A Note on Budget Balance under Interim Participation Constraints: The Case of Independent Types*

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Abstract

We provide a simple proof of the equivalence between ex ante and ex post budget balance constraints in Bayesian mechanism design with independent types when participation decisions are made at the interim stage. The result is given an interpretation in terms of efficient allocation of risk.

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1 Introduction

Resource constraints are important for many applications of mechanism design. Many authors require that resources are balanced in every state of the world, a constraint that we henceforth refer to as *ex post budget balance*. Others require that the expected budget deficit is zero, which henceforth will be referred to as *ex ante budget balance*. Ex post budget balance implies ex ante budget balance, but the converse is obviously not true. However, there are many results in the mechanism design literature that show that it often does not matter whether budget balance constraints are imposed ex ante or ex post. One group of such results assert that under certain conditions:

Claim 1 *For any mechanism that satisfies Bayesian incentive compatibility, interim individual rationality and ex ante budget balance there exists another mechanism with the same allocation rule that satisfies Bayesian incentive compatibility, interim individual rationality and ex post budget balance.*

Examples of papers with results along the lines of Claim 1 are Myerson and Satterthwaite (1983), Cramton, Gibbons and Klemperer (1987), and Mailath and Postlewaite (1990). These papers consider rather special problems. Claim 1 is mainly used as a step to simplify the constraint set in a constrained efficiency problem.

A second group of results concern the relationship between Vickrey-Groves-Clarke mechanisms and the AGV mechanism discovered by Arrow (1979) and d'Aspremont and Gérard-Varet (1979). In many cases it is then true that:

Claim 2 *If there exists an ex ante budget balanced and interim individually rational Vickrey-Groves-Clarke mechanism then there exists an ex post budget balancing Bayesian incentive compatible and interim individually rational mechanism with the same allocation rule.*

Makowski and Mezzetti (1994) establishes conditions for when Claim 2 is true, and Krishna and Perry (1998) contains a closely related result. These papers assume independent types and pure private values, but otherwise the analysis is rather general.

Finally, d'Aspremont et al (2004) establishes that, under quite general assumptions:[1]

Claim 3 *For any mechanism that satisfies Bayesian incentive compatibility and ex ante budget balance there exists another mechanism with the same allocation rule that satisfies Bayesian incentive compatibility and ex post budget balance.*

Claim 1 has previously only been proved for a few special models. Claims 2 and 3 have been established more generally, but in each case the claim is somewhat weaker than Claim 1. In Claim 2 only efficient allocation rules are covered, and in Claim 3 individual rationality is not addressed.

[1] See Lemma 1 and Theorem 2 in d'Aspremont et al (2004).

The main contribution of this note is to show that, if types are independent, then the statement in Claim 1 is true under otherwise rather general circumstance. This more general equivalence between ex ante and ex post budget balance in Bayesian mechanism design seems to be known by many researchers, but, to our knowledge, the result has never been put on record in the literature. We also offer a proof that provides an interpretation of the result in terms of efficient, incentive compatible, and interim individually rational allocations of risk.

Our proof works as follows. Agents are risk neutral in transfers and therefore willing to insure the mechanism designer at actuarially fair rates, just like an outside insurance provider. The only problem is that the ex post budget deficit depends on the type profile and that agents are privately informed about their own types. But, given that types are independently drawn, this problem can be overcome by a scheme that designates one agent as the "primary insurer", who provides insurance against deficits that cannot be predicted by the type of this agent. Some other agent, a "secondary insurer", provides insurance for the part that can be predicted by the type of the primary insurer. The independence assumption is used twice in our proof, firstly to show that the incentive compatibility constraint of the primary insurer holds, and secondly to show that the individual rationality constraint of the secondary insurer hold.

The assumption of independent types is of course restrictive. We indicate in the last section of this paper one way in which the assumption can be relaxed. If there exist two agents with signals that are *conditionally* independent given any realization of the other players' types, then our result continues to hold except that the transformation of transfer payments may alter the interim expected utility of one agent by a constant. This generalizes the main result in Crémer and Riordan (1985) slightly.

The remainder of this paper is organized as follows. In Section 2 we consider an insurance problem with privately informed insurance providers. We then use the analysis of the insurance problem to prove our main result in Section 3 for the case that agents' signals are independent. In Section 4 we prove the extension with the weakened independence assumption.

