Two-Sided Matching with Horizontally Differentiated Agents

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Abstract

We consider the problem of assigning sellers and buyers into stable matches in a market for differentiated goods. The match surplus function is decreasing in the distance between matching partners. We show that if it is also convex, then the matches that form under transferable utility (i.e. with bargaining) coincide with those under non-transferable utility (i.e. with fixed sharing rules). We characterize the set of stable matches and provide a surplus sharing rule that supports these matches when agents are allowed to bargain. We then consider the case in which the set of market participants is random. We provide a characterization of ex-ante match probabilities in terms of the so-called Catalan numbers, and prove several results concerning expected surpluses.

Keywords: Spatial differentiation, assignment game, two-sided matching.

JEL classification: C65, D83, D85, L14.

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1 Introduction

Industrial organization theorists have studied markets for horizontally differentiated goods for a long time. In the well-known Hotelling (1929) model of spatial differentiation, buyers and sellers are located along a line and face transportation costs, so that goods offered by a closeby seller are of higher value to a buyer than those offered by distant sellers. Salop (1979) extends Hotelling’s model to the circular case. The absence of a clear natural ordering of the product space is the defining feature which distinguishes horizontal from vertical differentiation, i.e. product type from product quality, and spatial models have become the workhorse for studying horizontally differentiated markets.

An equally large literature, started by Gale and Shapley (1962) and reviewed in Roth and Sotomayor (1990), studies bilateral trade in matching markets. These bilateral exchange environments are characterized by two key features. Firstly, agents face capacity constraints, for example a seller can supply at most one unit of the good. Secondly, the traded goods are indivisible. The most prominent applications of two-sided matching theory include the problems of assigning husbands to wives (the marriage problem), college applicants to colleges or medical interns to hospitals (the admissions problem), and workers to jobs (the job-matching problem).

The aim of this paper is to examine matching markets for horizontally differentiated goods. The assumption of this kind of differentiation is realistic in many two-sided matching situations. Consider the the marriage problem, for example, where (unlike wealth) beauty may well be “in the eye of the beholder.” In the job-matching problem workers may not only differ in their quality, which can be unanimously ranked, but also possess differing sets of skills that are valuable in some jobs but not in others. By using a spatial representation to model such preferences, a more detailed characterization of market outcomes can be derived than in more general matching models with heterogeneous agents (e.g. Crawford and Knoer [1981]).

The model we consider in the paper is a version of the assignment game first developed by Shaply and Shubik (1972). We assume that buyers and sellers of a good are located along a line. Each seller has either one or zero units of the good available, and each buyer demands either one or zero units of the good. The set of actual market participants consists therefore of all sellers and buyer with positive supply and demand quantities. If a seller-buyer pair from this set exchanges the good, surplus is generated that decreases with the distance between the two agents. We make the simplifying assumption that trade is uni-directional, so that a buyer derives the most utility from consuming the good produced by the seller immediately to her left side. The second-highest utility is derived from consuming the good produced by the seller two spots to her left, and so on. Sellers to the right of a buyer can be viewed as being located an infinite distance away from the buyer and have no value for the seller’s good. Alternatively on may think of goods being shipped on a one-way street so that it is impossible for a buyer to obtain a good from a seller who lives to the right of the buyer without violating traffic rules. This assumption

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1Roth (1984) describes in detail the National Intern Matching Program, a real-world matching market that operated the Gale-Shapley algorithm for some 11 years before publication of Gale and Shapley’s paper.
makes the model tractable so that a rich set of results can be obtained, but still captures the idea that goods or agents are horizontally differentiated.

The interplay of conflict and common interest is subtle in this environment. Observe that seller-buyer pairs in which the agents are located close to each other in principle have a common interest in matching, since these are pairs that can generate a relatively high surplus. In the presence of capacity constraints, however, a conflict of interests may still prevail. The reason is that in any given neighborhood of such a pair, there may be more buyers than sellers or more sellers than buyers. Hence there may exist either local excess demand or local excess supply of the goods, so that one side of the market will typically be rationed. The question then becomes how the allocation of goods is affected by the way match surpluses are shared. To address this question, we examine market outcomes under two assumptions on how surpluses are split, non-transferable utility and transferable utility. Under non-transferable utility, surpluses are divided by assuming a fixed sharing rule. This means that the match surplus generated when a seller and a buyer trade is split by allocating a fixed fraction of it to the seller, and the residual to the buyer. Transferable utility, on the other hand, means that buyers and sellers can bargain with each other over their shares of the surplus, or that a buyer is allowed bid for a seller’s good and a seller is allowed to bid for the right to supply a buyer. With transferable utility the surplus shares are determined endogenously and the existence of local excess demand will typically shift surpluses to the sellers’ side and vice versa.

The reason why we are interested in looking at these two cases is that the actual process by which sellers and buyers match can take on many different forms: Some markets, especially labor markets for entry level positions, are operated through a centralized matchmaking mechanism that leaves little room for endogenous wage determination. The market for clinical interns in U.S. hospitals is a well-known example. Other markets, however, lack such a matchmaker and agents themselves have to “sort out” who matches with whom by bargaining with each other over the terms of trade. The degree of centralization or decentralization of a market will then be reflected in market outcomes, i.e. allocations (the set of trades) and prices (the surplus shares). We provide a simple recursive characterization of the set of stable matches under non-transferable utility and show that for a convex match surplus function, this allocation is invariant to the possibility of bargaining. In particular, we show that for a convex surplus function the set of stable matches can be viewed in graph-theoretic terms as a collection of trees (i.e. a forest), regardless of whether agents obtain fixed surplus shares or bargain over their shares. The surplus share a trader will eventually obtain with transferable utility, however, adjusts to market conditions. That is, it is responsive to the demand and supply quantities of nearby agents. For the case of a convex match surplus function, we utilize the forest structure of the stable set to develop a simple bargaining rule by which a set of values can be computed that supports the stable set as an equilibrium under transferable utility.

We then go on to study random sets of market participants. To this end, we let the supply and demand quantities for goods become random variables, so that with some probability a seller has one unit of the good available, and with some probability a buyer has demand for one unit of the good. For each realization of this random variable, an
ex-post set of stable matches and values can be computed. We examine the ex-ante likelihood that two agents trade, and the ex-ante expected value for a buyer and a seller in this market. We demonstrate that finding the matching probability for a given pair of agents requires solving what is known as Catalan’s problem. This combinatorial problem has several representations, and a well-known one is the following: On a city grid consisting of \((n + 1) \times (n + 1)\) intersections, one wants to walk from the south-west corner to the north-east corner by taking a shortest possible path, i.e., one that consists of \(n\) steps north and \(n\) steps east. How many such paths do not pass through a coordinate \((k, l)\) where \(k > l\)? Put differently, how many ways can we walk through the city without entering the part below the main diagonal? The solution to this problem is an integer sequence known as the Catalan numbers that give the number of satisfactory walks for each \(n\). We derive the match probability in terms of this sequence. We also prove several results regarding the characterization of expected values under transferable and non-transferable utility.

The paper is organized as follows. In Section 2 we present the basic matching framework. In Section 3 we examine stable matches, and in Section 4 we examine stable values. We study random demand and supply in Sections 5 and 6. Section 7 concludes. Most proofs are contained in the appendix.

2 A Matching Market with Differentiated Goods

We formulate a stylized model of bilateral exchange between sellers on the one side and buyers on the other, and we will maintain these labels throughout the paper. The words sellers and buyers may suggest that a physical good is transferred between the agents. However the model applies equally to bilateral exchange between workers and firms (so the good is a labor service), managers and entrepreneurs (the good is a managerial service), or business consultants and clients (the good is advice). Even more generally, the model applies to any matching environment in which the input of two distinguishable sides of a market are needed in order for a successful match to form; the obvious example here is the marriage market.

2.1 Buyers and Sellers

Consider the following stylized model of a market with differentiated sellers and buyers. On the set of integers \(\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\), each \(i \in \mathbb{Z}\) denotes the location of one seller, and each \(j \in \mathbb{Z}\) denotes the location of one buyer. A sellers may or may not have a unit of her good available, and we let \(x_i \in \{0, 1\}\) denote seller \(i\)'s supply quantity. Similarly, buyer \(j\)'s demand quantity is \(y_j \in \{0, 1\}\). If \(x_i = y_j = 1\) for some pair \((i, j)\), then we say that \(i\) and \(j\) have coincidence of wants. If the seller’s good is transferred to the buyers, we say that \(i\) and \(j\) have matched. The net surplus generated from this match depends on the distance between the buyer and the seller. Specifically, we assume that match surplus is given by

\[
v(d) = \begin{cases} 
  u(d) - c & \text{if } d \equiv j - i > 0, \\
  -c & \text{otherwise},
\end{cases}
\]
where \( u : \{1, 2, \ldots\} \to (0, \infty) \) is a strictly decreasing match production function such that \( u(d) \to 0 \) as \( d \to \infty \), and \( c \geq 0 \) is a transaction cost. We assume \( c < u(1) \) (otherwise no match can produce positive net surplus) and define \( \hat{d} \equiv \sup\{d : u(d) \geq c\} \geq 1 \). If \( c = 0 \) then \( \hat{d} = \infty \), otherwise \( \hat{d} < \infty \). Let \( D \equiv \{(i, j) \in \mathbb{Z}^2 : i < j \leq i + \hat{d}\} \) denote the set of buyer-seller pairs that can obtain positive net surpluses if they trade.

Our framework is similar in spirit to several approaches taken in the dynamic random matching literature, such as the monetary search models of Kiyotaki and Wright (1989, 1991). A seller can profitably trade only with buyers to the right, and a buyer can profitably trade only with sellers to the left. In our model, match surplus decreases with the distance between the seller and the buyer. While the uni-directional nature of trading may seem restrictive, it facilitates tractability and, we believe, captures the idea of differentiation among buyers and sellers. A useful and intuitive metaphor for this model is to think of sellers being located on one side of a one-way street, and buyers being located on the other side of the same street. The goal is to ship a seller’s good to the nearest buyer who demands it without violating these rules. This interpretation is similar to Townsend’s (1980) “turnpike” that consists of two such one-way streets.

Let \( X = \{0, 1\}^\infty \) (throughout the paper the symbol \( \infty \) denotes countable infinity). The vector \( x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \in X \) is called a supply configuration, and

\[
y = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots) \in X
\]

is called a demand configuration. For \( a, b \in \mathbb{Z}, a < b \), we use the notation

\[
\bar{x}(a, b) = (x_a, x_{a+1}, \ldots, x_{b-2}, x_{b-1}) \in \{0, 1\}^{b-a}
\]

and

\[
\bar{y}(a, b) = (y_{a+1}, y_{a+2}, \ldots, y_{b-1}, y_b) \in \{0, 1\}^{b-a}.
\]

At the time matches are formed, all agents have full knowledge of the configurations \( x \) and \( y \). Which matches are formed in “equilibrium” (to be defined later) will obviously depend on how the potential surplus \( v(j - i) \) will be split between the parties if \( i \) and \( j \) match. We consider two ways by which surpluses can be shared: fixed-sharing rules (non-transferable utility) and bargaining (transferable utility).

### 2.2 Non-transferable utility

A fixed sharing rule is an instance of non-transferable utility, where match partners obtain exogenously given shares of \( v(j - i) \) and neither of them can transfer any fraction of that share to the other party. Under fixed sharing rule \( \alpha \in (0, 1) \), the seller \( i \) obtains \( \alpha v(j - i) \) and buyer \( j \) obtains \( (1 - \alpha) v(j - i) \). The value \( \alpha \) is an exogenously given sharing parameter. An agent who is not in a match obtains a zero surplus.

