(R)$ implies

$$\begin{array}{c}\exists L \in \mathbb{N}, y^0 \in X \text{ and } (T^1, y^1), \ldots, (T^L, y^L) \in \mathcal{A}_C \text{ such that} \\ y = y^0, y^{\ell-1} = f(T^\ell, y^\ell) \text{ and } (y^{\ell-1}, y^\ell) \in R_C \text{ for all } \ell \in \{1, \ldots, L\} \text{ and } y^L = z.\end{array} \tag{10}$$

Letting $M = K + L$, $z^0 = x^0$, $(U^m, z^m) = (S^m, x^m)$ for all $m \in \{1, \ldots, K\}$ and $(U^m, z^m) = (T^{m-K}, y^{m-K})$ for all $m \in \{K+1, \ldots, K+L\}$, (4) and (10) together imply

$$x = z^0, z^{m-1} = f(U^m, z^m) \text{ and } (z^{m-1}, z^m) \in R_C \text{ for all } m \in \{1, \ldots, M\} \text{ and } z^M = z. \tag{11}$$

Therefore, by definition of R, $(x, z) \in R$. Suppose we also have $(z, x) \in R$. This implies

$$(z, x) \in R_C \tag{12}$$

or

$$\begin{gathered}(x, z) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, z) \in \mathcal{A}_C \text{ and } f(S, z) = x \\ \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = z\end{gathered} \tag{13}$$

or

$$\begin{gathered}\exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\ z = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x.\end{gathered} \tag{14}$$

If (12) is true, (11) yields a contradiction to the property **RE**. (13) immediately contradicts (11). Finally, if (14) applies, in combination with (11), it implies a contradiction to the property **RI**. Thus, R is quasi-transitive.

To show that R is a G-rationalization of C, let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. This implies $(x, y) \in R_C \subseteq R$ for all $y \in S$ and, therefore, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Thus, $(S, x) \in \mathcal{A}_C$. Let $y = f(S, x)$ and suppose $(x, y) \in R$.

If $(x, y) \in R_C$, there exists $T \in \Sigma$ such that $y \in T$ and $x \in C(T)$. This contradicts the property **DRE**. If $(x, y) \in R_d$, we obtain a contradiction to the property **SI**. If (3) applies, it follows that there exists no $S \in \Sigma$ such that $(S, x) \in \mathcal{A}_C$ and $y = f(S, x)$, an immediate contradiction to our hypothesis. Finally, if (4) applies, we obtain a contradiction to the property **RI**. Thus, $(x, y) \notin R$ and hence $x \notin G(S, R)$. ■

Because the G-rationalization is not assumed to be complete in the above theorem, the construction of R is more complex—an absence of a weak preference for one of two distinct alternatives no longer implies a strict preference for the other. Thus, the components of R are constructed by including all pairs that are necessarily in this relation and then invoking the properties of f to ensure that R satisfies all of the requirements. In particular, as is the case whenever G-rationalizability is considered, the direct revealed preference relation R_C has to be respected. In addition, reflexivity implies that the diagonal relation R_d is contained in R. To avoid as many potential conflicts with quasi-transitivity as possible, any strict revealed preference is converted into an indifference whenever possible without contradiction. Finally, any chain of strict preference imposed by the conjunction of relationships imposed by f and by the direct revealed preference criterion has to be respected due to the quasi-transitivity requirement.

Our next task is to identify a necessary and sufficient condition for **G-0-Q**.

Theorem 7 *Suppose $C\colon \Sigma \rightarrow \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-0-Q** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE**, **RE** *and* **RI**.

Proof. Suppose C satisfies **G-0-Q** and let R be a quasi-transitive G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$ and consider any $(S, y) \in \mathcal{A}_C$. By definition, $S \in \Sigma$ and $y \in S \setminus C(S)$. The assumption that R is a G-rationalization of C implies the existence of $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that f satisfies **DRE**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To show that f satisfies **RE**, suppose $K \in \mathbb{N}$, $(S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C$, $S^0 \in \Sigma$ and $x^0 \in S^0$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Thus, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and the quasi-transitivity of R implies $(x^0, x^K) \in P(R)$. Therefore, $(x^K, x^0) \notin R$ and, because R is a G-rationalization of C, we obtain $x^K \notin C(S^0)$.

To establish the property **RI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and $(x^K, x^0) \in R_C$. By the definition of f, $(x^k, x^{k-1}) \notin R$ and by Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$. Therefore, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. R being quasi-transitive, it follows that $(x^0, x^K) \in P(R)$ and thus $(x^0, x^K) \in R$, which, by the definition of f, implies $f(S^0, x^0) \neq x^K$.

Now suppose that there exists $f \in \mathcal{F}_C$ satisfying **DRE**, **RE** and **RI**, provided that $\mathcal{A}_C \neq \emptyset$. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a quasi-transitive G-rationalization of C and we are done. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying the properties **DRE**, **RE** and **RI**. Define

$$
\begin{aligned}
R \;=\;& R_C \\
&\cup \{(x, y) \in X \times X \mid (y, x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y \\
&\qquad \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\
&\qquad y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x\} \\
&\cup \{(x, y) \in X \times X \mid \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\
&\qquad x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\}.
\end{aligned}
$$

To prove that R is quasi-transitive, suppose $x, y, z \in X$ are such that $(x, y) \in P(R)$ and $(y, z) \in P(R)$. By definition, $(x, y) \in R$ implies

$$(x, y) \in R_C \tag{15}$$

or

$$
\begin{gathered}
(y, x) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y \\
\text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\
y = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = x
\end{gathered} \tag{16}
$$

or

$$
\begin{gathered}
\exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1, x^1), \ldots, (S^K, x^K) \in \mathcal{A}_C \text{ such that} \\
x = x^0, x^{k-1} = f(S^k, x^k) \text{ and } (x^{k-1}, x^k) \in R_C \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y.
\end{gathered} \tag{17}
$$

Analogously, $(y,x) \notin R$ implies

$$(y,x) \notin R_C \tag{18}$$

and

$$\begin{gathered}(x,y) \notin R_C \text{ or } \exists S \in \Sigma \text{ such that } (S,y) \in \mathcal{A}_C \text{ and } f(S,y) = x \\ \text{or } \exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = y\end{gathered} \tag{19}$$

and

$$\begin{gathered}\nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ y = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = x.\end{gathered} \tag{20}$$

Because (18) must be true, (16) must be false. Therefore, it follows that (15) or (17) is true and that (19) is true. Because (15) and $(x,y) \notin R_C$ are incompatible, it follows that we must have

$$(15) \text{ and } \exists S \in \Sigma \text{ such that } (S,y) \in \mathcal{A}_C \text{ and } f(S,y) = x \tag{21}$$

or

$$(15) \text{ and } (17) \tag{22}$$

or (17). Clearly, (21) implies (17) and (22) implies (17) trivially. Thus, (17) follows in all possible cases. Analogously, $(y,z) \in P(R)$ implies

$$\begin{gathered}\exists L \in \mathbb{N}, y^0 \in X \text{ and } (T^1,y^1),\ldots,(T^L,y^L) \in \mathcal{A}_C \text{ such that} \\ y = y^0, y^{\ell-1} = f(T^\ell,y^\ell) \text{ and } (y^{\ell-1},y^\ell) \in R_C \text{ for all } \ell \in \{1,\ldots,L\} \text{ and } y^L = z.\end{gathered} \tag{23}$$