2 The Insurance Problem

Consider a risk-averse agent who faces the possibility of an accident. If an accident occurs, he has expenditures E. If no accident occurs, he has expenditure 0. The agent seeks insurance from N risk neutral insurance providers. Each insurance provider $i \in I = \{1, 2, ..., N\}$ privately observes the realization of a random variable $\tilde{s}_i$ that provides some information about the true probability of an accident. For $i \in I$ denote by $s_i \in S_i$ a generic realization of the i-th random variable, and define $s \equiv (s_1, s_2, ..., s_N)$ and $S \equiv \times_{i=1}^{N} S_i$. The probability of an accident is a function $\pi(s)$ of s. Define $x(s) \equiv E\pi(s)$ to be the expected expenditure conditional on the signals observed being s. As an example we might think of Lloyd's of London where members jointly provide insurance to a client, and members have private information about the client's true accident risk.

If the agent can commit to an insurance procurement mechanism, then the revelation principle implies that it is without loss of generality that the agent invites the insurance providers to report their private information, and that the rule by which the expenditure is shared depends on the agents' reports. The agent compensates the insurance provides with an up-front payment.[2]

Definition 1 *An insurance contract is a pair* $\langle p, m \rangle$ *, where* $p \in R^N$ *and* $m : S \to R^N$*.*

We write $p = (p_1, p_2, ..., p_N)$ for what can be interpreted as up-front payments and adopt the convention that a positive p_i is a transfer to agent i. Similarly, we write $m(s) = (m_1(s), ..., m_N(s))$ and treat a positive value $m_i(s)$ as a transfer from i to the agent that seeks insurance. Note that $m_i(s)$ is the *expected* contribution by insurance provider i to the accident costs. The actual expenditure by insurance provider i will be $m_i(s)/\pi(s)$.

Definition 2 *An insurance contract* $\langle p, m \rangle$

- *provides full insurance if*
$$\sum_{i=1}^{N} m_i(s) = x(s)$$
for every $s \in S$*;*
- *is actuarially fair if*
$$\sum_{i=1}^{N} p_i = \mathrm{E}\left[\sum_{i=1}^{N} m_i(\tilde{s})\right];$$
- *is non-manipulable if*
$$\mathrm{E}[m_i(s_i, \tilde{s}_{-i})|s_i] \geq \mathrm{E}[m_i(s_i', \tilde{s}_{-i})|s_i]$$
for each $i \in I$ *and all* $s_i, s_i' \in S_i$*;*
- *is interim individually rational if*
$$p_i \geq \mathrm{E}[m_i(\tilde{s})|s_i]$$
for each $i \in I$ *and* $s_i \in S_i$*.*

Full insurance is in our setting with one risk averse insured and several risk neutral insurance providers an efficiency condition. Actuarial fairness reflects that all bargaining power is on the insured's side: he makes a take-it-or-leave-it offer to the insurance providers that grants no expected rents. Finally, incentive compatibility and individual rationality are standard incentive constraints.

Proposition 1 below asserts the existence of insurance contracts that have all properties listed in Definition 2. Proposition 1 relies on the following condition.

[2] There is no loss of generality in the decomposition of transfers between insurers and insured into an upfront payment and ex post compensation. This decomposition only helps to make the exposition more concrete.

(IND) *The random variables $\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_N$ are independent.*[3]

Proposition 1 *If (IND) holds and $N \geq 2$, then there exist insurance contracts that are interim individually rational, provide full insurance, are actuarially fair, and are non-manipulable.*

To understand the construction in the proof of Proposition 1 below, it is useful to first consider an insurance scheme where $p_i = \mathrm{E}[x(\tilde{s})]$, $m_i(s) = x(s)$, and where $p_j = m_j(s) = 0$ for all $j \neq i$ and $s \in S$. That is, agent i provides full insurance at an actuarially fair rate and all other agents are inactive. The problem is that, since $x(s)$ in general varies in s_i, individual rationality and non-manipulability will generally fail to hold. But, s_i and s_{-i} are independent, so the obvious solution to this problem is to pick some other agent to insure agent i against the variability that is due to variation in s_i.