If \( x_i = y_j = 1 \) and \( (i, j) \in D \), then \( i \) and \( j \) have coincidence of wants and it is possible for these agents to match and receive positive payoff. However, either of the agents may
To formalize this idea, given supply and demand configurations $x$ and $y$ let $M \subseteq D$ be a set of matches $(i, j) \in D$ such that $\forall (i, j) \in M$, (i) $\not\exists k \neq i, j$ such that $(i, k) \in M$ or $(k, j) \in M$ (i.e. no agent is in more than one match), and (ii) $x_i = y_j = 1$ (i.e. only agents with coincidence of wants match). The set $M$ is called a match assignment. Define the value that any seller $i \in Z$ obtains under assignment $M$ as

$$V_S(i|M) = \begin{cases} \alpha v(j - i) & \text{if } \exists j \in Z \text{ s.t. } (i, j) \in M, \\ 0 & \text{otherwise}, \end{cases}$$

and similarly for a buyer’s value, $V_B(j|M)$. If $\exists (i, j) \in D$ such that $x_i = y_j = 1$ and $v(j - i) > \max\{V_S(i|M)/\alpha, V_B(j|M)/(1 - \alpha)\}$, we say that $(i, j)$ blocks $M$. We can now define an equilibrium of the matching market, given $x, y \in X$:

**Definition 1.** Given $(x, y) \in X^2$, a match assignment $M$ is stable if $M$ is not blocked by any $(i, j) \in D$.

Let $M$ be the space of all match assignments. A mapping $f : X^2 \rightarrow M$ is a stable matching function if $f(x, y)$ is a stable match assignment for all $(x, y)$.

### 2.3 Transferable utility

If match surplusses can be split between trading partners in any way they choose, we talk of transferable utility. In the current context, we also call this case bargaining: If an agent can leave a match and trade with someone else she can use this possibility as a “threat” to obtain a larger fraction of the match surplus. Conversely, an agent may offer her trading partner a larger fraction of the match surplus to induce the partner to stay the match and not trade with another agent. To model the case of such transferable utility, we proceed in similar fashion as above. Given $(x, y)$, let $M$ be a match assignment and let $V : M \rightarrow [0, \infty)^2$ be a function that assigns to each match $(i, j) \in M$ a pair of values $V(i, j) = (V_S(i), V_B(j))$ such that $V_S(i) + V_B(j) = v(j - i)$. $V$ is called a value assignment for $M$, and the pair $(M, V)$ is called a match-value assignment. The value assignment $V$ can then be extended naturally to all sellers and buyers, i.e. the value that any seller $i \in Z$ obtains under match-value assignment $(M, V)$ is simply

$$V_S(i|M, V) = \begin{cases} V_S(i) & \text{if } \exists j \in Z \text{ s.t. } (i, j) \in M, \\ 0 & \text{otherwise}, \end{cases}$$

and similarly for a buyer’s value, $V_B(j|M, V)$. We will most often drop the qualifiers $|M, V^M$ from the value notations, but this will not cause confusion. If $\exists (i, j) \in D$ such that $x_i = y_j = 1$ and $v(j - i) > V_S(i) + V_B(j)$, we say that $(i, j)$ blocks $(M, V)$. Similarly to the case of non-transferable utility, we can define an equilibrium of the matching market:

**Definition 2.** Given $(x, y) \in X^2$, a match-value assignment $(M, V)$ is stable if $(M, V)$ is not blocked by any $(i, j) \in D$. 


If \((M, V)\) is stable, then the matches in \(M\) cannot be broken up by bargaining such that both parties in the new match are better off compared to \((M, V)\). Let \(MV\) be the space of all match-value assignments. A mapping \(f \times g : X^2 \rightarrow MV\) is a \textit{stable bargaining function} if it assigns a stable match-value assignment to every configuration \((x, y)\).

### 3 Matching Equilibrium

In this section we examine the structure of the set of stable matches under both non-transferable and transferable utility. In the case of fixed surplus sharing we provide an algorithm for computing the unique stable match assignment for any given configuration \((x, y)\). If utility is transferable and agents bargain over match surplusses, then in principle different matches can obtain. We prove, however, that for a convex surplus function \(v\) the set of stable matches is invariant to the possibility of bargaining (Theorem 1). We then prove that the market will segment itself into “trading clusters” (Theorem 3). This result is reminiscent of a set of results from non-spatial matching theory that exchange takes place within non-overlapping matching sets only; however, the two models are very different: In our model differentiation is spatial, so what may be a good match for one agent is not a good match for another. We also do not consider dynamics and focus on the static assignment problem. Thus the similarity between the results is only superficial.

Consider the case of fixed surplus sharing. Given \((x, y) \in X^2\), we construct a function

\[
\varphi_{x,y} : \mathbb{Z}^2 \rightarrow \{0, 1\},
\]

where \(\varphi_{x,y}(i, j) = 1\) means that match \((i, j)\) is “clearable” given \(x\) and \(y\). We set \(\varphi_{x,y}(i, j) = 0\) if \((i, j) \notin D\). Now consider pairs \((i, j) \in D\), and suppose \(x_i = y_j = 1\). Then \(i\) and \(j\) have coincidence of wants and could trade with each other. Either agent, however, may decline the transaction if a closer match partner is available who agrees to enter the match. In this case, the new match would block the original assignment. For a blocking match to exist, this third agent must have positive demand or supply \textit{and} must not have an even closer potential match partner available herself, and so on. For instance, buyer \(j\) may decline to obtain the good from seller \(i < j\) if another seller \(k, i < k < j\), also supplies her good and does not herself decline the transaction with \(j\). If such an agent exists, then \(j\) would rather trade with \(k\) than with \(i\) since \((1 - \alpha)u(k, j) > (1 - \alpha)u(i, j)\), so that \((k, j)\) would be a blocking match.

To construct \(\varphi_{x,y}\), we will therefore work “from the inside out” and consider the closest possible matches first. Observe that if \(x_i = y_{i+1} = 1\) then \(i\) and \(i + 1\) must trade since \(v(1)\) is the maximal possible match value and \((i, i + 1) \in D\) by our assumption that \(c < u(1)\). Thus, for any \(\alpha \in (0, 1)\) and \(d > 1\), \(\alpha v(1) > \alpha v(d)\) and \((1 - \alpha)v(1) > (1 - \alpha)v(d)\). Therefore \(\varphi_{x,y}(i, i + 1) = 1\) if and only \(x_i = y_{i+1} = 1\). Whether a match is clearable between \(i\) and \(j\) with \(i + 1 \leq i + d\) is then determined recursively:

\[
\varphi_{x,y}(i, j) = 1 \iff \{ x_i = y_j = 1 \text{ and } \exists k, i < k < j, \text{ s.t. } \varphi_{x,y}(i, k) + \varphi_{x,y}(k, j) > 0 \}.
\] (1)

That is, \((i, j)\) is clearable if it is not blocked by a clearable match of shorter distance. Now
a stable match assignment \( M \) can be constructed from \( \varphi_{x,y} \):

\[(i, j) \in M \iff \varphi_{x,y}(i, j) = 1.\]

Note that by definition of \( \varphi \), \( M \) is well-defined and unique given \((x, y)\). Also observe that \( M \) does not depend on the particular value of \( \alpha \) that is used to split the surplus between a seller and a buyer. We summarize this construction of stable match assignments in the following result:

**Lemma 1.** Under any fixed sharing rule, there exists a unique stable matching function \( f : X^2 \to M \), given by

\[f(x,y) = \{(i, j) \in D : \varphi_{x,y}(i, j) = 1\},\]

where \( \varphi_{x,y}(i, j) \) is defined recursively by (1) and the terminal condition \( \varphi_{x,y}(i, i+1) = 1 \iff x_i = y_{i+1} = 1 \forall i. \)

Our next result shows that the set of stable matches is invariant to the possibility of bargaining with matching partners, provided \( u \) (and thus \( v \)) is convex.

**Theorem 1.** Fix \((x, y) \in X^2\), and let \( M^{NT} \) be a stable match assignment under non-transferable utility, and \((M^T, V^T)\) be a stable match-value assignment under transferable utility. If the match production function \( u \) is convex, then \( M^{NT} = M^T \).

The proof of Theorem 1 is in the Appendix. To show that in general \( M^T \neq M^{NT} \) if \( v \) fails to be convex, consider the following example:

**Example 1.** Let \( v(d) = 1 \) (.9, .7) for \( d = 1 \) (2, 3) and suppose \( x_1 = x_2 = y_3 = y_4 = 1 \), while all other agents have zero demand and supply quantities. In this case \( M^{NT} = \{(2,3), (1,4)\} \) by construction of \( \varphi \). However, the following is easily verified to be a stable match-value assignment:

\[M^T = \{(1,3), (2,4)\}, V^T(1,3) = (.36, .54), V^T(2,4) = (.54, .36).\]

Furthermore, it is impossible to find a stable match-value assignment \((M^T, V^T)\) such that \( M^T = M^{NT} \). It would need to be the case that \( V_S(1) + V_B(4) \geq v(2) = 0.9 \) and \( V_S(2) + V_B(3) \geq v(2) = 0.9 \), but \([V_S(1) + V_B(4)] + [V_S(2) + V_B(3)] = v(1) + v(3) = 1.7.\)

The example shows that within the economy’s feasibility set, the matching outcomes under fixed sharing rules may not be efficient: If \( \alpha = 1/2 \), for example, then agents 1 and 4 obtain values \( V_S(1|M^{NT}) = V_B(4|M^{NT}) = .35 \), while agents 2 and 3 obtain \( V_S(2|M^{NT}) = V_B(3|M^{NT}) = .5 \). These values are clearly dominated by the feasible values \( V^T \). On the other hand, \( V^T \) cannot be improved upon in this example. It is straightforward to establish the general efficiency of outcomes under transferable utility.

**Theorem 2.** Every stable match-value assignment is efficient.
Suppose that for a given configuration \((x, y), (M^T, V^T)\) is stable but not efficient. Then there exists another feasible assignment \((M', V')\) such that \(\forall i, j \in \mathbb{Z}, V_S(i|M', V') \geq V_S(i|M^T, V^T)\) and \(V_S(j|M', V') \geq V_S(j|M^T, V^T)\), with at least one such inequality strict. Take any match \((i, j) \in M'\) for which at least one agent obtains a strictly higher value in \((M', V')\) than he did in \((M^T, V^T)\). Then \(V_S(i|M', V') + V_B(j|M', V') > V_S(i|M^T, V^T) + V_B(j|M^T, V^T)\), but since \(V_S(i|M', V') + V_B(j|M', V') = v(j - i)\) we have a contradiction to the assumption that \((M^T, V^T)\) is stable.

In order to further characterize the structure of stable match assignments, we need to make a few definitions. Two matches \((i, j) \in D\) and \((i', j') \in D\) are said to be nested if \(i < i' < j < j'\) (we say \((i', j')\) is nested in \((i, j)\)) or \(i' < i < j < j'\) (we say \((i, j)\) is nested in \((i', j')\)). They are overlapping if \(i < i' < j < j'\) or \(i' < i < j' < j\), and side-by-side if \(j \leq i'\) or \(j' \leq i\). Figure 1 illustrates these definitions. Note that the two side-by-side matches in the right diagram, say \((i, j)\) and \((i', j')\), are such that \(j = i'\); by the weak inequalities used in the definition of side-by-side matches this is allowed.

![Figure 1: Relationships between matches](image)

Given a match assignment \(M, C \subset M\) is called a cluster if the following holds: For every \((i, j) \in C\) (1), if \((i', j') \in C\) then \((i, j)\) and \((i', j')\) are either nested, overlapping, or nested or overlapping with a common third match in \(C\), and (2) if \((i', j') \notin C\) then \((i, j)\) and \((i', j')\) are side-by-side. A cluster \(C\) is called a nested cluster if it does not contain overlapping matches. The size of a cluster \(C\), \(\delta(C)\) is defined as \(\delta(C) = \max_{(i, j), (i', j') \in C} \{j' - i\}\). Loosely speaking, a cluster is a set of matches that can be “separated” from \(M\), but is itself not further separable. Figure 2 illustrates these definitions. The sizes of the clusters on the right side are 2, 4, and 4.

With these definitions at hand, we can further characterize \(M^{NT}\) and \(M^T\). As the following theorem shows, if there is a transaction cost the market will be fragmented into clusters of bounded size.