Letting $M = K + L$, $z^0 = x^0$, $(U^m,z^m) = (S^m,x^m)$ for all $m \in \{1,\ldots,K\}$ and $(U^m,z^m) = (T^{m-K},y^{m-K})$ for all $m \in \{K+1,\ldots,K+L\}$, (17) and (23) together imply

$$x = z^0, z^{m-1} = f(U^m,z^m) \text{ and } (z^{m-1},z^m) \in R_C \text{ for all } m \in \{1,\ldots,M\} \text{ and } z^M = z. \tag{24}$$

Therefore, by the definition of R, $(x,z) \in R$. Suppose we also have $(z,x) \in R$. This implies

$$(z,x) \in R_C \tag{25}$$

or

$$\begin{gathered}(x,z) \in R_C \text{ and } \nexists S \in \Sigma \text{ such that } (S,z) \in \mathcal{A}_C \text{ and } f(S,z) = x \\ \text{and } \nexists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ x = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = z\end{gathered} \tag{26}$$

or

$$\begin{gathered}\exists K \in \mathbb{N}, x^0 \in X \text{ and } (S^1,x^1),\ldots,(S^K,x^K) \in \mathcal{A}_C \text{ such that} \\ z = x^0, x^{k-1} = f(S^k,x^k) \text{ and } (x^{k-1},x^k) \in R_C \text{ for all } k \in \{1,\ldots,K\} \text{ and } x^K = x.\end{gathered} \tag{27}$$

If (25) is true, (24) yields a contradiction to the property **RE**. (26) immediately contradicts (24). Finally, if (27) applies, combining it with (24), we are led to a contradiction to the property **RI**. Thus, R is quasi-transitive.

To show that R is a G-rationalization of C, let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. This implies $(x, y) \in R_C \subseteq R$ for all $y \in S$ and, therefore, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Thus, $(S, x) \in \mathcal{A}_C$. Let $y = f(S, x)$ and suppose $(x, y) \in R$.

If $(x, y) \in R_C$, there exists $T \in \Sigma$ such that $y \in T$ and $x \in C(T)$. This contradicts the property **DRE**. If (16) applies, it follows that there exists no $S \in \Sigma$ such that $(S, x) \in \mathcal{A}_C$ and $y = f(S, x)$, an immediate contradiction to our hypothesis. Finally, if (17) applies, we obtain a contradiction to the property **RI**. Thus, $(x, y) \notin R$ and hence $x \notin G(S, R)$. ■

The construction of R in this theorem is analogous to that of the previous result, except that R_d no longer appears because R is not required to be reflexive.

Our next theorem, which is due to Bossert, Sprumont and Suzumura (2002), characterizes the notions of rationalizability that are equivalent to **G-RC-A**. Again, see the original paper for a proof.

Theorem 8 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-RC-A**, **M-R-S**, **M-0-S**, **M-RC-A**, **M-R-A**, **M-C-A**, **M-0-A** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **II**.

Next, we turn to the rationalizability property **G-C-A**. We obtain the following new characterization result.

Theorem 9 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies* **G-C-A** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **DSI**.

Proof. First, suppose C satisfies **G-C-A** and let R be a complete and P-acyclical G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$. The assumption that R G-rationalizes C implies that, for any pair $(S, y) \in \mathcal{A}_C$, there exists $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$.

To prove that f satisfies **DRE**, suppose $(S, y) \in \mathcal{A}_C$, $T \in \Sigma$ and $x \in T$ are such that $f(S, y) = x$. By the definition of f, we obtain $(y, x) \notin R$. Because R is a G-rationalization of C, it follows that $y \notin C(T)$.

To establish the property **DSI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$ and, furthermore, $x^K \neq x^0$. By the definition of f, it follows that $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$. Because R is complete and $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$ by assumption, it follows that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. If $f(S^0, x^0) = x^K$, it follows that $(x^0, x^K) \notin R$ by definition and, by the assumption $x^K \neq x^0$ and the completeness of R, we obtain $(x^K, x^0) \in P(R)$. If $K = 1$, this contradicts the observation that $(x^K, x^0) \notin R$ which follows from the hypothesis $f(S^1, x^1) = x^0$ and the definition of f. If $K > 1$, we obtain a contradiction to the P-acyclicity of R. Therefore, $f(S^0, x^0) \neq x^K$.

We now prove the if part of the theorem. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a complete P-acyclical G-rationalization of C. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE** and **DSI**. Define

$$R = \{(x, y) \in X \times X \mid \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y\}.$$

We prove that R is complete. By way of contradiction, suppose $x, y \in X$ are such that $x \neq y$, $(x, y) \notin R$ and $(y, x) \notin R$. By the definition of R, this implies that there exist $S, T \in \Sigma$ such that $(S, x), (T, y) \in \mathcal{A}_C$, $f(S, x) = y$ and $f(T, y) = x$. Because $x \neq y$, this contradicts the property **DSI**. Thus, R is complete.

To show that R is P-acyclical, suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ are such that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. By the definition of R, this implies that there exist $S^1, \ldots, S^K \in \Sigma$ such that $(S^k, x^k) \in \mathcal{A}_C$ and $x^{k-1} = f(S^k, x^k)$ for all $k \in \{1, \ldots, K\}$. Moreover, for all $k \in \{1, \ldots, K\}$, there exists no $T^k \in \Sigma$ such that $(T^k, x^{k-1}) \in \mathcal{A}_C$ and $x^k = f(T^k, x^{k-1})$. This implies $x^{k-1} \neq x^k$ for all $k \in \{1, \ldots, K\}$. If $(x^K, x^0) \in P(R)$, there exists $S^0 \in \Sigma$ such that $(S^0, x^0) \in \mathcal{A}_C$ and $x^K = f(S^0, x^0)$. Furthermore, there exists no $T^0 \in \Sigma$ such that $(T^0, x^K) \in \mathcal{A}_C$ and $x^0 = f(T^0, x^K)$. This implies $x^0 \neq x^K$ and we obtain a contradiction to the property **DSI** and, thus, $(x^K, x^0) \notin P(R)$ and R is P-acyclical.

It remains to be shown that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. If there exist $y \in S$ and $T \in \Sigma$ such that $(T, x) \in \mathcal{A}_C$ and $f(T, x) = y$, we obtain a contradiction to the property **DRE**. Thus, by definition, $(x, y) \in R$ for all $y \in S$ and hence $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Let $y = f(S, x)$. By the definition of R, this implies $(x, y) \notin R$ and, therefore, $x \notin G(S, R)$. ■

The intuition underlying the definition of R in this result is quite straightforward. If $x = f(S, y)$, it follows that y cannot be at least as good as x and, because of completeness, this means that x must be better than y whenever $x \neq y$. Note that f is merely required to satisfy **DSI** rather than **II** (which is required in the previous theorem) and, therefore, R is not necessarily reflexive. The resulting relation has all the required properties as a consequence of the properties of f.