To see this, consider the decomposition,

$$x(s) = \underbrace{x(s) - \mathrm{E}[x(s_i, \tilde{s}_{-i}) \mid s_i]}_{\substack{\text{Difference between realized and} \\ \text{and conditional expected value}}} + \mathrm{E}\left[x(s_i, \tilde{s}_{-i})|s_i\right].$$

The idea of the insurance contract constructed below is that some agent i pays for the first term in this decomposition, while the second term is paid by someone else. These two agents are compensated by appropriate up-front lump sum payments, and all other agents pay nothing.

Proof: Let $\langle p^*, m^* \rangle$ be given by,

$$\begin{aligned} p_i^* &= 0 \\ m_i^*(s) &= x(s) - \mathrm{E}[x(s_i, \tilde{s}_{-i})|s_i] \\ p_j^* &= \mathrm{E}[x(\tilde{s})] \\ m_j^*(s) &= \mathrm{E}[x(s_i, \tilde{s}_{-i})|s_i] \\ p_k^* &= m_k^*(s) = 0 \end{aligned}$$

for every $k \neq i, j$ and every $s \in S$. By adding $m_i^*(s)$ and $m_j^*(s)$ we see that $\langle p^*, m^* \rangle$ provides full insurance. Actuarial fairness is satisfied since

$$\mathrm{E}\left[\sum_{k=1}^{N} m_i^*(\tilde{s})\right] = \mathrm{E}\left[m_i^*(\tilde{s}) + m_j^*(\tilde{s})\right] = \mathrm{E}[x(\tilde{s})] = \sum_{k=1}^{N} p_k^*.$$

[3]If (IND) holds and S_i is discrete, non-manipulability is a consequence of interim individually rationality and actuarial fairness. In general, individual rationality and actuarial fairness guarantee that non-manipulability is satisfied for almost all pairs (s_i, s_i'), but may fail on a set of measure zero. Since non-manipulability fails if just a single type earns a higher payoff than all others, we need the definition in general.

Non-manipulability is trivially satisfied for each agent $k \neq i, j$, as well as for agent j, since $m_j^*(s)$ is independent of s_j. If agent i observes signal realization s_i but reports s_i', her expected payment is:

$$\mathrm{E}\left[m_i^*(s_i', \tilde{s}_{-i})|s_i\right] = \mathrm{E}[x(s_i', \tilde{s}_{-i})|s_i] - \mathrm{E}[x(s_i', \tilde{s}_{-i})|s_i'] = 0,$$

where the second equality follows from the independence assumption (IND). Hence, there is no incentive for i to manipulate the contract.

Interim individual rationality holds trivially for agents $k \neq i, j$, and the above calculation demonstrates that it is satisfied for agent i. Agent j 's interim expected payment is:

$$\mathrm{E}\left[m_j^*(s_j', \tilde{s}_{-j})|s_j\right] = \mathrm{E}\left[\mathrm{E}[x(\tilde{s})|\tilde{s}_i]|s_j\right] = \mathrm{E}[x(\tilde{s})] = p_j^*,$$

where the second equality follows by (IND). Hence, individual rationality is satisfied also for agent j.

Q.E.D.

The proof of Proposition 1 uses an insurance contract where one agent acts as a "primary insurer" and a second agent insures the primary insurer against the variation that can be predicted by the agent providing the primary insurance. Obviously this implies that:

Remark 1 *If (IND) holds, $N \geq 2$, and i, j is a distinct pair of agents, there exist interim individually rational, non-manipulable and actuarially fair full insurance contracts such that: 1) agent i is the only agent for which the payment depends on his type, and; 2) agents other than i and j make zero payments in every state of the world.*

One can also construct contracts in which the role of the primary insurer is divided arbitrarily between the N agents, and in which the role to provide secondary insurance for i is divided arbitrarily among the remaining $N-1$ agents. Formally, let $\alpha \in R^N$ be a vector which may be thought of as "primary insurance weights" in that we require that $\sum_i \alpha_i = 1$, but where any particular α_i can be any real number. In the same way, for each $i \in I$, let $\beta_i \in R^{N-1}$ be a vector of "secondary insurance weights", that is the shares in the insurance against the variation that can be predicted by i. As with the primary weight, $\sum_{j \neq i} \beta_{ij} = 1$, but a particular β_{ij} may be negative. Now, for each $i \in I$ and every $s \in S$ let:

$$\begin{aligned} p_i^* &= \sum_{j \neq i} \beta_{ji} \alpha_j \mathrm{E}[x(\tilde{s})] \\ m_i^*(s) &= \alpha_i \left(x(s) - \mathrm{E}[x(s_i, \tilde{s}_{-i})|s_i]\right) + \sum_{j \neq i} \beta_{ji} \alpha_j \left(\mathrm{E}[x(s_j, \tilde{s}_{-j})|s_j]\right) \end{aligned}$$

We leave it to the reader to verify that an insurance scheme of this form provides full insurance, and is actuarially fair, non-manipulable, and interim individually rational.