**Theorem 3.** Suppose \(c > 0\). Given \((x, y) \in X^2\), let \(M^{NT}\) be a stable match assignment under non-transferable utility and let \((M^T, V^T)\) be a stable match-value assignment under transferable utility.

(i) \(M^{NT}\) can be partitioned into a family \(\mathcal{C}\) of nested clusters of maximal size \(\hat{d}\), i.e. \(\delta(C) \leq \hat{d} \ \forall C \in \mathcal{C}\). The same holds for \(M^T\) if \(u\) is convex.
If $u$ fails to be convex, then there exists a constant $\eta < \infty$, independent of $(x,y)$, such $M^T$ can be partitioned into a family $\mathcal{C}'$ of (not necessarily nested) clusters of maximal size $\eta$.

The proof of Theorem 3 is in the Appendix. Part (i) of the result is intuitive: If there is a positive transaction cost, matches between agents that are too dissimilar will not be worthwhile undertaking, so that $\hat{d}$ is an upper bound on the distance between match partners. The nested nature of stable matches incorporated in the recursive definition of $\varphi$ then implies the clustering result as there cannot be overlapping matches under non-transferable utility. By Theorem 1 this structure carries over to the transferable utility case if $u$ is convex. Part (ii) on the other hand is less intuitive, as Example 1 has demonstrated that with a non-convex match surplus function there may be overlapping stable matches. However, we demonstrate in the proof that chains of such overlapping matches have to end eventually if there is a positive transaction cost.
4 Construction of a Bargaining Function

While the previous section was concerned with a characterization of a stable matching function, this section looks at bargaining functions. Unlike the stable match assignment, however, a stable value assignment is typically not unique: For a given \(x, y\), \((M, V)\) and \((M, V')\) can be two stable match-value assignments with \(V \neq V'\). In particular, a well-known result by Shapley and Shubik (1972) shows that the set of all stable value assignments will be a complete lattice (under the usual partial ordering on utility vectors). The role of \(g\), the second component in the bargaining function \(f \times g\), is to determine the transfers each match partner receives. In what follows we will assume that \(u\) is convex, so \(M = f(x, y)\) is the set of stable matches at \((x, y)\) under both transferable and non-transferable utility, where \(f\) is defined by in Lemma 1. Our goal is to find \(g\) so that \(f \times g\) is a stable bargaining function, i.e. \((M, V) = (f(x, y), g(x, y))\) is a stable match-value assignment at every \((x, y) \in X^2\).

4.1 The graph structure of \(M\)

Fix some configuration \((x, y) \in X^2\) and let \(M = f(x, y)\). We will express \(M\) in graph-theoretic terms as a forest, and each nested cluster in \(M\) as a tree, and then utilize this structure to assign values \(V = g(x, y)\).

Given any set of matches \(C \subset D\), define \(s(C) = (s_1(C), s_2(C)) \in C\) by

\[
s_1(C) = \min \{i' : (i', j') \in C\}, \quad s_2(C) = \max \{j' : (i', j') \in C\}.
\]

If \(C\) is a nested cluster, then \(s(C) \in C\); we call \(s(C)\) the spanning match of \(C\). Call \(C' \subset C\) a nested subcluster of \(C\) if \(s(C') \in C\), and if \((i, j) \in C\setminus C'\) then \(\neg[s_1(C') < i < j < s_2(C')]\). If \(C'\) is a subcluster of \(C\), then \(C\) is a supercluster of \(C'\). A nested cluster \(C\) can then be expressed in terms of its spanning match and a set of nested subclusters: \(C = s(C) \cup R(C)\), where \(R(C)\) is a family of nested subclusters of \(C\) such that \(C' \not\subset C''\) for all \(C', C'' \in R(C)\), and if \(C' \in R(C)\) and \(C'\) is a subcluster of any nested cluster \(C''\), then \(s(C') \in C''\). Thus \(R(C)\) is the set of largest subclusters of \(C\) of \(C\). Each nested subcluster \(C' \in R(C)\) can similarly be expressed in terms of its spanning match \(s(C')\) and a family of nested subclusters \(R(C')\). Similarly, for each nested subcluster \(C'\), we let \(\rho(C')\) denote the smallest supercluster of \(C'\).

As an illustration, consider the middle graph on the left side of Figure 2. The three depicted matches, taken together, form a nested cluster \(C\). The largest match (which connects the two “gray” agents) is the spanning match \(s(C)\), and the two matches that are in the shaded region form a nested subcluster \(C'\) of \(C\) which is also the single element of \(R(C)\). The innermost match in the graph is a nested subcluster of both \(C\) and \(C'\), and it is also the single element of \(R(C')\).

Generally, any nested cluster \(C\) can be viewed as a tree whose nodes correspond to the elements of \(C\): The spanning match of \(C\) is the initial node in the tree, the spanning matches of the subclusters \(C' \in R(C)\) are its successor nodes (or children), and so forth. Likewise, \(s(\rho(C'))\) is the parent node of \(s(C')\). By Theorem 3 \(M\) can be partitioned into a family \(\mathcal{C}\) of nested clusters of maximal size \(d\). Then each \(C \in \mathcal{C}\) is a tree, and \(M\) itself is a
forest as the $C \in \mathcal{C}$ are not nested in each other by definition of a nested cluster. Figure 3 illustrates how a single nested cluster can be represented as a tree.

In $M$ there will be a countable number of such trees. Suppose $C', C''$ are nested clusters in $M$. We make two observations: First, if $C'' \not\subseteq C'$ and $C' \not\subseteq C''$, then $(i', j') \in C'$ and $(i'', j'') \in C''$ must be side-by-side matches of $M^T$. Second, if $C'' \subset C'$, then $(i'', j'') \in C''$ is nested in $s(C')$.

**4.2 Construction of a local value assignment**

Recall that the set of stable matches $M$ was generated recursively “from the inside out” (i.e. starting with matches between adjacent agents and working outwards) through the function $\varphi$, resulting in a collection of nested clusters $\mathcal{C}$. For each $C \in \mathcal{C}$, we now generate a set of values $V^C$ recursively “from the outside in,” i.e. starting with the spanning match of a nested cluster in $\mathcal{C}$ and working inwards. The resulting pair $(C, V^C)$ is called a local match-value assignment $(C, V^C)$. The global assignment $(M, V)$ is then defined as the union of the local assignments over all nested clusters $C$, and shown to be stable further below.

To construct a local assignment, fix any $C \in \mathcal{C}$. Let $C'$ be a nested subcluster of $C$, let $(i', j') = s(C')$, and suppose that values have already been assigned to the match


(i'', j'') = s(ρ(C')). Then we assign values to (i', j') as follows. First, define

\[ \bar{w}_S(i') = v(j'' - i') - V_S^C(j''), \quad \bar{w}_B(j') = v(j' - i'') - V_B^C(i''). \]  

Then the values \( V_C^C(i', j') \) for \((i', j')\) are given by

\[
\left( V_S^C(i'), V_B^C(j') \right) = \left( \frac{1}{2}v(j' - i') + \frac{1}{2}[\bar{w}_S(i') - \bar{w}_B(j')], \right. \\
\left. \frac{1}{2}v(j' - i') + \frac{1}{2}[\bar{w}_B(j') - \bar{w}_S(i')] \right) .
\]  

This procedure can be used to iteratively assign values to all matches in \( C \), working from the outside in. What is left to do is to assign starting values to the spanning match of \( C \). To do this, define the pseudo-span of \( C \), \((i^p, j^p)\), as follows. Let

\[ i^p = \max\{i' : x_{i'} = 1 \text{ and } \exists j' \text{ s.t. } (i', j') \in M\} \]  

if such an agent exists; otherwise set \( i^p = -\infty \). Similarly,

\[ j^p = \min\{j' : y_{j'} = 1 \text{ and } \exists i' \text{ s.t. } (i', j') \in M\} \]  

if this agent exists; otherwise set \( j^p = \infty \). Note that if \( i^p \) and \( j^p \) are both finite, then the fact that \( i^p \) and \( j^p \) have coincidence of wants but are not matched implies \( u(j^p - i^p) \leq c \), and thus \( v(j^p - i^p) \leq 0 \). Let

\[ \bar{w}_S^p(i) = \max\{v(j^p - i), 0\}, \quad \bar{w}_B^p(j) = \max\{v(j - i^p), 0\}. \]  

Then \( V_C(i, j) \) is given by

\[
\left( V_S^C(i), V_B^C(j) \right) = \left( \frac{1}{2}v(j - i) + \frac{1}{2}[\bar{w}_S^p(i) - \bar{w}_B^p(j)], \right. \\
\left. \frac{1}{2}v(j - i) + \frac{1}{2}[\bar{w}_B^p(j) - \bar{w}_S^p(i)] \right) .
\]

It is easily seen that \( V_C \) is a well-defined set of values, i.e. \( \forall (i, j) \in C \), \( V_S^C(i), V_B^C(j) > 0 \) and \( V_S^C(i) + V_B^C(j) = v(j - i) \). Now construct \( V_C \) for each \( C \in \mathcal{C} \) in exactly the same fashion, and let the match-value assignment \((M, V^M)\) be given by

\[ (M, V) = \bigcup_{C \in \mathcal{C}} (C, V^C). \]

The matching function \( f \) (defined through \( \varphi \)) together with the described procedure to assign values gives rise to a bargaining function \( f \times g \). We will now show that this bargaining function is stable.
4.3 Internal stability of \((C, V^C)\)

We first show that each local assignment \((C, V^C)\) is “internally stable” in the sense that \((C, V^C)\) is not blocked by any feasible trade in \(C\). In the section to follow, we then use this result to prove that \((M, V) = \cup_C(C, V^C)\) is stable in the sense defined in section 2.

For any two pairs \((i, j) \in M\) and \((i', j') \in M\) and a given value assignment \(V\), call \((i, j)\) and \((i', j')\) relatively stable if \(V_S(i) + V_B(j') \geq v(j' - i)\) and \(V_S(i') + V_B(j) \geq v(j - i')\). It is easily seen that for a match-value assignment \((M, V)\) to be stable, as per Definition 2, it is necessary that all \((i, j) \in M\) and \((i', j') \in M\) are relatively stable. Furthermore, relative stability of matches in \(M\) in conjunction with the requirements that all matched agents obtain a non-negative value, and that \((M, V)\) not be blocked by any \((i, j)\) where at least one of \(i\) and \(j\) is unmatched is equivalent to stability.

We start with two results whose proofs are in the Appendix.

**Lemma 2.** If the match production function \(u\) is convex, then the following holds for any value match-value assignment \((M, V)\). If \((i, j)\) and \((i', j')\) are relatively stable side-by-side matches, and if \((i'', j'')\) is nested in \((i', j')\) and \((i'', j'')\), \((i', j')\) are relatively stable, then \((i'', j'')\) and \((i, j)\) are relatively stable.

**Lemma 3.** If the match production function \(u\) is convex, then for each \(C \in \mathcal{C}\) the following properties hold for the local assignment \((C, V^C)\) constructed in section 4.2. Let \(C_0\) be any nested subcluster of \(C\).

(i) For all nested subclusters \(C'\) of \(C_0\), \(s(C')\) and \(s(C_0)\) are relatively stable.

(ii) For all \(C', C'' \in R(C_0)\), \(s(C')\) and \(s(C'')\) are relatively stable.

The following result shows \((C, V^C)\) to be internally stable. The proof uses the two preceding lemmata and is given below.

**Lemma 4.** Suppose the match production function \(u\) is convex and \(c > 0\). For \(C \in \mathcal{C}\), let \((C, V^C)\) be a local match-value assignment constructed in section 4.2. Then for all \((i', j') \in C\) and \((i'', j'') \in C\), are relatively stable.