To characterize the rationalizability properties **G-R-A** and **G-0-A**, we replace the property **DSI** with the property **RI** in the axiom employed in the previous theorem. Again, the result is new.

Theorem 10 *Suppose $C \colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-R-A**, **G-0-A** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **RI**.

Proof. By Theorem 3, it is sufficient to treat the case of **G-0-A**.

Suppose C satisfies **G-0-A** and let R be a P-acyclical G-rationalization of C. Suppose $\mathcal{A}_C \neq \emptyset$. The assumption that R G-rationalizes C implies that, for any pair $(S, y) \in \mathcal{A}_C$, there exists $x \in S$ such that $(y, x) \notin R$. Define $f(S, y) = x$. That f satisfies **DRE** follows as in the previous theorem.

To establish the property **RI**, suppose $K \in \mathbb{N}$ and $(S^0, x^0), \ldots, (S^K, x^K) \in \mathcal{A}_C$ are such that $f(S^k, x^k) = x^{k-1}$ and $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and, moreover, $(x^K, x^0) \in R_C$. By the definition of f, we obtain $(x^k, x^{k-1}) \notin R$ for all $k \in \{1, \ldots, K\}$. By Theorem 2, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ and, thus, $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. If $f(S^0, x^0) = x^K$, it follows that $(x^0, x^K) \notin R$ by definition. Because $(x^K, x^0) \in R_C$ implies $(x^K, x^0) \in R$ by Theorem 2, we obtain $(x^K, x^0) \in P(R)$. If $K = 1$, this contradicts the observation that $(x^K, x^0) \notin R$, which follows from the hypothesis $f(S^1, x^1) = x^0$ and the definition of f. If $K > 1$, we obtain a contradiction to the P-acyclicity of R. Therefore, $f(S^0, x^0) \neq x^K$ and **RI** is satisfied.

We now prove the if part of the theorem. If $\mathcal{A}_C = \emptyset$, $R = X \times X$ is a P-acyclical G-rationalization of C. If $\mathcal{A}_C \neq \emptyset$, there exists a function $f \in \mathcal{F}_C$ satisfying **DRE** and **RI**. Define

$$\begin{aligned} R \;=\; & R_C \cup \{(x, y) \in X \times X \mid (y, x) \in R_C \text{ and} \\ & \quad \nexists S \in \Sigma \text{ such that } (S, x) \in \mathcal{A}_C \text{ and } f(S, x) = y\}. \end{aligned}$$

To demonstrate that R is P-acyclical, suppose $K \in \mathbb{N} \setminus \{1\}$ and $x^0, \ldots, x^K \in X$ are such that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$. Consider any $k \in \{1, \ldots, K\}$. By definition,

$$\begin{aligned} \{(x^{k-1}, x^k) \in R_C \quad \text{or} \quad & [(x^k, x^{k-1}) \in R_C \text{ and } \nexists T^k \in \Sigma \text{ such that} \\ & (T^k, x^{k-1}) \in \mathcal{A}_C \text{ and } f(T^k, x^{k-1}) = x^k]\} \end{aligned}$$

and

$$\begin{aligned} (x^k, x^{k-1}) \notin R_C \quad \text{and} \quad & [(x^{k-1}, x^k) \notin R_C \text{ or } \exists S^k \in \Sigma \text{ such that} \\ & (S^k, x^k) \in \mathcal{A}_C \text{ and } f(S^k, x^k) = x^{k-1}]. \end{aligned}$$

Because $(x^k, x^{k-1}) \notin R_C$ must be true,

$$(x^k, x^{k-1}) \in R_C \text{ and } \nexists T^k \in \Sigma \text{ such that } (T^k, x^{k-1}) \in \mathcal{A}_C \text{ and } f(T^k, x^{k-1}) = x^k$$

cannot be true. Therefore, $(x^{k-1}, x^k) \in R_C$ must be true, which, in turn, implies that $(x^{k-1}, x^k) \notin R_C$ cannot be true. Therefore, it follows that

$$(x^{k-1}, x^k) \in R_C \text{ and } \exists S^k \in \Sigma \text{ such that } (S^k, x^k) \in \mathcal{A}_C \text{ and } f(S^k, x^k) = x^{k-1}.$$

Using the same argument, it follows that $(x^K, x^0) \in P(R)$ implies that $(x^K, x^0) \in R_C$ and there exists $S^0 \in \Sigma$ such that $(S^0, x^0) \in \mathcal{A}_C$ and $f(S^0, x^0) = x^K$. This contradicts the property **RI** and, thus, $(x^K, x^0) \notin P(R)$ and R is P-acyclical.

We complete the proof by showing that R is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose $x \in C(S)$. This implies $(x, y) \in R_C$ and, by definition of R, $(x, y) \in R$ for all $y \in S$. Hence, $x \in G(S, R)$.

Now suppose $x \notin C(S)$. Thus, $(S, x) \in \mathcal{A}_C$. Let $y = f(S, x)$. If $(x, y) \in R_C$, there exists $T \in \Sigma$ such that $y \in T$ and $x \in C(T)$. Because $y \in S$, this contradicts the property **DRE**. Therefore, $(x, y) \notin R_C$ and, together with the observations $(S, x) \in \mathcal{A}_C$ and $y = f(S, x)$, it follows that $(x, y) \notin R$ and hence $x \notin G(S, R)$. ■

As usual, any G-rationalization R has to respect the direct revealed preference relation R_C. Furthermore, the construction of the above theorem converts all strict direct revealed preferences into indifferences whenever this is possible without conflicting with the interpretation of the function f. This is done to reduce the potential for conflicts with P-acyclicity as much as possible. That the resulting relation satisfies the required properties follows again from the properties of f.

Finally, we provide a characterization of the rationalizability requirement **G-RC-0** and the properties that are equivalent to it according to Theorem 3. The result is due to (and proven in) Bossert, Sprumont and Suzumura (2002).

Theorem 11 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with an arbitrary non-empty domain $\Sigma \subseteq \mathcal{X}$. C satisfies any of* **G-RC-0**, **G-C-0**, **M-RC-0**, **M-R-0**, **M-C-0**, **M-0-0** *if and only if, whenever $\mathcal{A}_C \neq \emptyset$, there exists $f \in \mathcal{F}_C$ satisfying* **DRE** *and* **DRI**.

For the sake of easy reference, the characterization theorems of this section are summarized in Table 1. Each row corresponds to a rationalizability property and each column except for the last (which identifies the relevant theorem) represents a property of a function f as defined earlier. An asterisk in a cell means that the corresponding property of f is used in the characterization of the corresponding rationalizability requirement.

Table 1: Quasi-Transitive and P-Acyclical Rationalizability

	SE	**DRE**	**RE**	**DSE**	**IE**	**SI**	**DRI**	**RI**	**DSI**	**II**	**Theorem**
G-RC-Q					*					*	**4**
G-C-Q	*			*					*		**5**
G-R-Q		*	*			*		*			**6**
G-0-Q		*	*					*			**7**
G-RC-A		*								*	**8**
G-C-A		*							*		**9**
G-R-A		*						*			**10**
G-RC-0		*					*				**11**

5 Domain-Closedness Properties

The theory of rationalizable choice functions on the arbitrary domains is general enough to be applicable to whatever choice contexts we may care to specify. However, this generality is secured at the cost of abstracting away whatever structural conditions that may be imposed on the choice domain if we pay due attention to the specific choice context we are focusing on.