3 Ex Ante and Ex Post Budget Balance

Assume again that there are N agents $i \in I = \{1, 2, ..., N\}$. They have to choose one decision a from a set A of possible collective decisions. Each agent i privately receives a signal $\tilde{s}_i$ with realizations in S_i. We now interpret $\tilde{s}_i$ as agent i's type. Preferences are defined over A and a numéraire good called "money". Let $t_i \in R$ be the transfer of money from agent i. Each agent i has a quasi-linear von Neumann Morgenstern utility given by

$$u_i(a, s) - t_i$$

where $u_i : A \times S \to R$. As in the previous section, each agent observes s_i, but remains uninformed about s_{-i}. Finally, there is a resource constraint: implementing decision $a \in A$ costs $r(a, s) \in R$ units of the numéraire good. Note the generality of our model. Each agent i's signal potentially affects agent i's and other agents' preferences as well as the resource requirements.

Definition 3 *A* mechanism *is a pair $\langle f, t\rangle$, where*

- $f : S \to A$ *is the* allocation rule. *For every $s \in S$ the decision $f(s)$ is implemented when s is announced.*
- $t : S \to R^N$ *is the* payment rule. *We define $t(s) = (t_1(s), t_2(s), ..., t_N(s))$, and for every $i \in I$ and $s \in S$ the value $t_i(s)$ is the transfer from agent i when s is announced.*

By the revelation principle it is without loss of generality to restrict attention to truthful equilibria in direct mechanisms. For simplicity of notation, we only consider *pure* direct revelation mechanisms, that is, mechanisms that pick some alternative in A with probability 1, conditional on the agents' announcements, but our argument is extendable to the case of random decisions.

Definition 4 *A mechanism $\langle f, t\rangle$ is*

- ex post budget balanced *if, for each $s \in S$ we have:*

$$\sum_{i=1}^{N} t_i(s) = r(f(s), s);$$

- ex ante budget balanced *if*

$$\mathrm{E}\left[\sum_{i=1}^{N} t_i(s)\right] = \mathrm{E}[r(f(s), s)];$$

When budget balance is imposed as an ex ante constraint it is implicitly assumed that a risk neutral outside party, who is unaffected by the choice of allocation, is willing to provide insurance against the ex post deficit. In some applications it is not intuitively plausible that such an outside party exists. More generally, one might argue that if an insurer is available then this agent should be explicitly incorporated into the model. The set up with ex post budget constraint does not rule out the presence of such a risk neutral agent that acts as an insurer. In this sense, the set up with ex post budget constraint is generally applicable.

It now follows almost immediately from Proposition 1 that given an ex ante budget balancing mechanism there exists an ex post budget balancing mechanism with the same allocation rule for which interim expected payments are unchanged for all agents, assuming that all agents announce truthfully. That is:

Proposition 2 *Suppose (IND) holds and* $N \geq 2$. *For every ex ante budget balanced mechanism* $\langle f, t \rangle$ *there is an ex post budget balanced mechanism* $\langle \hat{f}, \hat{t} \rangle$ *such that:*

- *The allocation rule is unchanged:*
  $\hat{f}(s) = f(s)$ *for every* $s \in S$, *and ;*

- *The interim expected payments are unchanged for all agents:*
  $\mathrm{E}[\hat{t}_k(s'_k, \tilde{s}_{-k}) \mid s_k] = \mathrm{E}[t_k(s'_k, \tilde{s}_{-k}) \mid s_k]$ *for every* $k \in I$ *and all* $s_k, s'_k \in S_k$.