**Proof of Lemma 4.** Take any two matches \((i', j') \in C\) and \((i'', j'') \in C\). Let \(C'\) and \(C''\) denote the subclusters of \(C\) such that \((i', j') = s(C')\), and \((i'', j'') = s(C'')\). If one of \(C'\) and \(C''\) is a subcluster of the other, then Lemma 3 (i) implies that \((i', j')\) and \((i'', j'')\) are relatively stable. If neither of \(C'\) and \(C''\) is a subcluster of the other, then \((i', j')\) and \((i'', j'')\) must be side-by-side matches. Let \(C_0\) be the smallest nested subcluster of \(C\) such that \(C'\) and \(C''\) are both nested subclusters of \(C_0\). Then there are \(D', D'' \in R(C_0)\) such that \(C'\) is a subcluster of \(D'\) and \(C''\) is a subcluster of \(D''\). By Lemma 3 (ii), \(s(D')\) and \(s(D'')\) are relatively stable; also notice that \(s(D')\) and \(s(D'')\) are side-by-side matches. Furthermore, by Lemma 3 (i) \(s(C')\), \(s(D')\) are relatively stable and \(s(C''), (D'')\) are relatively stable. Thus by Lemma 2 (i) \(s(C'')\) and \(s(D')\) are relatively stable. But then \(s(C'')\) and \(s(D')\) are side-by-side matches, so applying Lemma 2 (i) again, \(s(C'')\) are \(s(C')\) relatively stable. □
4.4 Stability of \((M, V^M)\)

The global match-value assignment \((M, V^M)\) is the union of the local assignments \((C, V^C)\), taken over all nested clusters \(C \in \mathcal{C}\). This assignment is stable, in the sense of Definition 2:

**Theorem 4.** Suppose the match production function \(u\) is convex and \(c > 0\). For each \(C \in \mathcal{C}\), let \((C, V^C)\) be a local match-value assignment constructed in section 4.2. Then the global match-value assignment \((M, V) = \bigcup_{C \in \mathcal{C}} (C, V^C)\) is stable.

We give the outline of the proof here in the main text. A detailed proof of Theorem 4 is contained in the Appendix. Note that each matched agent, by construction, obtains a positive payoff, so no pairs will break up because an agent is better off in “autarchy” than in her assigned match. Then stability is equivalent to (a) relative stability of all matched pairs, and (b) the non-blocking condition for seller-buyer pairs involving one or two unmatched agents. For (a), if two matches belong to the same nested cluster, then by Lemma 4 they are relatively stable. If they belong to different clusters, then by Lemma 2 and Lemma 3 (i) they are relatively stable if the spanning matches of these two clusters are relatively stable, which is shown in the proof. For (b), note that any two agents who have coincidence of wants but are not matched must be so far apart that a positive surplus cannot be generated. Thus, all we need to check is if some matched agent wants to leave her currently assigned trade partner and enter a match with a currently unmatched agent. The construction of a cluster’s pseudo-span enables us to check only if the agents in a given match prefer a match with an agent in the pseudo-span of the cluster containing the current match. In the proof we also show that this is not the case. Therefore \((M, V)\) must be stable.

Intuitively, the stable value assignment we constructed simply took a convex combination (with equal weights) of the buyer-optimal stable payoff and seller-optimal stable payoff for the spanning match of a nested cluster, temporarily disregarding the subclusters. Given these values, we assigned a convex combination of the buyer-optimal stable payoff and seller-optimal stable payoff to the children nodes of this initial match, again temporarily disregarding any further subclusters, and so on. Our proofs then boil down to showing that for a convex match production function, these averages form an overall stable assignment.

5 Matching Probabilities

We now assume that the supply and demand quantities \(x_i\) and \(y_j\) are random variables and examine the ex-ante probabilities that two agents trade (this will be done in the current section) as well as the ex-ante expected surplus an agent obtain in the market (this will be done in the next section). We let \(p\) denote the probability that \(x_i = 1\), \(\forall i\), and \(q\) denote the probability that \(y_j = 1\), \(\forall j\), and assume that all supply and demand quantities are independently distributed.

We will focus on the equivalent cases of non-transferable utility and transferable utility with a convex match production function. Then \(M = f(x, y)\) is the unique set of stable
matches at configuration \((x, y) \in X^2\). Before \((x, y)\) is realized, let
\[
\mu(d) = Pr[\varphi_{x,y}(i, i + d) = 1] = Pr[(i, i + d) \in f(x, y)]
\] (8)
be the probability that a transaction takes place between a seller and a buyer who are a distance \(d\) apart. This probability takes into account both the fact that there may or may not be coincidence of wants between \(i\) and \(i + d\) (i.e. that \(x_i = y_{i+d} = 1\), and the fact that if there is coincidence of wants the transaction must still be clearable. \(\mu(d)\) is well-defined: Let the map \(\hat{\varphi}_{i,j}: X^2 \to \{0, 1\}\) be given by \(\hat{\varphi}_{i,j}(x, y) = 1 \iff \varphi_{x,y}(i, j) = 1\) and observe that \(\hat{\varphi}^{-1}_{i,j}(\cdot) \in \mathcal{P}\) by construction of \(\varphi\), where \(\mathcal{P}\) is the product algebra on \(X^2\) (a formal definition if given in section 6.1). In particular, \(\varphi_{x,y}(i, i + d)\) depends only on \(x(i, i + d)\) and \(y(i, i + d)\), so it is admissible to rewrite \(\varphi\) in terms of \(x(i, j)\) and \(y(i, j)\) only:
\[
\varphi_{\pi(i,j),\bar{\pi}(i,j)}(i, j) = 1 \iff \varphi_{x,y}(i, j) = 1.
\]

We will now find an explicit formula for \(\mu\), a task that involves two steps. To simplify notation, we fix \(i = 0\) and let \(j = d\). Whether \(i\) and \(j\) trade depends on \(\pi(0, d) = (x_0, \ldots, x_{d-1})\) and \(\bar{\pi}(0, d) = (y_1, \ldots, y_d)\) only. These are vectors of length \(d\), and if it is clear what \(d\) is we shorten notation and write \(\pi, \bar{\pi}\) instead. Now define
\[
T(d) = \{ (\pi \in \{0, 1\}^d, \bar{\pi} \in \{0, 1\}^d) \mid \varphi_{\pi,\bar{\pi}}(0, d) = 1 \}.
\]
The first step is to identify the set \(T(d)\), which will require some work. The second step is then straightforward: Compute the probability of each element in \(T(d)\) and add them up.

### 5.1 Some examples
We use notation
\[
\langle \bar{\pi}, \pi \rangle = \begin{pmatrix} y_1 & y_2 & \cdots & y_d \\ x_0 & x_1 & \cdots & x_{d-1} \end{pmatrix}
\]
to express supply and demand configurations. For 0 and \(d\) to trade, there must be coincidence of wants between 0 and \(d\), that is \(x_0 = y_d = 1\). This condition will be sufficient if \(d = 1\), i.e. there are no agents located between 1 and 2 so that clearability is always satisfied. Hence \(\pi = (1)\) and \(\bar{\pi} = (1)\), and
\[
T(1) = \{ \langle 1 \rangle \}.
\]
If \(d = 2\), we need to consider clearability. Suppose \(x_0 = y_2 = 1\), but also \(x_1 = 1\). Then 2 will not choose to trade with 0, since \((1, 2)\) blocks \((0, 2)\). Similarly, if \(y_1 = 1\), then \((0, 1)\) blocks \((0, 2)\). Clearability therefore requires that \(x_1 = y_1 = 0\), and
\[
T(2) = \{ \langle 0 1 \rangle \}.
\]
When \(d > 2\), things turn out to be more intricate: Even when agents located between 0 and \(d\) have positive actual supplies or demands, as long as all of these intermediate agents can form clearable matches among themselves, nothing stands in the way of a successful
match between 0 and \(d\). Obviously this requires that there be as many positive entries in
\((x_1, \ldots, x_{d-1})\) as there are in \((y_1, \ldots, y_{d-1})\). However, that condition is not sufficient for
\((0, d)\) to clear. Consider the following example with \(d = 3\): If \(\mathbf{x} = (1, 0, 1)\) and \(\mathbf{y} = (1, 0, 1)\),
the matches that clear are \((0, 1)\) and \((2, 3)\). Hence, 0 does not trade with 3—although there
is one unit of the good supplied and one unit demanded by agents located between 0 and 3,
the configurations are such that 1 and 2 cannot trade this unit among themselves (they do
not have coincidence of wants). On the other hand, suppose \(\mathbf{x} = (1, 1, 0)\) and \(\mathbf{y} = (0, 1, 1)\).
In this case the trade \((1, 2)\) does clear, allowing \((0, 3)\) to clear as well. Consequently, we have
\[
T(3) = \{ \langle 001 \rangle_{100}, \langle 011 \rangle_{110} \},
\]
so there are two possible values for \((\mathbf{x}, \mathbf{y})\) that result in trade between seller 0 and buyer
3. Given \(T(d)\), it is straightforward to find the probability that \((\mathbf{x}, \mathbf{y}) \in T(d)\): Every zero
entry in \(\mathbf{x}\) has probability \(1 - p\), and every positive entry has probability \(p\). Likewise, every
zero entry in \(\mathbf{y}\) has probability \(1 - q\), and every positive entry has probability \(q\). Since
all agents’ demands and supplies are independent, simply multiplying these probabilities
yields the overall probability of a given pair \((\mathbf{x}, \mathbf{y})\). For example if \(p = q\), then the
probabilities of the events \(T(d), d = 1, 2, 3\), are given by
\[
\begin{align*}
\mu(1) &= Pr[T(1)] = p^2, \\
\mu(2) &= Pr[T(2)] = p^2(1 - p)^2, \\
\mu(3) &= Pr[T(3)] = p^2(1 - p)^4 + p^4(1 - p)^2.
\end{align*}
\]
When we go to even larger distances between agents, the possibilities for trades among
the intermediate agents become numerous, and the sets \(T(d)\) become large. For instance,
if \(d = 4\) there are 5 elements in \(T(4)\),
\[
T(4) = \{ \langle 0001 \rangle_{1000}, \langle 0100 \rangle_{1100}, \langle 0010 \rangle_{1010}, \langle 0001 \rangle_{1100}, \langle 0111 \rangle_{1110} \}.
\]
For \(d = 5\) there are 14 elements in \(T(5)\),
\[
T(5) = \{ \langle 00001 \rangle_{10000}, \langle 01001 \rangle_{11000}, \langle 00101 \rangle_{10100}, \langle 00011 \rangle_{11000}, \langle 00111 \rangle_{11100}, \langle 00011 \rangle_{11110}, \\
\langle 01101 \rangle_{11100}, \langle 00111 \rangle_{11010}, \langle 01011 \rangle_{11110}, \langle 00111 \rangle_{11100}, \langle 01011 \rangle_{11110}, \langle 00111 \rangle_{11110} \}.
\]
The reader may check that these are indeed all the possibilities for agents 0 and \(d\) to form
a successful match. To identify the sets \(T(d)\) in a systematic way, consider the following,
seemingly unrelated problems:

5.2 Aside: Catalan’s Problem

How many ways can a major league baseball team obtain a break-even record at the end
of a season while not having a losing record at any point during the season? Given a
convex polygon with \(n + 2\) sides, how many ways can its vertices be joined to triangulate
it by non-intersecting diagonals? Lemma 5 provides the answer to both.
Lemma 5. Let $n$ be a positive integer, and let $m_l \in \{-1, 1\}$ for $l = 1, \ldots, 2n$ be a sequence satisfying $\sum_{l=1}^{2n} m_l = 0$. There are
\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \]
ways to order $m_1, \ldots, m_{2n}$ such that $\sum_{l=1}^{z} m_l \geq 0$ for $z < 2n$.

The numbers $C_n$ are called the Catalan numbers. A direct count of the number of satisfactory sequences $\{x_l\}$ is surprisingly difficult. For an indirect proof of Lemma 5, see Martin (2001, Chapter 6).

The answer to the first problem posed above is given by the 81st Catalan number. To have a break-even record the team needs to win 81 games and lose 81 games, as a team plays 162 games per season. There are $\binom{162}{81}$ ways to order these wins and losses, but it turns out that only one out of every 82 of these sequences is such that the team has not lost more games than it has won at any point, hence $C_{81}$ is the number of satisfactory sequences of wins and losses. The second problem is Eugene Catalan’s original question, and its answer is $C_n$. There are many other ways in which one can think of the Catalan numbers, for example: On a city grid of $n+1$ East-West streets and $n+1$ North-South streets, suppose the diagonal from the South-West corner to the North-East corner divides the city in a safe and an unsafe part, with the diagonal belonging to the safe part. Then there are $C_n$ shortest walks from the South-West corner to the North-East corner that do not enter the unsafe part of the city.