An important class of domain restrictions that has been explored quite extensively in the literature on rational choice is based on set-theoretic *closedness* properties of the domain Σ. The two properties that have received the most attention are *closedness under union* and *closedness under intersection.* Closedness under union requires that, for any collection of sets in $\mathcal{X}$ each of which belongs to Σ, their union is a member of Σ as well. Analogously, closedness under intersection demands that, whenever the members of a collection of sets in $\mathcal{X}$ belong to Σ and their intersection is non-empty, then this intersection is an element of Σ as well.

These domain restrictions are of importance because they are plausible for several economic choice models. For example, closedness under intersection is satisfied by the domain consisting of all compact, convex and comprehensive subsets of $\mathbb{R}^n_+$, the non-negative orthant of Euclidean n-space for $n \in \mathbb{N}$. This domain is relevant in axiomatic models of bargaining where the elements of $\mathbb{R}^n_+$ are interpreted as possible utility or payoff distributions among $n \in \mathbb{N}$ agents that can be achieved through a bargaining process, and the origin represents the outcome that results if the agents fail to reach an agreement—the disagreement outcome. Feasible sets of utility distributions are usually assumed to be compact, convex and comprehensive. Compactness is a standard regularity condition in economic models, convexity accommodates the possibility of considering probability distributions in addition to certain outcomes, and comprehensiveness formalizes the commonly-imposed free-disposal assumption.

Rationalizability plays an important role in the literature on axiomatic approaches dealing with the bargaining problem. The most important bargaining solution is the Nash (1950) solution. It selects, for each compact, convex and comprehensive subset of $\mathbb{R}^n_+$, the unique utility distribution that maximizes the product of the agents' utilities. Clearly, by its very definition, this solution is rationalizable: it selects the best elements according to the ordering on $X = \mathbb{R}^n_+$ that ranks utility vectors on the basis of the product of their components. More general rationalizable solutions to the bargaining problem are studied in Lensberg (1987; 1988), Thomson and Lensberg (1989), Peters and Wakker (1991), Bossert (1994) and Sanchez (2000); see also Thomson (2005) for a survey of axiomatic models of bargaining.

Closedness under union is satisfied, for example, by the domain that consists of all compact and comprehensive (but not necessarily convex) subsets of $\mathbb{R}^n_+$ (or subsets of $\mathbb{R}^n$ with suitably modified definitions of comprehensiveness or a weakening of compactness to make the two properties compatible). A prominent example of an application of this domain is Donaldson and Weymark (1988) who study rationalizable solutions to social choice problems in economic environments.

Up to now, the application of these types of domain restrictions has been limited to the study of full rationalizability—rationalizability by an ordering (recollect that in the case of an ordering, it is not necessary to specify whether greatest-element rationalizability or maximal-element rationalizability is considered because the two notions coincide). The usefulness of these domain assumptions lies in the fact that they allow us to work with properties that are weaker (and, thus, easier to justify) than others on arbitrary domains but turn out to be equivalent on suitably specified domains. In particular, the assumption of closedness under union implies that rationalizability by an ordering can be obtained as a consequence of a property that is, on arbitrary domains, considerably weaker than the requisite necessary and sufficient condition for that type of rationalizability. This property is *Arrow's choice axiom;* see Arrow (1959). Furthermore, while closedness under intersection is not sufficient to obtain the equivalence of

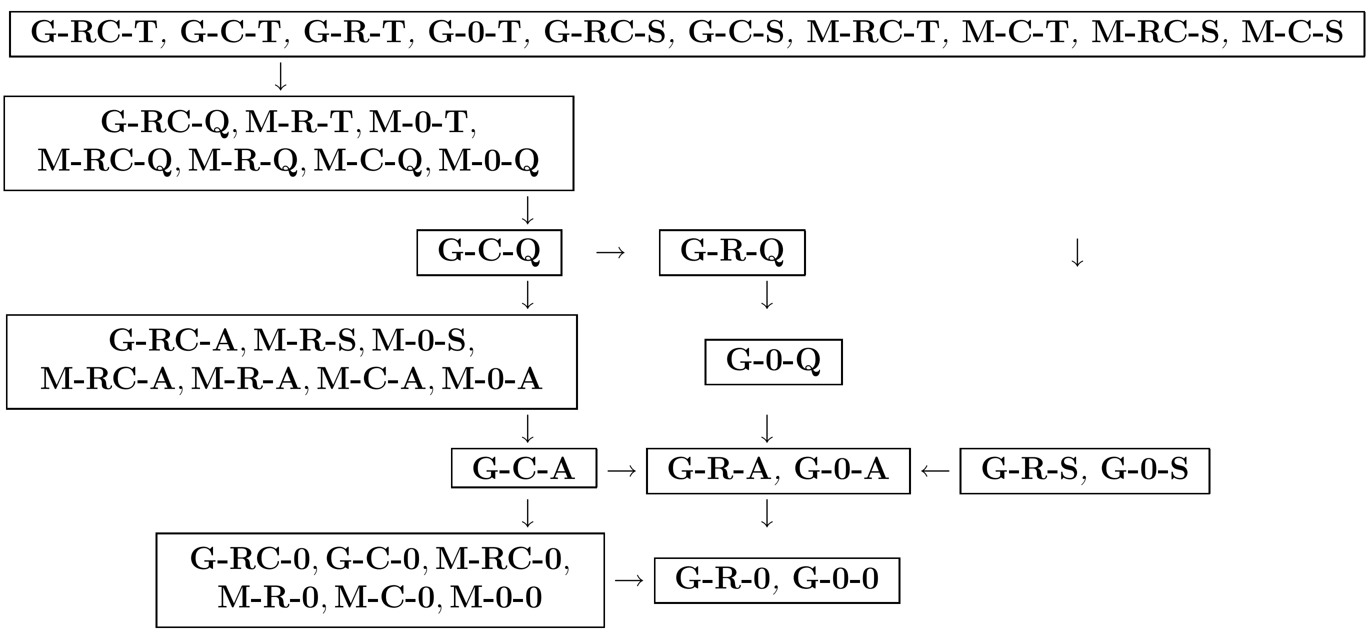
Arrow's choice axiom and full rationalizability, it does imply that Arrow's choice axiom has some additional strength as compared to its power on arbitrary domains.

The purpose of this section is to examine whether new logical relationships, in addition to those summarized in Section 3, emerge under one or the other of these closedness properties. It turns out that closedness under union leads to an additional implication, namely, that **G-C-Q** implies **G-RC-A**. It is worth mentioning that, unlike most of the earlier literature, we employ a closedness property with respect to set-theoretic unions that is not restricted to *finite* unions. The reason is that the argument employed to establish the above-mentioned new implication does not go through if attention is restricted to finite unions; this will become clear in the proof (which relies on the axiom of choice). In contrast, closedness under intersection does not give us any additional results: the same logical relationships as those of Section 3 are valid.