Proof: Consider a mechanism $\langle f, t \rangle$ that is ex ante budget balanced. For every state $s \in S$ define

$$x(s) = r(f(s), s) - \sum_{i=1}^{N} t_i(s).$$

That is, $x(s)$ is the ex post deficit under $\langle f, t \rangle$. Now let $\langle p, m \rangle$ be an insurance contract for this ex post deficit that has the properties listed in Proposition 1. Non-manipulability implies that for every $i \in I$ there exists some real number k_i such that $p_i - \mathrm{E}[m_i(s'_i, \tilde{s}_{-i})|s_i] = k_i$ for every $s_i, s'_i \in S_i$. To satisfy interim individual rationality, k_i must be weakly positive, and if any k_i were strictly larger than 0 a violation of actuarial fairness would be implied. We conclude that $p_i - \mathrm{E}[m_i(s'_i, \tilde{s}_{-i})|s_i] = 0$ for every $i \in I$ and every $s_i, s'_i \in S_i$. Consider the transfer scheme $\hat{t}$ defined as $\hat{t}_i(s) = t_i(s) - p_i + m_i(s)$ for every $i \in I$ and $s \in S$. Since $p_i = \mathrm{E}[m_i(s'_i, \tilde{s}_{-i})|s_i]$ for every $s_i, s'_i \in S_i$ it is immediate that $\mathrm{E}[\hat{t}_i(s'_i, \tilde{s}_{-i})|s_i] = \mathrm{E}[t_i(s'_i, \tilde{s}_{-i})|s_i]$ for every $s_i, s'_i \in S_i$. As the scheme provides full insurance, the mechanism with allocation rule f and transfer rule $\hat{t}$ is ex post budget balanced.

Q.E.D.

Proposition 2 does not require either incentive compatibility or individual rationality. However, if truth telling is a Bayesian equilibrium, the same holds true in the ex post budget balanced mechanism, and, if the original mechanism satisfied an interim individual rationality constraint for any set of agents agent, then the same will be true in the ex post budget balanced mechanism. Hence we have shown that:

Corollary 1 *Suppose (IND) holds and* $N \geq 2$. *Then, for every Bayesian incentive compatible, interim individually rational and ex ante budget balanced mechanism* $\langle f, t \rangle$ *there exists a Bayesian* *incentive compatible, interim individually rational and ex post budget balanced mechanism* $\langle \hat{f}, \hat{t} \rangle$ *such that* $f(s) = \widehat{f}(s)$ *for every* s.

Corollary 1 seems to be known by many researchers, but we are not aware of any paper in the literature that contains the result. Makowski and Mezzetti (1994) show that Corollary 1 holds for every *surplus maximizing* decision rules. In many settings, surplus maximizing rules are not implementable, either because of interactions with budget and participation constraints or because of interdependencies in values (Jehiel and Moldovanu, 2001). In such settings our result, that does not restrict attention to surplus maximizing rules, becomes important. If, for example, one wishes to characterize second best mechanisms, one needs a simple characterization of such mechanisms.[4]

Even closer to our result is Lemma 1 and Theorem 2 in d'Aspremont et al (2004), where a more general condition than (IND) is considered, but where individual rationality is not taken into consideration. This is proved using the exact same construction as in our proof, which is to add an incentive compatible insurance scheme to an existing mechanism. Incentive compatibility, but not individual rationality, is preserved under their condition. More precisely, they show that, in the absence of interim participation constraints, efficient decision rules can always be implemented if, for every real-valued function $r(s)$, there exists a transfer rule t such that $\sum_i t_i(s) = r(s)$ for every s and

$$\mathrm{E}\left[t_i(s_i, s_{-i}) \,|s_i\right] \geq \mathrm{E}\left[t_i\left(s_i', s_{-i}\right) |s_i\right]$$

for every agent i and every pair $s_i, s_i' \in S_i$. In our language this condition is simply saying that for every uncertain expenditure that the mechanism designer might have there exists a non-manipulable full insurance agreement between the agents and the mechanism designer. While the condition it is not expressed in terms of primitives, there are several known sufficient conditions, one being stochastic independence. Our contribution may thus be viewed as clarifying that this construction preserves interim incentives to participate under the assumption of independent types.[5]

Independence is arguably a very special case and the main objective in d'Aspremont et. al. (2004) is to establish conditions for when the results in Crémer and McLean (1985, 1988) can be extended, so that arbitrary decision rules can be implemented by incentive compatible and ex post budget balanced mechanisms. They offer a somewhat indirect condition in their Theorem 4, and Kosenok and Severinov (2004) have subsequently established a result that is more transparently expressed in terms of conditions on the agent's beliefs. Although these conditions rule out independent types, Kosenok and Severinov argue that, for fixed type spaces, the conditions are generically satisfied. Somewhat weaker earlier results along these lines may be found in Fudenberg et. al. (1995).