5.3 Computation of match probabilities

We now return to our original problem of developing a systematic way of assessing the probability of $T(d)$, i.e. the probability that agents 0 and $d$ trade. For the time being suppose $d \geq 2$. Given $\overline{x}$ and $\overline{y}$, construct a new vector $m \in \{-1, 0, 1\}^d$ by setting
\[ m_l = x_{l-1} - y_l, \quad l = 1, \ldots, d. \]

From $m$ construct a second vector $\sigma \in \mathbb{Z}^d$ as follows:
\[ \sigma_l = \sum_{\kappa=1}^{l} m_{\kappa}, \quad l = 1, \ldots, d. \]  

For example, the element $\langle 0, 1, 0, 1, 0 \rangle$ in $T(5)$ will result in $m = (1, 0, 1, -1, -1)$ and $\sigma = (1, 1, 2, 1, 0)$. We then have the following result:

Lemma 6. Fix $(x, y)$ and $d \geq 2$. Then $\varphi_{x,y}(0,d) = 1$ if and only if $\sigma_l \geq 1$ for $l = 1, \ldots, d-1$, and $\sigma_d = 0$.

In understanding the relationship between the Catalan numbers and the sets $T(d)$, which we will exploit further below, the proof of Lemma 6 is illustrative and therefore given in the text.
We define a sequence of sets \( \{B_l\}_{l=1}^{d}, B_l \subseteq \{1, \ldots, d\} \). The sets \( B_l \) will represent the “supply stack,” that is, \( B_l \) will contain those sellers among \( 0, \ldots, l-1 \) who have not been assigned to a buyer among agents \( 1, \ldots, l \). Construct this sequence as follows. Set \( B_0 = \emptyset \). For \( l \geq 1 \), consider the following three cases.

1. If \( m_l = 1 \), i.e. \( x_{l-1} = 1 \) and \( y_l = 0 \), then seller \( l-1 \) has positive supply but cannot trade with the buyer immediately to the right, \( l \). Hence we add \( l \) to \( B_{l-1} \) and set \( B_l = B_{l-1} \cup \{l\} \).

2. If \( m_l = -1 \), i.e. \( x_{l-1} = 0 \) and \( y_l = 1 \), then buyer \( l \) has positive demand but cannot obtain it from the seller immediately to the left, \( l-1 \). If \( B_{l-1} \) is non-empty, however, \( l \) can obtain the good from any seller in \( B_{l-1} \), and the most preferred of these is the one closest to \( l \). Hence we delete this agent from \( B_{l-1} \) and set \( B_l = B_{l-1} \setminus \max B_{l-1} \).

3. If \( m_l = 0 \), then either \( x_{l-1} = y_l = 1 \) or \( x_{l-1} = y_l = 0 \). In the first case neither does seller \( l-1 \) have positive supply nor does buyer \( l \) have positive demand. In the second case seller \( l-1 \) supplies the good but it is immediately transferred to buyer \( l \). In either case, the set \( B_{l-1} \) is unaffected, so we set \( B_l = B_{l-1} \).

By construction of \( \{B_l\}_{l=1}^{d} \), \( |B_l| = \max\{\sigma_l, 0\} \). Now suppose \( x_0 = 1 \), so \( m_1 \geq 0 \). If \( \sigma_l = 0 \) for the first time for some \( l < d \), then \( B_l = \emptyset \) and seller 0 has supplied the good to buyer \( l < d \). On the other hand, if \( \sigma_{l'} \geq 1 \forall l' \leq l \) then \( 0 \in B_l \). It follows that for 0 to trade with \( d \), \( \sigma_l \geq 1 \) for \( 1 \leq l < d \), and \( \sigma_d = 0 \).

The vector \( m \) (equivalently \( \sigma \)) determines if 0 and \( d \) trade. For this match to be successful, \( m_1 = 1 \) and \( m_d = -1 \). The remaining elements form a vector \( \tilde{m} = (m_2, \ldots, m_{d-1}) \) of length \( d - 2 \). By Lemma 6, \( \tilde{m} \) must satisfy two requirements: First, there must be as many positive entries as there are negative entries, for otherwise \( \sigma_d \neq 0 \). Second, going from left to right, the partial sum \( \sigma_l = \sum_{\kappa=2}^{l} m_\kappa \) must be non-negative for all \( 2 \leq l \leq d-1 \), for otherwise \( \sigma_l < 1 \) for some \( l < d \). Hence, by means of the associated vector \( \sigma \) we have identified a way to determine if \((\pi, \bar{\pi}) \in T(d)\).

It will now be convenient to break up the vector \( \tilde{m} \) into \( 2n \leq d-2 \) non-zero entries and \( d-2(n+1) \geq 0 \) zero entries. The number of ways to arrange zero and non-zero entries is given by the binomial coefficient \( \binom{d-2}{2n} \). Out of the \( 2n \) non-zero entries, recall that \( n \) must be equal to \( +1 \) and \( n \) must be equal to \( -1 \). To satisfy the partial sum condition for \( \tilde{m} \), that condition must be satisfied for the non-zero entries (the zero entries have no effect on the partial sums). By Lemma 5 there are exactly \( C_n \) orderings of the \( n \) positive and \( n \) negative entries that are such that all partial sums are non-negative. In total, then, there are

\[
\binom{d-2}{2n} = \frac{(d-2)!}{n!(n+1)!((d-2(n+1))!}
\]  

possible vectors \( \tilde{m} \in \{-1, 0, 1\}^{d-2} \) with \( n \) positive and \( n \) negative entries that allow for trade between 1 and \( 1+d \), conditional on \( m_1 = 1 \) and \( m_d = -1 \). Hence there are as many vectors \( m \in \{-1, 0, 1\}^d \) that allow for trade between 1 and \( 1+d \).
For given $n$, the probability of encountering a vector $m \in \{-1, 0, 1\}^d$ with $n$ positive and $n$ negative entries is

$$Pr[|\{l: m_l = 1\}| = |\{l: m_l = -1\}| = n] = C_n\left(\frac{d-2}{2n}\right)r^{n+1}s^{d-2(n+1)},$$

where $r = pq(1-p)(1-q)$ and $s = pq + (1-p)(1-q)$ are the probabilities of a $(+1, -1)$ pair and a zero entry in $m$, respectively. Since $n$ can take on the values $n = 0, \ldots, \lceil \frac{d-2}{2} \rceil$ (the largest integer smaller or equal to $\frac{d-2}{2}$), to find the probability of $T(d)$, $d \geq 2$, we need to sum over this range:

$$\mu(d) = \sum_{n=0}^{\lceil \frac{d-2}{2} \rceil} C_n\left(\frac{d-2}{2n}\right)r^{n+1}s^{d-2(n+1)}. \quad (11)$$

Finally, if $d = 1$ then $i = 0$ and $d = 1$ trade with probability $pq$, as argued above, and if $d > \hat{d}$ then trade between $i = 0$ and $j = d$ results in a net loss and thus will be blocked.

The following theorem summarizes the result:

**Theorem 5.** Assume either non-transferable utility, or transferable utility and a convex match production function $u$. Then the probability $\mu(d)$ that seller $i$ and buyer $j = i + d$ will trade is given by (11) for $1 < d \leq \hat{d}$, $\mu(d) = pq$ for $d = 1$, and $\mu(d) = 0$ for $d > \hat{d}$.

## 6 Expected Surpluses

This section examines the expected surplus an agent receives in the matching market, i.e. before she learns her supply or demand quantities. Using Theorem 5, one can easily compute the expected payoffs for sellers and buyers under non-transferable utility:

$$W_{NT}^S(\alpha) \equiv \alpha \sum_{d=1}^{\hat{d}} \mu(d)v(d) \quad \text{and} \quad W_{NT}^B(\alpha) \equiv (1 - \alpha) \sum_{d=1}^{\hat{d}} \mu(d)v(d).$$

Note that when $\alpha = 1/2$, sellers and buyers expect the same expected surplus, even though $p$ may not equal $q$. Also note that these values are independent of the locations $i$ and $j$ for sellers and buyers, which is of course due to the assumptions that all sellers (all buyers) have identical probabilities of positive supply (demand), that the match surplus function $v$ was defined in terms of the distance between a buyer and a seller only.

Computing corresponding expected payoffs under transferable utility is less straightforward. If $u$ is convex, stable matches are identical to those under non-transferable utility. In particular $\varphi_{x,y}(i, j)$ indicates if the match $(i, j) \in D$ is ex-post clearable by Theorem 1, and by Theorem 5 $\mu(j-i)$ is the ex-ante probability of this event. However, as already indicated in Section 4, bargaining functions are not unique. Moreover, there is no guarantee that an expected surplus exists for every stable bargaining function. For these reasons, we will restrict attention to bargaining functions that satisfy certain technical restrictions. The restrictions we use are measurability and symmetry, which we will now define. The bargaining function constructed in section 4 satisfies these conditions.
6.1 Measurability

We start with a definition of measurability, for which we endow $X^2$ with the product algebra: A subset $A \subset X^2$ is a cylinder if there exists a finite integer $n$ and a set $H \subseteq (\{0,1\}^{2n})^2$ such that

$$A = \{(x,y) : (x-n,i+n), (y-n,i+n)) \in H\}.$$

The product algebra $P$ is then the $\sigma$-field generated by the cylinders in $X^2$. We make the following definition:

**Definition 3.** A bargaining function $f \times g : X^2 \to MV$ is (product) measurable if for all $(i,j) \in D$ and all $v', v'' \in \mathbb{R}$,

$$\{(x,y) \in X^2 : (i,j) \in f(x,y) \text{ and } v' \leq g^S_i(x,y) \leq v''\} \in P$$

and

$$\{(x,y) \in X^2 : (i,j) \in f(x,y) \text{ and } v' \leq g^B_j(x,y) \leq v''\} \in P.$$

An intuitive interpretation of measurability as defined above is the following. Suppose two agents trade with each other and divide the match surplus in a certain way. Then this division may be affected by the supply and demand quantities of sufficiently close agents, but it is unaffected by the supply and demand quantities of agents located a sufficiently large distance away. From a technical perspective, measurability means that expected surplusses can be computed under the bargaining function $g$: If $p$ is the probability that $x_i = 1$ and $q$ is the probability that $y_j = 1$, then a well-defined probability exists for each event in $P$, called the product measure on $X^2$ (see Billingsley [1995] Section 2). We formally define this measure in the proof of Theorem 6) in the appendix.

6.2 Symmetry

Next we define two symmetry properties. For $d \in \mathbb{Z}$ define a mapping $\sigma_d : X^2 \to X^2$ by

$$(x', y') = \sigma_d(x,y) \iff \forall i \in \mathbb{Z} : (x'_i, y'_i) = (x_{i-d}, y_{i-d}).$$

Thus, under transformation $\sigma_d$, both the supply and the demand configuration is “shifted” to the right by $d$ spots. For $i, j \in \mathbb{Z}$, define another mapping $\pi_{i,j} : X^2 \to X^2$ by

$$(x', y') = \pi_{i,j}(x,y) \iff \forall d \in \mathbb{Z} : (x'_{i+d}, y'_{i+d}) = (y_{j-d}, x_{i-d}).$$

Under transformation $\pi_{i,j}$, the supply and demand configurations are “pivoted” around the pair $(i,j)$: If a seller located any $d$ spots to the left of seller $i$ has positive supply, then under $\pi_{i,j}$ the buyer located $d$ spots to the right of buyer $j$ now has positive demand, and vice versa. We make the following definitions:

**Definition 4.** A bargaining function $f \times g : X^2 \to MV$ is shift-symmetric if for all $d \in \mathbb{Z}$ and all $(x,y) \in X^2$,

(a) $(i,j) \in f(x,y) \iff (i+d,j+d) \in f(\sigma_d(x,y))$ and
(b) \( g^S_i(x, y) = g^S_{i+d}(\sigma_d(x, y)) \) and \( g^R_j(x, y) = g^R_{j+d}(\sigma_d(x, y)) \),

Definition 5. A bargaining function \( f \times g : X^2 \to MV \) is pivot-symmetric if for all \( i, j \in \mathbb{Z} \) and all \( (x, y) \in X^2 \),

(a) \( (i, j) \in f(x, y) \iff (\hat{i} + \hat{j} - j, \hat{i} + \hat{j} - i) \in f(\pi_{i,j}(x, y)) \) and

(b) \( g^S_i(x, y) = g^R_j(\pi_{i,j}(x, y)) \).