We now introduce the formal definitions of our closedness assumptions. A non-empty domain $\Sigma \subseteq \mathcal{X}$ is *closed under union* if and only if, for all non-empty collections $\mathcal{S}$ of elements of Σ, $\cup_{S \in \mathcal{S}} S \in \Sigma$. Analogously, Σ is *closed under intersection* if and only if, for all non-empty collections $\mathcal{S}$ of elements of Σ, if $\cap_{S \in \mathcal{S}} S \neq \emptyset$, then $\cap_{S \in \mathcal{S}} S \in \Sigma$.

Requiring Σ to be closed under union produces a new implication. In particular, it is now the case that **G-C-Q** implies **G-RC-A** so that we obtain the following theorem.

Theorem 12 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a choice function with a non-empty domain $\Sigma \subseteq \mathcal{X}$ which is closed under union. Then*



Proof. First, note that all the implications established in Theorem 3 remain valid because they apply to any domain (and, thus, in particular, to domains that are closed under union). Therefore, it remains to show that, provided that Σ is closed under union, **G-C-Q** implies **G-RC-A** and that no implications other than those resulting from the theorem statement are valid.

(a) Suppose Σ is closed under union, and let R be a complete and quasi-transitive G-rationalization of C. Define

$$\mathcal{N} = \{(x, S) \mid (x, x) \notin R \text{ and } S \in \Sigma \text{ and } (x, y) \in R \text{ for all } y \in S \setminus \{x\}\}.$$

If $\mathcal{N} = \emptyset$, it follows immediately that $R \cup R_d$ is a reflexive, complete and quasi-transitive (and thus P-acyclical) G-rationalization of C and we are done.

Now suppose $\mathcal{N} \neq \emptyset$. Let $\mathcal{Z}$ denote the set of all non-empty subsets of $X \times \Sigma$, and define a function $\tau \colon \Sigma \to \mathcal{Z}$ by $\tau(S) = \{(z, S) \mid z \in C(S)\}$ for all $S \in \Sigma$. Furthermore, let

$$\mathcal{T} = \{\tau(S) \mid S \in \Sigma\}.$$

Let Φ be the set of all functions $\varphi \colon \mathcal{T} \to \cup_{T \in \mathcal{T}} T$ such that $\varphi(T) \in T$ for all $T \in \mathcal{T}$. By the axiom of choice, $\Phi \neq \emptyset$.

For all $\varphi \in \Phi$, define a function $g_\varphi \colon \mathcal{N} \to X$ by

$$g_\varphi(x, S) = z \;\Leftrightarrow\; \varphi(\tau(S)) = (z, S)$$

for all $(x, S) \in \mathcal{N}$. Let

$$\mathcal{G} = \{g \colon \mathcal{N} \to X \mid \text{ there exists } \varphi \in \Phi \text{ such that } g = g_\varphi\}.$$

The set $\mathcal{G}$ is non-empty because Φ is non-empty.

For all $g \in \mathcal{G}$, let

$$R_g = R \cup R_d \setminus \{(x, y) \mid \text{ there exists } S \in \Sigma \text{ such that } (x, S) \in \mathcal{N} \text{ and } y = g(x, S)\}.$$

Clearly, R_g is reflexive because $R_d \subseteq R$.

By the definition of $\mathcal{N}$ and $\mathcal{G}$, $(x, y) \in I(R)$ for all pairs (x, y) such that there exists $S \in \Sigma$ with $(x, S) \in \mathcal{N}$ and $y = g(x, S)$. Thus, $(x, y) \in P(R_g)$ for all such pairs which, together with the completeness of R, implies that R_g is complete. This argument also establishes that $P(R) \subseteq P(R_g)$.

Next, we show that R_g is a G-rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $x \in C(S)$. Because R is a G-rationalization of C, $(x, y) \in R$ for all $y \in S$ and in particular $(x, x) \in R$. By the definition of $\mathcal{N}$, this implies $(x, S) \notin \mathcal{N}$ which entails $(x, y) \in R_g$ for all $y \in S$. Thus, $x \in G(R_g, S)$.

Now suppose $x \in G(R_g, S)$. If there exists $y \in S \setminus \{x\}$ such that $(x, y) \notin R$, the completeness of R implies that $(y, x) \in P(R) \subseteq P(R_g)$, contradicting the hypothesis $x \in G(R_g, S)$. If $(x, x) \notin R$, the definition of R_g implies that $(x, g(x, S)) \notin R_g$, again contradicting the hypothesis $x \in G(R_g, S)$ because, by definition, $g(x, S) \in S$. Thus, $(x, y) \in G(R, S)$ and $x \in C(S)$ follows from the assumption that R G-rationalizes C.

We have now established that R_g is a reflexive and complete G-rationalization of C for all $g \in \mathcal{G}$. To complete the proof, we show that there must exist a $g \in \mathcal{G}$ such that R_g is P-acyclical.

A P-cycle in R_g can be written as a pair $c = \left(K_c, \{x_c^0, \ldots, x_c^{K_c}\}\right)$ where $K_c \in \mathbb{N} \setminus \{1\}$ and $x_c^k \in X$ for all $k \in \{0, \ldots, K_c\}$ such that $(x_c^{k-1}, x_c^k) \in P(R_g)$ for all $k \in \{1, \ldots, K_c\}$ and $(x_c^{K_c}, x_c^0) \in P(R_g)$. Without loss of generality, we can assume that the x_c^k are pairwise distinct.

Let $\mathcal{C}_g$ be the set of all P-cycles in R_g. We need to show that there exists $g \in \mathcal{G}$ such that $\mathcal{C}_g = \emptyset$. By way of contradiction, suppose that $\mathcal{C}_g \neq \emptyset$ for all $g \in \mathcal{G}$. Consider any $g \in \mathcal{G}$ and any $c \in \mathcal{C}_g$.

If there exists $k \in \{1, \ldots, K_c - 1\}$ such that $(x_c^{k-1}, x_c^k) \in P(R)$ and $(x_c^k, x_c^{k+1}) \in P(R)$, the quasi-transitivity of R implies $(x_c^{k-1}, x_c^{k+1}) \in P(R)$. The same reasoning implies that if $(x_c^{K_c-1}, x_c^{K_c}) \in P(R)$ and $(x_c^{K_c}, x_c^0) \in P(R)$, we must have $(x_c^{K_c-1}, x_c^0) \in P(R)$. Thus, because K_c is finite, we can without loss of generality assume that, in any P-cycle, there are never two consecutive instances of strict preference according to $P(R)$.

If there exists $k \in \{1, \ldots, K_c - 1\}$ such that $(x_c^{k-1}, x_c^k) \in P(R_g) \setminus P(R)$ and $(x_c^k, x_c^{k+1}) \in P(R_g) \setminus P(R)$, the definition of R_g implies that there exist $S, T \in \Sigma$ such that $(x_c^k, S) \in \mathcal{N}$, $x_c^{k-1} = g(x_c^k, S)$, $(x_c^{k+1}, T) \in \mathcal{N}$ and $x_c^k = g(x_c^{k+1}, T)$. By definition of $\mathcal{N}$ and g, this implies $(x_c^k, x_c^k) \notin R$ and $x_c^k \in C(T)$, a contradiction to the assumption that R is a G-rationalization of C. Thus, there cannot be two consecutive instances of strict preference according to $P(R_g) \setminus P(R)$ either.