[4] Fang and Norman (2006, discussion of Corollary 1) use the result of this paper to prove the possibility for implementing *near* efficient outcomes in the case with private values.

[5] There are also some minor formal differences. We allow for interdependent values and infinite type spaces.

We close this section by observing that (see Remark 1) we may pick an insurance scheme with only two agents active when transforming the ex ante budget balanced mechanism to an ex post budget balanced mechanism. Since the added payments for one of these agents (the secondary insurer) depend only on the type of the primary insurer it follows that:

Remark 2 *If (IND) holds, $N \geq 2$, and $i, j \in I$ is an arbitrary (distinct) pair, then there exists an ex post budget balanced mechanism $\langle \hat{f}, \hat{t} \rangle$ satisfying the conditions in Proposition 2 that also has the following properties:*

- *The change in agent j's payment does not depend on agent j's signal:* $\hat{t}_j(s_j, s_{-j}) - t_j(s_j, s_{-j}) = \hat{t}_j(s'_j, s_{-j}) - t_j(s'_j, s_{-j})$ *for any* $s_j, s'_j \in S_j$;
- *The payment rule for agents other than i and j is unchanged:* $\hat{t}_k(s) = t(s)$ *for every* $k \in I$ *with* $k \neq i, j$ *and for every* $s \in S$.

Hence, if truth telling is a dominant strategy in the original ex ante budget balanced mechanism $\langle f, t \rangle$ we can create an ex post budget balanced mechanism where truth telling is dominant for all but a single agent.

4 An Extension

In this Section we relax the independence assumption. The weaker condition that we are considering is called "Condition S" in Crémer and Riordan (1985). We call it (CIND).

(CIND) *There are two agents $i, j \in I$, where $i \neq j$, such that conditional on every realization of the other agents' signals, the signals $\tilde{s}_i$ and $\tilde{s}_j$ of agents i and j are independent.*

Condition (CIND) implies that $N \geq 2$, so Proposition 3, unlike Proposition 1, does not explicitly mention this condition.

Proposition 3 *Suppose that condition (CIND) holds for some agents i, j. Then there is an insurance contract that provides full insurance, is actuarially fair, non-manipulable, and is interim individually rational for every agent except possibly agent j.*

Proof: Let $\langle p^*, m^* \rangle$ be given by,

$$\begin{aligned}
p^*_i &= 0 \\
m^*_i(s) &= x(s) - \mathrm{E}[x(\tilde{s}_j, s_{-j}) | s_{-j}] \\
p^*_j &= \mathrm{E}[x(\tilde{s})] \\
m^*_j(s) &= \mathrm{E}[x(\tilde{s}_j, s_{-j}) | s_{-j}] \\
p^*_k &= m^*_k(s) = 0 \ \forall k \neq i, j.
\end{aligned}$$

It is immediate that $\langle p^*, m^*\rangle$ provides full insurance and is actuarially fair. To check incentive compatibility and individual rationality for agent 1 assume that agent 1 observes s_i but reports s_i'. Define $s_{-i,j} = (s_k)_{k\in N, k\neq i,j}$. Using the law of iterated expectations and (CIND) we have that

$$\begin{aligned} \mathrm{E}\left[x(s_i', \tilde{s}_{-i})|s_i\right] &= \mathrm{E}[\mathrm{E}[x(s_i', \tilde{s}_j, s_{-i,j})|s_i, s_{-i,j}]|s_i] \\ &= \mathrm{E}[\mathrm{E}[x(s_i', \tilde{s}_j, s_{-i,j})|s_i', s_{-i,j}]|s_i], \end{aligned}$$

which implies that

$$\mathrm{E}\left[m_i^*(s_i', \tilde{s}_{-i})|s_i\right] = \mathrm{E}\left[x(s_i', \tilde{s}_{-i})|s_i\right] - \mathrm{E}\left[\mathrm{E}[x(s_i', \tilde{s}_j, s_{-i,j})|s_i', s_{-i,j}]|s_i\right] = 0.$$

We conclude that the contract is non-manipulable and individually rational for agent i. Non-manipulability for agent j is trivial, since $m_j^*(s)$ is independent of s_j (but $\mathrm{E}\left[m_j^*(s_j, \tilde{s}_{-j})|s_j\right]$ is in general not constant in s_j, so individual rationality may be violated for agent j). Trivially, the contract is non-manipulable and individually rational for each $k \neq i, j$.