A bargaining function that satisfies both Definition 4 and Definition 5 is simply called symmetric. Symmetry has the following intuitive interpretation. Suppose an agent faces exactly the same environment in configuration \( x, y \) as does another agent in \( x', y' \), in the sense that any arbitrarily large neighborhood of buyers and sellers around the first agent is identical, in terms of supply and demand, to the corresponding neighborhood around the second agent. Then under a shift-symmetric bargaining function the first agent matches with another agent at \( (x, y) \) if and only the second agent does at \( (x', y') \), and if they match then the distance to their match partners is the same. Furthermore the first agent obtains the same payoff at \( (x, y) \) as the second agent at \( (x', y') \). Thus, Definition 4 can be regarded as an anonymity property, saying that the “name” of an agent does not affect her bargaining possibilities embodied in \( f \times g \). Next suppose we interchange the roles of buyers and sellers in that the supply quantities offered by any neighborhood of agents around some seller become the demand quantities offered by the corresponding neighborhoods of agents around some buyer, and vice versa. Then under a pivot-symmetric bargaining function the, match partner and payoffs of the agents are interchanged in a pivotal fashion as well. This simply means that it is immaterial which side of the market is labeled “sellers” and which side is labeled “buyers” as long as for a match to be successful one needs the input of one agent from each side, and Definition 5 then says that the way payoffs are split should reflect this independence of labels.

6.3 Characterization of expected surpluses

Let \( W^T_S(f \times g) \) and \( W^T_B(f \times g) \) denote the expected payoffs to sellers and buyers if utility is transferable and match surplus is split according to \( g \), where \( f \times g \) is a stable, symmetric, and measurable bargaining functions. We can then state a condition under which agents receive the same expected surplus in the transferable utility case as they do under the equal-split rule:

Theorem 6. Let \( f \times g \) be any stable, symmetric, and measurable bargaining function, and suppose \( p = q \). If the match production function \( u \) is convex, the ex-ante expected payoffs in the matching market satisfy

\[
W^T_S(f \times g) = W^T_B(f \times g) = W^{NT}(1/2) = W^{NT}(1/2).
\]

Note that the bargaining function from section 4, defined through \( \phi \) and \( \psi \), is symmetric and measurable. For both \( \phi \) and \( \psi \), an agent’s location matters only in so far as its relation to other agents is concerned, hence symmetry is satisfied. Measurability is also assured:
For $\varphi$ this is obvious if $c > 0$. For $\vartheta$, take \((i, j) \in Z\) such that \((i, j)\) is not the spanning match of any cluster $C \in \mathcal{C}$. Then the values $\vartheta(i, j) = (V_S(i), V_B(j))$ depend only on $\pi(s(C'))$ and $\overline{\pi}(s(C'))$, where $C'$ is the cluster in $\mathcal{C}$ containing $(i, j)$. Since $c > 0$ all clusters in $\mathcal{C}$ are of size $\hat{d} < \infty$ or less. Now if $(i, j) \in Z$ is the spanning match of some cluster $C \in \mathcal{C}$, the values $V_S(i)$ and $V_B(j)$ may depend on $x_{ip}$ and $y_{jp}$ where $(ip, jp)$ is the pseudo-span of $C$, given by (4)–(5). However, by (6) this requires that $v(j - ip) > 0$ or $v(jp - i) > 0$, and for this to be the case it is necessary that $j - ip \leq \hat{d}$ and $jp - i \leq \hat{d}$.

Now let $W_T(f \times g)$ denote the expected surplus of a seller under bargaining function $f \times g$. Define the expected surplusses for buyers in the same way. We then have the following relationship between expected surplusses with transferable utility and with non-transferable utility:

**Theorem 7.** Consider the case of convex $u$ and $c > 0$. For the bargaining function $f \times g$ constructed in section 4, the following holds: There exists a mapping $\chi : (0, 1)^2 \rightarrow (0, 1)$ such that

$$W_T^S(f \times g) = W_{NT}^S(\chi(p, q)) \quad \text{and} \quad W_T^B(f \times g) = W_{NT}^B(\chi(p, q)).$$

Furthermore, $\chi$ is continuous, strictly decreasing in its first argument, strictly increasing in its second argument, and satisfies $\chi(p, q) = 1 - \chi(q, p)$.

The claims in Theorem 7 are intuitive. In particular, a stochastic increase in the supply of goods (in terms of a higher value for $p$) lowers the expected surplus of sellers relative to buyers. Similarly a stochastic increase in demand lowers the expected surplus of buyers relative to sellers.

### 7 Conclusion

We have provided a fairly extensive characterization of market outcomes in a simple spatial matching market. Allowing trade to take place in both directions instead of only one would add realism to the model, but would also complicate the analysis of the market considerably. In particular, the characterization of stable matches in terms of nested clusters, and of match probabilities in terms of the Catalan number sequence, would no longer hold. Even with the assumption of uni-directional trade, however, the model gives rise to interesting applications and extensions. One we are currently pursuing in a companion paper (Klumpp [2004]) is to add a “networking stage” to the model. Suppose sellers and buyers have to know each other in order to trade, that is, they must be linked in a social network. Suppose further that these networks are costly and must be established prior to the realization of the supply and demand shocks $x$ and $y$. This extension of our model is best thought of in terms of a job networking problem: With some probability a worker may become unemployed and be looking for a job. Similarly, with some probability a firm may have a job opening. Investing in social relationships to other agents will then provide access to trade opportunities should a worker be unemployed are should a firm have an opening. One can show that agents invest in regular networks, that is, they will be linked to their $k$ nearest neighbors. The characterization of match probabilities and expected
match surpluses we provide in this paper can then be used to calculate the expected value of any such network. These computations, in turn, can be used to investigate the size of the equilibrium network that maximizes for each agent the difference between the expected network value and its cost.

Appendix: Proofs

We will make use of the following property of convex functions:

**Lemma 7.** Let $h : \mathbb{R} \to \mathbb{R}$ be decreasing and convex. If $x, y > z$ then $h(x) + h(y) - h(z)$, and if $x > z > y$ then $h(x) + h(y) - h(z)$. 

**Proof.** For $a_0 = x, b_0 = z, c_0 = z$, we have

$$h(a_0 + b_0 - c_0) = h(x) = h(a_0) + h(b_0) - h(c_0). \tag{12}$$

Let $\Delta = y - z > (\leq) 0$, and suppose we increase (decrease) $b_0$ from $z$ to $z + \Delta = y$. This increases (decreases) $a_0 + b_0 - c_0$ from $x$ to $x + \Delta$. Since $h$ is decreasing, both sides in (12) decrease (increase). Since $x > z$ and $h$ convex, the right-hand side decreases (increases) by more than the left-hand side. \hfill \square

**Proof of Theorem 1**

To prove Theorem 1, fix $(x, y)$, and suppose that $M^{NT}$ is a stable match assignment and $(M^T, V^T)$ is a stable value assignment. We need to show that $\forall (i, j) \in D, (i, j) \in M^{NT} \iff (i, j) \in M^T$. The proof is by induction on the match distance $d = j - i$. Notice that if $u$ is convex so is $v$.

**Step 1.** Let $d \in \{1, \ldots, d-1\}$ and assume that $\forall (i, j) \in D, (i, j) \in M^{NT} \iff (i, j) \in M^T$ (the induction hypothesis). We show that $\forall i \in \mathbb{Z}, (i, i+d+1) \in M^{NT} \iff (i, i+d+1) \in M^T$.

To prove necessity, suppose $(i, i+d+1) \notin M^{NT}$. Then either $x_i + y_{i+d+1} < 2$ and thus $(i, i+d+1) \notin M^T$, or (by construction of $\varphi$) at least one of the following holds: $(i, k) \in M^{NT}$ for some $i < k < i+d+1$ or $(k', i+d+1) \in M^{NT}$ for some $i < k' < i+d+1$. Then by the induction hypothesis $(i, k) \in M^T$ or $(k', i+d+1) \in M^T$ or both, and since each agent can be in at most one match, $(i, i+d+1) \notin M^T$.

To show sufficiency, we first prove that $(i, i+d+1) \in M^{NT}$ implies that $(i, k) \in M^T$ for some $k$ and $(k', i+d+1) \in M^T$ for some $k'$, and then that $k = i + d + 1$ and $k' = i$.

**Step 1a.** Take any $(i, i+d+1) \in M^{NT}$. If $(i, i+d+1) \in M^T$ then we are done. If $(i, i+d+1) \notin M^T$ at least one of the following must hold in order for $(i, i+d+1)$ not to block $(M^T, V^T)$: (i) $(i, k) \in M^T$ for some $k > i$, or (ii) $(k', j) \in M^T$ for some $k' < i + d + 1$. We show that both (i) and (ii) hold. Suppose (i) is true. By the induction hypothesis we can rule out $k < i + d + 1$: If $(i, k) \in M^T$ for $i < k < i + d + 1$ then $(i, k) \in M^{NT}$ and therefore $(i, i+d+1) \notin M^{NT}$ since each agent can be in at most one match. Hence $k \geq i + d + 1$. If $k = i + d + 1$ then $(i, i+d+1) \in M^T$ and we are
done, so suppose \( k > i + d + 1 \). Then \( V_S^T(i) \leq v(k - i) < v(d + 1) \). This implies that \( V_B^T(i + d + 1) \geq v(d + 1) - V_S^T(i) > 0 \), for otherwise \( (i, i + 1) \) would block \((M^T, V^T)\). But \( V_B^T(i + d + 1) > 0 \) implies \( (k', i + d + 1) \in M^T \) for some \( k' < i + d + 1 \), so (ii) must be true as well, and invoking the induction hypothesis shows that \( k' < i \). A parallel argument establishes that (ii) \( \Rightarrow \) (i).

**Step 1b.** Now suppose that \((i, k) \in M^T \) and \((k', i + d + 1) \in M^T \) but \( k > i + d + 1 \) and \( k' < i \). It must be that \( x_{k'} = y_k = 1 \). The sum of utilities obtained by \( k \) and \( k' \) can be written as

\[
V_S^T(k') + V_B^T(k) = v(i + d + 1 - k') + v(k - i) - V_B^T(i + d + 1) - V_S^T(i) \tag{13}
\]

For \((i, i + d + 1)\) not to block \((M^T, V^T)\), it is necessary that

\[
v(d + 1) - V_B^T(i + d + 1) \leq V_S^T(i) \tag{14}
\]

Substituting (14) into (13) yields

\[
V_S^T(k') + V_B^T(k) \leq v(i + d + 1 - k') + v(k - i) - v(d + 1). \tag{15}
\]

Then if \( v \) is convex \( v(i + d + 1 - k') + v(k - i) - v(d + 1) < v(k - k') \) by Lemma 7, which together with (15) implies that \((k', k)\) blocks \((M^T, V^T)\). Thus \((M^T, V^T)\) cannot be stable, a contradiction, and therefore \((i, i + d + 1) \in M^{NT} \Rightarrow (i, i + d + 1) \in M^T \).