Because R is quasi-transitive and therefore P-acyclical, every P-cycle must contain at least one instance of strict preference according to $P(R_g) \setminus P(R)$. Because there cannot be any two consecutive instances of strict preference according to $P(R_g) \setminus P(R)$ and $K_c \geq 2$ for any P-cycle c by assumption, it follows that every P-cycle must also contain at least one instance of strict preference according to $P(R)$.

Combining the findings of the previous three paragraphs, it follows that we can, without loss of generality, assume that all P-cycles alternate between instances of strict preference according to $P(R_g) \setminus P(R)$ and those according to $P(R)$. Thus, for all $g \in \mathcal{G}$ and for all P-cycles $c = \big(K_c, \{x_c^0, \ldots, x_c^{K_c}\}\big) \in \mathcal{C}_g$, we assume that $(x_c^{k-1}, x_c^k) \in P(R_g) \setminus P(R)$ for all odd $k \in \{1, \ldots, K_c\}$, $(x_c^{k-1}, x_c^k) \in P(R)$ for all even $k \in \{1, \ldots, K_c\}$, and $(x_c^{K_c}, x_c^0) \in P(R)$. Note that this implies that K_c must be an odd number greater than or equal to three. Furthermore, for all odd $k \in \{1, \ldots, K_c\}$, there exists $S_c^k \in \Sigma$ such that $(x_c^k, S_c^k) \in \mathcal{N}$ and $x_c^{k-1} = g(x_c^k, S_c^k)$.

Let Υ be the set of all alternatives in $C(\Sigma)$ that do not appear in any P-cycle. That is, $x \in C(\Sigma)$ is an element of Υ if and only if, for all $g \in \mathcal{G}$, for all $c \in \mathcal{C}_g$ and for all odd $k \in \{1, \ldots, K_c\}$, $x_c^{k-1} \neq x$. Now define $\Sigma' = \{S \in \Sigma \mid C(S) \cap \Upsilon = \emptyset\}$, that is, Σ' is composed of those sets S in Σ such that all elements of $C(S)$ appear in some P-cycle. Note that $\Sigma' = \Sigma$ is possible. Let, for all sets $S \in \Sigma \setminus \Sigma'$, y_S be an alternative in $C(S)$ that does not appear in any P-cycle. Let $\mathcal{G}'$ be the subset of $\mathcal{G}$ containing all functions g such that, for all $(x, S) \in \mathcal{N}$ with $S \in \Sigma \setminus \Sigma'$, $g(x, S) = y_S$. Clearly, $\mathcal{G} \neq \emptyset$. If $\Sigma' = \emptyset$, any $g \in \mathcal{G}'$ is such that $\mathcal{C}_g = \emptyset$, a contradiction. Therefore, $\Sigma' \neq \emptyset$. Let

$$T = \bigcup_{g \in \mathcal{G}'} \bigcup_{c \in \mathcal{C}_g} \bigcup_{k \in \{1, \ldots, K_c\}} S_c^k. \tag{28}$$

Because $\mathcal{G}'$ is non-empty and $\mathcal{C}_g$ is non-empty for all $g \in \mathcal{G}'$, $T \neq \emptyset$. Because Σ is closed under union, $T \in \Sigma$. By definition, all sets S_c^k in (28) are elements of Σ'. Because C is G-rationalizable by R, it follows that, for all $y \in S_c^k \setminus C(S_c^k)$, there exists $x \in S_c^k \subseteq T$ such that $(y, x) \notin R$. Again invoking the G-rationalizability of C by R, this implies that

$$C(T) = \bigcup_{g \in \mathcal{G}'} \bigcup_{c \in \mathcal{C}_g} \bigcup_{k \in \{1, \ldots, K_c\}} C(S_c^k). \tag{29}$$

Consider any $g \in \mathcal{G}'$, $c \in \mathcal{C}_g$ and $k \in \{1, \ldots, K_c\}$, and let $x \in C(S_c^k)$. Because $S_c^k \in \Sigma'$, there exists a P-cycle containing x. Thus, there exists $g' \in \mathcal{G}'$, $c' \in \mathcal{C}_g$ and $y \in X$ such that $(x, y) \in P(R_g) \setminus P(R)$. Furthermore, there exist $z, w \in X$ such that $(z, w) \in P(R_g) \setminus P(R)$ and $(w, x) \in P(R)$. By definition, there exists $S \in \Sigma'$ such that $w \in S \subseteq T$ and, because R is a G-rationalization of C, $x \notin C(T)$. This argument applies to all elements in the union on the right side of (29) and, therefore, $C(T) = \emptyset$, a contradiction.

To prove that no further implications other than those resulting from the arrows depicted in the theorem statement are valid, it is sufficient to provide examples showing that (b) **G-RC-Q** does not imply **G-0-S**; (c) **G-0-S** does not imply **G-0-Q**; (d) **G-0-S** does not imply **G-C-0**; (e) **G-RC-A** does not imply **G-0-Q**; (f) **G-C-A** does not imply **G-RC-A**; (g) **G-R-Q** does not imply **G-C-0**; (h) **G-0-Q** does not imply **G-R-Q**; and (i) **G-C-O** does not imply **G-0-A**. All of these examples are such that Σ is closed under union and under intersection.

(b) To show that **G-RC-Q** does not imply **G-0-S**, consider the following example. Let $X = \{x, y, z\}$ and $\Sigma = \mathcal{X}$. Define the choice function C by letting $C(\{x\}) = \{x\}$, $C(\{x, y\}) = \{x\}$, $C(X) = \{x, z\}$, $C(\{x, z\}) = \{x, z\}$, $C(\{y\}) = \{y\}$, $C(\{y, z\}) = \{y, z\}$ and $C(\{z\}) = \{z\}$. This choice function is G-rationalizable by the reflexive, complete and quasi-transitive relation

$$R = \{(x, x), (x, y), (x, z), (y, y), (y, z), (z, x), (z, y), (z, z)\}.$$

Suppose R' is a G-rationalization of C. Because $y \in C(\{y, z\})$, we have $(y, y) \in R'$. Therefore, $y \notin C(\{x, y\})$ implies $(x, y) \in P(R')$. Because $y \in C(\{y, z\})$, we have $(y, z) \in R'$ and, analogously, because $z \in C(\{x, z\})$, we have $(z, x) \in R'$. This implies that R' cannot be consistent.

(c) We now prove that **G-0-S** does not imply **G-0-Q**. Let $X = \{x, y, z, w\}$ and $\Sigma = \{X, \{x, y, w\}, \{y\}, \{y, z\}, \{z\}\}$. Define the choice function C by letting $C(X) = \{w\}$, $C(\{x, y, w\}) = \{x, w\}$, $C(\{y\}) = \{y\}$, $C(\{y, z\}) = \{y\}$ and $C(\{z\}) = \{z\}$. This choice function is G-rationalizable by the consistent relation

$$R = \{(x, x), (x, y), (x, w), (y, y), (y, z), (z, z), (w, x), (w, y), (w, z), (w, w)\}.$$

Suppose C is G-rationalizable by a quasi-transitive relation R'. Because $y \in C(\{y, z\})$, we have $(y, y) \in R'$. Therefore, $y \notin C(\{x, y\})$ implies $(x, y) \in P(R')$. Analogously, $z \in C(\{z\})$ implies $(z, z) \in R'$ and, therefore, $z \notin C(\{y, z\})$ implies $(y, z) \in P(R')$. Because R' is quasi-transitive, it follows that $(x, z) \in P(R')$ and hence $(x, z) \in R'$. Furthermore, because R' is a G-rationalization of C and $x \in C(\{x, y, w\})$, we must have $(x, x) \in R'$, $(x, y) \in R'$ and $(x, w) \in R'$. Thus, $x \in G(X, R')$ and, because R' is a G-rationalization of C, it follows that $x \in C(X)$, contradicting the definition of C.