Q.E.D.

Just like the proof of Proposition 1, the proof of Proposition 3 used a scheme with only two active agents. It follows that we get a close parallel with Remark 1, where the only difference is that we can no longer assure that the scheme is individually rational for both agents:

Remark 3 *Suppose that (CIND) holds for some agents i, j. Then, there is a non-manipulable, actuarially fair insurance contract providing full insurance that is interim individually rational for every agent except possibly agent j in which all agents other than i and j make zero payments in every state $s \in S$ and where j's payment is independent of her type.*

Next, in the same way in which Proposition 2 follows from Proposition 1, we can use Proposition 3 to conclude:

Proposition 4 *Suppose that condition (CIND) holds for some agents i, j. Then, for every ex ante budget balanced mechanism $\langle f, t\rangle$, there is an ex post budget balanced mechanism $\langle \hat{f}, \hat{t}\rangle$ such that:*

- *The allocation rule is unchanged:*
  $\hat{f}(s) = f(s)$ *for every* $s \in S$;
- *The interim expected payments are unchanged for all agents $k \neq j$*
  $\mathrm{E}\left[\hat{t}_k\left(s_k', \tilde{s}_{-k}\right)|s_k\right] = \mathrm{E}\left[t_k\left(s_k', \tilde{s}_{-k}\right)|s_k\right]$ *for all* $k \in I\setminus\{j\}$ *and all* $s_k, s_k' \in S_k$;

Proposition 4 is true regardless of whether incentive compatibility or individual rationality are satisfied or not. However, if truth telling is a Bayesian equilibrium in the original mechanism, then the same will be true in the ex post budget balanced mechanism. Moreover, if for any of the agents other than agent i, truth telling was a dominant strategy in the original mechanism, the same will be true in the ex post budget balanced mechanism. Finally, if the original mechanism satisfied an interim individual rationality constraint for any agent, then the same will be true in the ex post budget balanced mechanism, except possibly for agent j.

Next, just like in the case with independent types we can use Remark 3 to conclude that:

Remark 4 *If (CIND) holds for agents $i, j \in I$, then there exists an ex post budget balanced mechanism $\langle \hat{f}, \hat{t} \rangle$ satisfying the conditions in Proposition 4 and the following additional properties:*

- *The change in agent j's payment does not depend on agent j's signal:* $\hat{t}_j(s_j, s_{-j}) - t_j(s_j, s_{-j}) = \hat{t}_j(s'_j, s_{-j}) - t_j(s'_j, s_{-j})$ *for any* $s_j, s'_j \in S_j$;
- *The payment rule for agents other than i and j is unchanged:* $\hat{t}_k(s) = t(s)$ *for every* $k \in I$ *with* $k \neq i, j$ *and for every* $s \in S$.

An implication of Proposition 4 and Remark 4 is as follows. Consider the case that there are private values, i.e. every agent's utility depends only on his own type, but not on the other agents' types. Suppose also that the resource costs are a linear function of agents types: $r(a, s) = \bar{r} + \sum_{i=1}^{N} r_i(a, s_i)$. Let f^* be an allocation rule that maximizes social surplus, i.e. the sum of utilities minus resource costs, in every state. A Vickrey-Clarke-Groves payment rule will ensure that truthful reporting of type is a dominant strategy. By adding appropriate constants, we can ensure that the mechanism is ex ante budget balanced. By Remark 4 we can make sure that truth telling remains a dominant strategy for $N - 1$ agents when we construct an ex post budget balanced mechanism, and Proposition 4 guarantees that truth telling is optimal for the remaining agent provided that the other $N-1$ agents tell the truth. This is Theorem 2 in Crémer and Riordan (1985). Our analysis strengthens Crémer and Riordan's result because it applies not only to surplus maximizing allocation rules but to other allocation rules as well.

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