**Step 2.** We now prove that \( \forall i \in \mathbb{Z}, (i, i + 1) \in M^{NT} \iff (i, i + 1) \in M^T \). Necessity is obvious given the construction of \( \phi \) above: If \((i, i + 1) \notin M^{NT} \), then it must be true that \( x_i = 0 \) or \( y_{i+1} = 0 \) or both, and so \((i, i + 1) \notin M^T \). To show sufficiency, we first prove that \((i, i + 1) \in M^{NT} \) implies that \((i, k) \in M^T \) for some \( k \) and \((k', j) \in M^T \) for some \( k' \), and then that \( k = i + 1 \) and \( k' = i \).

**Step 2a.** Take any \((i, i + 1) \in M^{NT} \). If \((i, i + 1) \in M^T \), we are done. If \((i, i + 1) \notin M^T \), then at least one of the following must hold in order for \((i, i + 1) \) not to block \((M^T, V^T): (i) \ (i, k) \in M^T \) for \( k > i + 1 \), or (ii) \((k', j) \in M^T \) for \( k' < i \). We show that (i) and (ii) both hold. If (i) is true then \( V_S^T(i) \leq v(k - i) < v(1) \), and this implies that \( V_B^T(i + 1) > 0 \) for otherwise \((i, i + 1) \) would block \((M^T, V^T)\). But \( V_B^T(i + 1) > 0 \) implies \((k', i + 1) \in M^T \) for some \( k' < i + 1 \), and since \( k' \neq i \), \( k' < i \), so (ii) must be true as well. A parallel argument shows that (ii) \( \Rightarrow \) (i).

**Step 2b.** This step is identical to step 1a with \( d = 0 \). Since step 1a did not rely on the induction hypothesis, it follows that \((i, i + 1) \in M^{NT} \Rightarrow (i, i + 1) \in M^T \).

**Proof of Theorem 3**

Recall that \((i, j) \notin M^{NT} \subset D \) if \( j - i > \hat{d} \), so to prove part (i) simply observe if \((i, j) \in M^{NT} \) and \((i', j') \in M^{NT} \), then \((i, j) \) and \((i', j') \) are not overlapping by (1). It follows \( M^{NT} \) can be partitioned into a family of nested clusters of size \( \hat{d} \) or less. By Theorem 1, if \( u \) is convex the same holds for \( M^T \).
If $u$ is not convex, then it is possible that $M^T \neq M^{NT}$, in particular $M^T$ may contain overlapping matches (see example 1). To prove (ii), therefore, we demonstrate that there exists a constant $\hat{m} \geq 1$ such that no sequence \{(it, jt)\}_{t=0,1,\ldots,m} \subset M^T$ exists for which $m > \hat{m}$ and $i_t < i_{t+1} < j_t < j_{t+1}$, $0 \leq t < m$. Letting $\eta = \hat{m}d$, it follows that $M^T$ can be partitioned into a family of clusters of size $\eta$ or less.

Since $c > 0$, $j_t - i_t \leq 0 < \delta \forall t$; this implies that there exists a constant $\epsilon < 0$ such that $v(j_t - i_t) - v(j_{t-1} - i_t) \leq \epsilon \forall m$. Because $M^T$ consists of stable matches under transferable utility, it must be the case that

$$V_S^T(i_1) \geq v(j_0 - i_1) - V_B^T(j_0),$$

and thus

$$V_B^T(j_1) = v(j_1 - i_1) - V_S^T(i_1) \leq v(j_1 - i_1) - v(j_0 - i_1) + V_B^T(j_0). \quad (16)$$

Also by stability

$$V_S^T(i_2) \geq v(j_1 - i_2) - V_B^T(j_1) \geq v(j_1 - i_2) - v(j_1 - i_1) + v(j_0 - i_1) - V_B^T(j_0),$$

where the second inequality is by (16). Thus

$$V_B^T(j_2) = v(j_2 - i_2) - V_S^T(i_2) \leq v(j_2 - i_2) - v(j_1 - i_2) + v(j_1 - i_1) - v(j_0 - i_1) + V_B^T(j_0).$$

Continuing in this fashion we can write

$$V_B^T(j_m) \leq \sum_{t=1}^{m}[v(j_t - i_t) - v(j_{t-1} - i_t)] + V_B^T(j_0) \leq m\epsilon + V_B^T(j_0).$$

Since $\epsilon < 0$ there exists $\hat{m} \geq 1$ such that $V_B^T(j_m) < 0 \forall m > \hat{m}$, and thus $(i_m, j_m) \notin M^T$. \hfill \qed

**Proof of Lemma 2**

Since $(i'', j'')$ is nested in $(i', j')$ we have $i' < i'' < j'' < j'$. Without loss of generality, assume $(i, j)$ is to the left of $(i', j')$, i.e. $i < j \leq i' < i'' < j'' < j'$. We need to verify that $V_S(i) + V_B(j'') \geq v(j'' - i)$. Since $(i', j')$ and $(i, j)$ are relatively stable, $V_S(i) + V_B(j') \geq v(j' - i)$. Since $(i'', j'')$ and $(i', j')$ are relatively stable, $V_S(i') + V_B(j'') \geq v(j'' - i')$. Hence

$$V_S(i) + V_B(j'') \geq v(j' - i) - V_B(j') + v(j'' - i') - V_S(i')$$

$$= v(j' - i) - v(j'' - i') + v(j' - i') > v(j'' - i),$$

where the second inequality follows from Lemma 7. \hfill \qed
Proof of Lemma 3

Throughout the proof, for any \((i, j)\) we will use the notation \(V^s_B(j) = \bar{w}_B(j)\) and \(V^s_B(i) = v(j - i) - \bar{w}_B(j)\), and \(V^b_s(i) = \bar{w}_s(i)\) and \(V^b_B(j) = v(j - i) - \bar{w}_s(i)\) (\(\bar{w}_s(i)\) and \(\bar{w}_B(j)\) are defined in (2) and depend on the parent node of \((i, j)\)). If \((i, j)\) is assigned values \((V_s(i), V_B(j))\) by (3), then these can be written as \(V_s(i) = \frac{1}{2}(V^b_s(i) + V^b_B(i))\) and \(V_s(j) = \frac{1}{2}(V^b_B(j) + V^b_B(j))\).

Proof of property (i). The proof is in two steps. In Step 1 we show that if \(C' \in R(C_0)\), then \(s(C')\) and \(s(C_0)\) are relatively stable. In Step 2 we show that if \(C'\) is a nested subcluster of \(C_0\) such that \(s(C')\) and \(s(C_0)\) are relatively stable, then for every \(C'' \in R(C')\), \(s(C'')\) and \(s(C_0)\) are relatively stable. The two results together imply property (i).

Step 1. Take \(C' \in R(C_0)\), let \((i, j) = s(C_0)\) and \((i', j') = s(C')\). Note that \(i < i' < j' < j\). We have \(V_s(i) + V^b_B(j') = v(j' - i)\) by definition of \(\bar{w}_B(j')\)

\[V^s_s(i') + V_B(j) = v(j' - i') - v(j' - i) + V_s(i) + V_B(j) = v(j' - i') - v(j' - i) + v(j - i) > v(j - i')\]  

by Lemma 7. A parallel argument shows that \(V^b_s(i') + V^b_B(j) = v(j - i')\) and \(V^s(i) + V^b_B(j) > v(j' - i')\) by applying the same steps as above. Since \(V^s(i') = \frac{1}{2}(V^b_s(i') + V^b_B(i'))\) we have

\[V_s(i') + V_B(j) = \frac{1}{2}(V^b_s(i') + V_B(j)) + \frac{1}{2}(V^b_B(i') + V_B(j)) > v(j - i').\]

Similarly \(V^b_B(j') = \frac{1}{2}(V^b_B(j') + V^b_B(j')\) and therefore \(V_s(i) + V_B(j') > v(j - i')\). Thus \((i, j)\) and \((i', j')\) are relatively stable.

Step 2. Take any nested subcluster \(C'\) of \(C_0\). Denote \((i, j) = s(C_0)\), \((i', j') = s(C')\), and \((i'', j'') = s(C'')\) for \(C'' \in R(C')\). Note that \(i < i' < i'' < j'' < j' < j\). If \((i', j')\) and \((i, j)\) are relatively stable, \(V_s(i) + V_B(j') \geq v(j' - i)\) and \(V_s(i') + V_B(j) \geq v(j - i')\). Then

\[V^s_B(i''') = V_s(i) + v(j'' - i') - V_s(i') = V_s(i) + v(j'' - i') - v(j' - i') + V_B(j') \geq v(j' - i') - v(j' - i') + v(j'' - i') > v(j'' - i),\]

where the first inequality is because \((i', j')\) and \((i, j)\) are relatively stable, and the second is by Lemma 7. Similarly

\[V^s(i''') + V^b_B(j) = V_B(j) + v(j'' - i') - v(j'' - i') + V^b_B(i') \geq v(j - i') + v(j'' - i') - v(j'' - i') > v(j - i').\]

Using the values \(V^b_s(i'')\) and \(V^b_B(j'')\) in a parallel argument and proceeding analogous to Step 1 shows that \((i, j)\) and \((i'', j'')\) are relatively stable. \(\square\)

Proof of property (ii). Take \(C', C'' \in R(C_0)\), let \((i, j) = s(C_0)\), \((i', j') = s(C')\), \((i'', j'') = s(C'')\). Note that \(s(C')\) and \(s(C'')\) are side-by-side matches, so without loss of generality assume \(C'\) is to the left of \(C'': i < i' < j' \leq i'' < j'' < j\). By Lemma 7

\[V^s_B(i') + V^b_B(j'') = v(j' - i') - v(j' - i) + V_s(i) + v(j'' - i) - V_s(i) = v(j' - i') - v(j' - i) + v(j'' - i) > v(j'' - i').\]
A parallel argument applies using the values $V_S^k(i')$, etc. Hence proceeding as in the proof of property (i) and (ii), we have $(i', j')$ and $(i'', j'')$ relatively stable.

**Proof of Theorem 4**

We first examine whether any match involving an unmatched agent can block $(M, V^M)$. Let $i'$ be any unmatched seller with $x_{i'} = 1$, and let $j'$ be any unmatched buyer with $y_{j'} = 1$. By construction of the set $M$, it follows that $v(j' - i') < 0$, so $(i', j')$ does not block $(M, V^M)$. We now show that $(M, V^M)$ is also not blocked by $(i', j)$ or $(i, j')$ for some $(i, j) \in M$. Let $C \in \mathcal{C}$ be the nested cluster containing $(i, j)$. Analogous to property (i) of Lemma 3 it can be shown that if $(i^p, j^p)$ is the pseudo-span of $C \in \mathcal{C}$, then $v(j - i^p) < V_j$ and $v(j^p - i) < V_i$. By construction of $i^p$ and $j^p$, it follows that $v(j - i') \leq v(j - i^p)$ and $v(j' - i') \leq v(j^p - i)$, and therefore $(M, V^M)$ cannot be blocked by $(i', j)$ or $(i, j')$.

What remains to be shown is that all $(i, j), (i', j') \in M$ are relatively stable. Let $C \in \mathcal{C}$ be the nested cluster containing $(i, j)$ and $C' \in \mathcal{C}$ be the nested cluster containing $(i', j')$. If $C = C'$, then by Lemma 4 $(i, j)$ and $(i', j')$ are relatively stable; hence assume $C \neq C'$. We show that $s(C)$ and $s(C')$ are relatively stable; then by Lemma 2 global stability follows.

Note that $(i, j)$ and $(i', j')$ are side-by-side matches, and without loss of generality assume $i < j \leq i' < j'$. Hence we need to show that $V_S(i) + V_B(j') \geq v(j' - i')$. We denote by $(i^p, j^p)$ the pseudo-span of $C$ and by $(i'^p, j'^p)$ the pseudo-span of $C'$. Regarding the order of the four pairs $(i, j), (i', j'), (i^p, j^p)$, and $(i'^p, j'^p)$, there are five mutually exhaustive cases to consider:

**Case 1.** The pseudo-spans are side-by-side, i.e. $i^p < i < j \leq j^p \leq i'^p < i' < j' < j'^p$. Since $v(j' - i^p) \leq V_B(j')$ and $i < i^p$, it must be that $v(j' - i) \leq V_B(j') \leq V_S(i) + V_B(j')$.