(d) Next, we prove that **G-0-S** does not imply **G-C-0**. Let $X = \{x, y, z\}$ and $\Sigma = \{\{x\}, \{x, y\}, X, \{x, z\}\}$. Define the choice function C by letting $C(\{x\}) = \{x\}$, $C(\{x, y\}) = \{x, y\}$, $C(X) = \{x\}$ and $C(\{x, z\}) = \{x, z\}$. C is G-rationalizable by the consistent relation

$$R = \{(x, x), (x, y), (x, z), (y, x), (y, y), (z, x), (z, z)\},$$

but it does not have a complete G-rationalization. By way of contradiction, suppose R' is such a relation. By completeness, we must have $(y, z) \in R'$ or $(z, y) \in R'$. Suppose first that $(y, z) \in R'$. Because R' is a G-rationalization of C and $y \in C(\{x, y\})$, it follows that $(y, x) \in R'$

and $(y,y) \in R'$. Together with $(y,z) \in R'$ and the assumption that R' is a G-rationalization of C, we obtain $y \in C(X)$, contradicting the definition of C. Now suppose $(z,y) \in R'$. Because R' is a G-rationalization of C and $z \in C(\{x,z\})$, it follows that $(z,x) \in R'$ and $(z,z) \in R'$. Together with $(z,y) \in R'$ and the assumption that R' is a G-rationalization of C, we obtain $z \in C(X)$, again contradicting the definition of C.

(e) To prove that **G-RC-A** does not imply **G-0-Q**, let $X = \{x,y,z\}$ and $\Sigma = \mathcal{X}$. Define the choice function C by $C(\{x\}) = \{x\}$, $C(\{x,y\}) = \{x\}$, $C(X) = \{x\}$, $C(\{x,z\}) = \{x,z\}$, $C(\{y\}) = \{y\}$, $C(\{y,z\}) = \{y\}$ and $C(\{z\}) = \{z\}$. C is G-rationalizable by the reflexive, complete and P-acyclical relation

$$R = \{(x,x),(x,y),(x,z),(y,y),(y,z),(z,x),(z,z)\}.$$

By way of contradiction, suppose R' is a quasi-transitive G-rationalization of C. Because $y \in C(\{y,z\})$, $z \in C(\{z\})$ and $z \notin C(\{y,z\})$, the assumption that R' is a G-rationalization of C implies $(y,z) \in P(R')$. Furthermore, because $x \in C(\{x,y\})$ and $y \notin C(\{x,y\})$, we obtain $(x,y) \in P(R')$. Because R' is quasi-transitive, it follows that $(x,z) \in P(R')$. This, in turn, implies $(z,x) \notin R'$ and, because R' is a G-rationalization of C, $z \notin C(\{x,z\})$, contradicting the definition of C.

(f) To show that **G-C-A** is not sufficient to imply **G-RC-A**, let $X = \{x,y,z,w\}$ and $\Sigma = \{X,\{x,y,w\},\{x,w\},\{y\},\{y,z\},\{y,z,w\},\{y,w\},\{w\}\}$. Define a choice function C by letting $C(X) = \{w\}$, $C(\{x,y,w\}) = \{x,w\}$, $C(\{x,w\}) = \{x,w\}$, $C(\{y\}) = \{y\}$, $C(\{y,z\}) = \{y\}$, $C(\{y,z,w\}) = \{y,w\}$, $C(\{y,w\}) = \{y,w\}$ and $C(\{w\}) = \{w\}$. This choice function is G-rationalizable by the complete and P-acyclical relation

$$R = \{(x,x),(x,y),(x,w),(y,y),(y,z),(y,w),(z,x),(z,y),(w,x),(w,y),(w,z),(w,w)\}.$$

Suppose R' is a reflexive, complete and P-acyclical G-rationalization of C. Reflexivity implies that we must have $(z,z) \in R'$. This, in turn, implies $(y,z) \in P(R')$ because $C(\{y,z\}) = \{y\}$. Furthermore, because $C(\{y,w\}) = \{y,w\}$, we must have $(y,y) \in R'$ and $(y,w) \in R'$. Thus, because $y \notin C(\{x,y,w\})$, it must be the case that $(y,x) \notin R'$ and, because R' is complete, $(x,y) \in P(R')$. Because $x \in C(\{x,y,w\})$, it follows that $(x,x) \in R'$, $(x,y) \in R'$ and $(x,w) \in R'$. Thus, $x \notin C(X)$ implies $(x,z) \notin R'$. The completeness of R' implies $(z,x) \in P(R')$, contradicting the P-acyclicity of R'.

(g) To see that **G-R-Q** does not imply **G-C-0**, consider the following example. Let $X = \{x,y,z\}$, $\Sigma = \{\{x\},\{x,y\},X,\{x,z\}\}$, $C(\{x\}) = \{x\}$, $C(\{x,y\}) = \{x,y\}$, $C(X) = \{x\}$ and $C(\{x,z\}) = \{x,z\}$. C is G-rationalizable by the reflexive and quasi-transitive relation

$$R = \{(x,x),(x,y),(x,z),(y,x),(y,y),(z,x),(z,z)\},$$

but it does not have a complete G-rationalization. By way of contradiction, suppose R' is a complete G-rationalization of C. Completeness implies that we must have $(y,z) \in R'$ or $(z,y) \in R'$. Suppose first that $(y,z) \in R'$ is true. Because R' is a G-rationalization of C and $y \in C(\{x,y\})$, it follows that $(y,x) \in R'$ and $(y,y) \in R'$. Together with $(y,z) \in R'$ and the definition of G-rationalizability, we obtain $y \in C(X)$, contradicting the definition of C. Now suppose $(z,y) \in R'$. Because R' is a G-rationalization of C and $z \in C(\{x,z\})$, it follows that

$(z, x) \in R'$ and $(z, z) \in R'$. Together with $(z, y) \in R'$ and the definition of G-rationalizability, we obtain $z \in C(X)$, again contradicting the definition of C.

(h) To show that **G-0-Q** does not imply **G-R-Q**, consider the following example. Let $X = \{x, y, z, w\}$ and $\Sigma = \{\{x, y\}, X, \{x, y, w\}, \{y\}, \{y, z, w\}, \{y, w\}\}$, and define the choice function C by letting $C(\{x, y\}) = \{y\}$, $C(X) = \{w\}$, $C(\{x, y, w\}) = \{y, w\}$, $C(\{y\}) = \{y\}$, $C(\{y, z, w\}) = \{z, w\}$ and $C(\{y, w\}) = \{y, w\}$. This choice function is G-rationalizable by the quasi-transitive relation

$$R = \{(x, y), (x, w), (y, x), (y, y), (y, w), (z, y), (z, z), (z, w), (w, x), (w, y), (w, z), (w, w)\}.$$

Suppose R' is a reflexive and quasi-transitive G-rationalization of C. By reflexivity, $(x, x) \in R'$ and, because $x \notin C(\{x, y\})$ and $y \in C(\{x, y\})$, we must have $(y, x) \in P(R')$ and $(y, y) \in R'$. Because $y \in C(\{x, y, w\})$, we have $(y, w) \in R'$. Hence, $y \notin C(\{y, z, w\})$ implies $(y, z) \notin R'$ because R' is a G-rationalization of C. Because $z \in C(\{y, z, w\})$, the assumption that R' is a G-rationalization of C implies $(z, y) \in R'$ and, thus, $(z, y) \in P(R')$. R' being quasi-transitive, we obtain $(z, x) \in P(R')$. Because $z \in C(\{y, z, w\})$, it follows that $(z, y) \in R'$, $(z, z) \in R'$ and $(z, w) \in R'$. Together with $(z, x) \in P(R') \subseteq R'$ and the assumption that R' is a G-rationalization of C, we obtain $z \in C(X)$, which contradicts the definition of C.

(i) We now show that **G-C-0** does not imply **G-0-A**. Let $X = \{x, y, z, w\}$ and $\Sigma = \{X, \{x, y, w\}, \{x, z, w\}, \{x, w\}, \{y, z, w\}, \{y, w\}, \{z, w\}, \{w\}\}$. Define the choice function C by letting $C(X) = \{w\}$, $C(\{x, y, w\}) = \{x, w\}$, $C(\{x, z, w\}) = \{z, w\}$, $C(\{x, w\}) = \{x, w\}$, $C(\{y, z, w\}) = \{y, w\}$, $C(\{y, w\}) = \{y, w\}$, $C(\{z, w\}) = \{z, w\}$ and $C(\{w\}) = \{w\}$. C is G-rationalizable by the complete relation R given by

$$\{(x, x), (x, y), (x, w), (y, y), (y, z), (y, w), (z, x), (z, z), (z, w), (w, x), (w, y), (w, z), (w, w)\},$$

but it does not have a P-acyclical G-rationalization. Suppose R' is a G-rationalization of C. Because $C(\Sigma) = X$ and R' G-rationalizes C, it follows that $R_d \subseteq R'$. Because $y \notin C(\{x, y, w\})$ and $y \in C(\{y, z, w\})$, the assumption that R' is a G-rationalization of C implies that we must have $(x, y) \in P(R')$. Analogously, $[x \notin C(\{x, z, w\})$ and $x \in C(\{x, y, w\})]$ implies $(z, x) \in P(R')$, and $[z \notin C(\{y, z, w\})$ and $z \in C(\{x, z, w\})]$ implies $(y, z) \in P(R')$. Therefore, R' cannot be P-acyclical. ■

On the other hand, closedness under intersection does not change the results of Section 3 at all. To see this, note first that all the implications established in Section 3 remain true because they are valid on any domain (and, thus, in particular, on domains that are closed under intersection). Furthermore, the examples employed in the proof of Theorem 3 are all defined on domains that are closed under intersection and, in addition, **G-C-Q** does not imply **G-RC-A** even if Σ is closed under intersection. To see that this is the case, consider the following example, which establishes the claim. Let $X = \{x, y, z, w\}$ and $\Sigma = \{\{x, y, w\}, \{x, w\}, \{y\}, \{y, z\}, \{y, z, w\}, \{y, w\}, \{w\}\}$, and define $C(\{x, y, w\}) = \{w\}$, $C(\{x, w\}) = \{w\}$, $C(\{y\}) = \{y\}$, $C(\{y, z\}) = \{y\}$, $C(\{y, z, w\}) = \{y\}$, $C(\{y, w\}) = \{y, w\}$ and $C(\{w\}) = \{w\}$. This choice function is G-rationalizable by the complete and quasi-transitive relation

$$R = \{(x, y), (x, z), (x, w), (y, y), (y, z), (y, w), (z, x), (z, y), (z, w), (w, x), (w, y), (w, w)\}.$$

Suppose R' is a reflexive, complete and P-acyclical G-rationalization of C. Because $C(\{y, z\}) = \{y\}$ and R' is reflexive, we obtain $(y, z) \in P(R')$. Analogously, because $C(\{x, w\}) = \{w\}$ and R' is reflexive, we must have $(w, x) \in P(R')$. Because $y \in C(\{y, w\})$ and $y \notin C(\{x, y, w\})$, we must have $(y, x) \notin R'$ and, because R' is complete, it follows that $(x, y) \in P(R')$. Analogously, because $w \in C(\{y, w\})$ and $w \notin C(\{y, z, w\})$, we must have $(w, z) \notin R'$ and, because R' is complete, it follows that $(z, w) \in P(R')$. Therefore, we have established that $(x, y) \in P(R')$, $(y, z) \in P(R')$, $(z, w) \in P(R')$ and $(w, x) \in P(R')$, contradicting the P-acyclicity of R'.

6 Concluding Remarks

The conditions employed in our axiomatizations involve existential clauses. This is sometimes seen as a shortcoming, but this objection, by itself, does not stand on solid ground: there is nothing inherently undesirable in an axiom involving existential clauses. If the argument is that existential clauses are difficult to verify in practice, this is easily countered by the observation that universal quantifiers are no easier to check algorithmically than existential quantifiers. At least, in the case of existential clauses, a search algorithm can terminate once one object with the desired property is found. In this respect, our conditions compare rather favorably with those that are required for many forms of rationalizability where universal quantifiers play a dominant role.

We suspect that a major reason behind the reluctance to accept existential clauses in the context of rational choice may be that conditions involving existential requirements are seen as being ‘too close’ to the rationalizability property itself, because the desired property is expressed in terms of the existence of a rationalization. This is (except for obvious cases) a matter of judgement, of course. Our view is that the combinations of the axioms employed in the characterizations of the weak forms of rationalizability represent an interesting and insightful way of separating the properties which are necessary and sufficient for each class of weak rationalizability. Furthermore, the axioms we use appear to be rather clear and the roles they play in the respective results have very intuitive interpretations. Finally, we should observe that the mathematical structures encountered here are similar to those appearing in dimension theory, which addresses the question of how many orderings are required to express a quasi-ordering as the intersection of those orderings. Consequently, closely related complexities cannot but arise. In fact, existential clauses appear in many of the characterization results in that area; see, for example, Dushnik and Miller (1941).

In concluding this paper, some remarks on further problems to be explored are in order. Because we do not impose any restrictions on the domain of a choice function (other than non-emptiness), our results are extremely general. As a result, our theorems can be of relevance in whatever context of rational choices as purposive behavior we may care to specify, which is an obvious merit of our general approach. Note, however, that this approach may overlook some meaningful further directions to explore by being insensitive to the structural properties of the domain which may make perfect sense in the specific contexts on which we are focusing. We examined some consequences of an important example of such structural properties, namely, closedness under union, but there are many others to be examined in future work.

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