**Case 2.** The pseudo-spans overlap, i.e. $i^p < i < j \leq i^p < j^p \leq i' < j' < j'^p$. As in case 1, $v(j' - i^p) \leq V_B(j')$ and $i < i^p$ imply that $v(j' - i) \leq V_B(j') < V_S(i) + V_B(j')$.

**Case 3.** The buyer in both pseudo-spans is identical, i.e. $i^p < i < j \leq i^p \leq i' < j' < j^p$. As in case 1, $v(j' - i^p) \leq V_B(j')$ and $i < i^p$ imply that $v(j' - i) \leq V_B(j') < V_S(i) + V_B(j')$.

**Case 4.** The seller in both pseudo-spans is identical, i.e. $i^p = i'^p < i < j \leq j^p \leq i' < j' < j'^p$. Since $v(j^p - i) \leq V_S(i)$ and $j' > i^p$, it must be that $v(j' - i) \leq V_S(i) + V_B(j')$.

**Case 5.** The buyer and the seller in both pseudo-spans are identical, i.e. $i^p = i'^p < i < j \leq i' < j' < j^p = j'^p$. The proof is analogous to property (ii) of Lemma 3.

**Proof of Theorem 6**

By Definition 4 it follows that $W^T_S(f \times g)$ and $W^T_B(f \times f)$ are independent of the identity of the seller and buyer, so without loss of generality focus on seller $i = 0$. Let $\xi : \mathcal{P} \rightarrow [0, 1]$ be the product measure on $\mathcal{P}$: $\xi(\emptyset) = 0$ and $\xi(X^2) = 1$, and if $H \subseteq (\{0, 1\}^{2n})^2$ (some $n$)
and $A = \{(x, y) : (\bar{x}(i-n, i+n), \bar{y}(i-n, i+n)) \in H\}$, then
\[
\xi(A) = \sum_{(\bar{x}, \bar{y}) \in H} \prod_{i=n}^{i+n} [\bar{x}_i p + (1 - \bar{x}_i)(1 - p)] \times [\bar{y}_i q + (1 - \bar{y}_i)(1 - q)].
\]
\(\xi(A)\) is then the probability of event $A \in \mathcal{P}$, and it can be shown that $\xi$ is finitely additive (in fact countably additive; see Billingsley [1995], Theorem 2.3).

Suppose now that $u$ is convex. Given a stable bargaining function $f \times g$ (whose first component coincides then with $f$ under non-transferable utility), an integer $d > 0$, and numbers $0 < a < b < 1$, define
\[
g(d, a, b) = \{(x, y) \in X^2 : (0, d) \in f(x, y) \text{ and } g_y^S(x, y) \in [av(d), bv(d)]\}.
\]
Since $f \times g$ is measurable, $g(d, a, b) \in \mathcal{P}$. In particular, observe that $\xi(g(d, 0, 1)) = \mu(d)$. Let
\[
w_S(d) = \mu(d)E[g^S_y(x, y) | \varphi_{x,y}(0, d) = 1]
\]
denote the part of the expected value $W^T_S(f \times g)$ that is generated for seller 0 by trading with buyer $d$. If $r$ is a positive integer, then by the monotone convergence theorem we have
\[
w_S(d) = \lim_{r \to \infty} v(d) \sum_{t=1}^{r} \left(1 - \frac{1}{r}\right) \xi(g(d, 1 - \frac{t}{r}, 1 - \frac{t}{r})).
\]
Now extend the transformation $\pi$ defined in Section 6.2 to subsets of $A \subset X^2$, i.e.
\[
\pi_{i,j}(A) = \{\pi_{i,j}(x, y) \in X^2 : (x, y) \in A\}.
\]
Observe that if $p = q$, then for any $A \in \mathcal{P}$ we have $\xi(A) = \xi(\pi_{i,j}(A))$. Furthermore, it follows from the construction of $\varphi$ that
\[
\varphi_{x,y}(i, j) \Leftrightarrow \varphi_{\pi_{i,j}(x,y)}(i, j).
\]
Thus, if $f \times g$ satisfies Definition 5, then $\xi(g(d, a, b)) = \xi(g(d, 1 - b, 1 - a))$. Making this replacement in (19) yields
\[
w_S(d) = \lim_{r \to \infty} v(d) \sum_{t=1}^{r} \left(1 - \frac{t}{r}\right) \xi(g(d, 1 - \frac{t}{r}, 1 - \frac{t}{r})).
\]
But this is just $\mu(d)v(d) - W_S(d)$ by (18) and $\xi$ finitely additive, so it must be that $w_S(d) = \frac{1}{2} \mu(d)v(d)$. Since $d$ was arbitrary, we immediately obtain the result:
\[
W^T_S(f \times g) = \sum_{d=1}^{\hat{d}} w_S(d) = \frac{1}{2} \sum_{d=1}^{\hat{d}} \mu(d)v(d) = W^{N^T}_S(1/2),
\]
and similarly for the buyer’s value. □
Proof of Theorem 7

The fact that some function $\xi : (0,1)^2 \rightarrow (0,1)$ exists that satisfies the equality in the theorem is obvious: Since with convex $u$ the same matches form under fixed sharing rules and bargaining, the same total surpluses are created. Hence, any expected surplus under bargaining has to be equal to some fraction of the total surplus. Symmetry is also obvious, given the symmetric construction of values in (2)–(7). We now establish the monotonicity property.

For a given pair $(i_0, j_0) \in D$, define a family of sets of matches

$$
L(i_0, j_0) = \left\{ \{ (i_1, j_1), \ldots, (i_m, j_m) \} \subseteq D : 
2 \leq m \leq \left\lfloor \frac{d - (j - i)}{2} \right\rfloor, i_0 > i_1 > \ldots > i_m, j_0 < j_1 < \ldots < j_m \right\}.
$$

If $(i_0, j_0)$ is not the spanning match of some $C \in \mathcal{C}$, then it will be the spanning match of a nested subcluster of some $C \in \mathcal{C}$. Recall that the matches in $C$ can be represented as nodes in a tree, and the family $L(i_0, j_0)$ contains all possible paths that could connect the node $(i_0, j_0)$ with the initial node $s(C)$. Let

$$
\ell = ( (i_0, j_0), \ldots, (i_m, j_m) )
$$

represent a typical element of $L(i_0, j_0)$. From $\ell$ one can compute the vector

$$
d = (d_0, \ldots, d_m) = (j_0 - i_0, \ldots, j_m - i_m)
$$

that represents the distances between the seller and buyer in each of the matches in $\ell$. Note that a given $d$ can be derived from various chains $\ell$. Conditioning on $d$, we can write the expected surplus of seller $i_0$ under the bargaining function constructed in section 4 as follows:

$$
E[V_S(i_0, j_0)|d_0, d_1, \ldots] = \frac{1}{2} v(d_0) + \frac{1}{2} \sum_{k=1}^{d_1 - d_0 - 1} P_r[i_0 - i_1 = k]
\times \{ v(d_1 - k) - E[V_B(i_1, j_1)|d_1, d_2, \ldots] - v(d_0 + k) + E[V_S(i_1, j_1)|d_1, d_2, \ldots]\}.
$$

To understand (20), observe that in (3) the values assigned to a match depend on the values of the parent match in its nested cluster as well as on the location of the match within its parent. Thus, what we have done in (20) is simply to sum over all possible locations of the match $(i_0, j_0)$ within its parent match $(i_1, j_1)$ and compute the induced values for the agents in match $(i_0, j_0)$. Notice that we have used the fact that the values assigned to the outer match $(i_1, j_1)$ do not depend on the inner match $(i_0, j_0)$.

Now observe first that $P_r[i_0 - i_1 = k] = P_r[j_1 - j_0 = k]$. This follows from the fact that for each $\pi(i_1, j_1)$ and $\bar{\pi}(i_1, j_1)$, there is a “mirror image” $\pi' = (y_{j_1}, y_{j_1-1}, \ldots, y_{i_1+1})$ and
\( y_j = (x_{j_1-1}, x_{j_1-2}, \ldots, y_{i_1}) \). If a pair of agents with distance \( d_0 \) trades in the configuration \( (\pi, \gamma) \), then there will be a pair if agents with this distance trading in configuration \( (\pi', \gamma') \), and the distance between the buyer in this new match and \( j_1 \) will be the same as the distance between the seller in the old match and \( i_1 \). Furthermore, since there are as many positive entries in \( \pi(i_1, j_1) \) as there are in \( \gamma(i_1, j_1) \) (otherwise \( (i_1, j_1) \) would not match), there is exactly this number number of positive entries in \( \pi' \) and \( \gamma' \), so observing the pair of configurations \( (\pi', \gamma') \) is exactly as likely as observing the original pair.

Further observe that \( j_1 - j_0 = k \) implies \( i_0 - i_1 = d_1 - d_0 - k \) and vice versa. Hence we can express (20) as

\[
E[V_S(i_0, j_0)|d_0, d_1, \ldots] = \frac{1}{2}v(d_0) + \frac{1}{2} \sum_{k=1}^{d_1-d_0-1} Pr[i_0 - i_1 = k] \times \{-E[V_B(i_1, j_1)|d_1, d_2, \ldots] + E[V_S(i_1, j_1)|d_1, d_2, \ldots]\}
= \frac{1}{2}v(d_0) + \frac{1}{2}(E[V_S(i_1, j_1)|d_1, d_2, \ldots] - E[V_B(i_1, j_1)|d_1, d_2, \ldots])
= \frac{1}{2}v(d_0) - \frac{1}{2}v(d_1) + E[V_S(i_1, j_1)|d_1, d_2, \ldots]. \tag{21}
\]

Since \( E[V_S(i_1, j_1)|d_1, d_2, \ldots] \) can be expressed similarly as

\[
E[V_S(i_1, j_1)|d_1, d_2, \ldots] = \frac{1}{2}v(d_1) - \frac{1}{2}v(d_2) + E[V_S(i_2, j_2)|d_2, d_3, \ldots],
\]

and so on, we can write

\[
E[V_S(i_0, j_0)|d_0, d_m] = \frac{1}{2}(v(d_0) - v(d_m)) + E[V_S(i_m, j_m)|i_m, j_m] = s(C), \ C \in \mathcal{C}. \tag{22}
\]

(22) expresses the value of any seller, conditional on the seller belonging to a match of length \( d_0 \), and this match belonging to a nested cluster whose spanning match is of length \( d_m \). Now let \( A_S(d, p, q) = E[V_S(i, i + d)|i, i + d] = s(C), \ C \in \mathcal{C}, p, q \); and let \( A_B(d, p, q) \) denote the corresponding expectation for a buyer. Proving that \( \xi(p, q) \) decreases in \( p \) and increases in \( q \) hence amounts to showing that \( A_S(d, p, q) \) decreases in \( p \) and increases in \( q \) for every \( d \).

Define \( f_{S}^{p,q} : \{1, 2, \ldots \rightarrow [0, 1] \} \) by

\[
f_{S}^{p,q}(z) = Pr[i - i^p = z|(i, i+d) = s(C), C \in \mathcal{C}, p, q].
\]

Define \( f_{B}^{p,q} \) similarly by

\[
f_{B}^{p,q}(z) = Pr[j^p - (i + d) = z|(i, i+d) = s(C), C \in \mathcal{C}, p, q].
\]

Notice that for every \( q, p \) and \( p' > p \), \( f_{S}^{p,q} \) first-order stochastically dominates \( f_{S}^{p',q} \) and \( f_{B}^{p,q} \) dominates \( f_{B}^{p,q} \). Similarly, for every \( p, q \) and \( q' > q \), \( f_{S}^{p,q} \) dominates \( f_{S}^{p,q} \) and \( f_{B}^{p,q} \) dominates \( f_{B}^{p,q} \). Let \( (i, j) \) be the spanning match of some nested cluster. The the seller’s value in increases in the distance \( i - i^p \) and decreases in \( j^p - j \), and the buyer’s value increases in \( j^p - j \) and decreases in \( i - i^p \). Therefore \( A_S(d, p, q) \) decreases in \( p \) and increases in \( q \).
References


