A General Equilibrium Model of Statistical Discrimination*

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Abstract

We consider a general equilibrium model with endogenous human capital formation in which *ex ante* identical groups may be treated differently in equilibrium due to informational externalities. Unlike earlier models of statistical discrimination, group inequalities may arise even if the corresponding model with a single group has a unique equilibrium. The dominant group gains from discrimination, rationalizing why a majority may be reluctant to eliminate discrimination. The model is also consistent with "reverse discrimination" as a remedy against discrimination since it may be necessary to decrease the welfare of the dominant group to achieve parity.

1 Introduction

This paper studies a competitive model that can rationalize group inequalities as a result of statistical discrimination. Two distinguishable groups have identical distributions of productive characteristics, but may in equilibrium specialize. An equilibrium where groups specialize is characterized by differences in human capital investments, average wages and job assignments.

Unlike the previous literature on statistical discrimination there is a conflict of interest between groups in our model. Discrimination may be interpreted as one group exploiting the other by designating them as "cheap labor" in an unskilled job, which under quite general circumstances increases the average productivity of workers in the dominant group.

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While thinking of discrimination in terms of exploitation seems natural to us, the previous literature on statistical discrimination has followed another path. Models differ a lot in details, but discrimination between identical groups is usually rationalized as a coordination failure. To generate discrimination in this way it suffices to construct a model with multiple equilibria. Discrimination is then explained as one group coordinating on a bad equilibrium and the rest of the economy being in a better equilibrium.

When discrimination is explained as pure coordination, it does not matter whether groups are competing for jobs in the same labor market or are living on separate "islands". That is, groups can be treated separately. This modelling strategy has been so dominant that separability between groups sometimes is taken to be a defining feature of the theory of statistical discrimination.

Models where statistical discrimination is a coordination problem are very tractable, an obvious advantage. However, the tractability comes at a cost of some implausible consequences. The dominant group would have nothing to lose if the disadvantaged group could solve the coordination failure, suggesting that economic policies aimed at excluding groups from certain professions (as in the pre-civil rights era in the US, South Africa during the apartheid regime and in many Southeast Asian countries today) would be irrational. Moreover, since parity can be achieved without harm to the dominant group one wonders how reverse discrimination would arise in a world where the problem is coordination.

While our model in many ways is closely related to other models of statistical discrimination, it is *not* a model of different groups coordinating on different equilibria. Discrimination can occur also if the model has a unique symmetric equilibrium. There is still an element of a self-confirming prophesy in that the roles of the groups may be reversed in different equilibria and that there always exists a symmetric equilibrium. The difference is that, in an equilibrium with group inequalities, the disadvantaged group cannot re-coordinate on a better equilibrium without a simultaneous re-coordination (on a worse outcome) by the other group.

The dominant group always gains from discrimination, explaining resistance towards measures intended to eliminate economic discrimination as well as why it may be in the self-interest of a dominant group to institutionalize discrimination. We show in a parametric example that such incentives exist even if discrimination is not self-enforcing as an equilibrium, thus rationalizing discriminatory policies as more than a coordination device.

1.1 Related Literature

There is a large literature on statistical discrimination following the seminal contributions by Arrow [4] and Phelps [17]. One strand assumes exogenous differences in the precision of information, which creates a rationale for firms to use "irrelevant" group characteristics.[1] The other major strand assumes no exogenous differences. Instead, non-trivial choices by workers, typically pre-market investments in human capital, are introduced which generates a rationale to condition on group identity if workers from different groups behave differently in equilibrium.[2] Our work falls into this second category.

Our model borrows some properties from Arrow [4] and Coate and Loury [5]. Like in Arrow's model (but unlike Coate and Loury's) the labor market is competitive. Arrow, however, does not explicitly derive how incentives to invest depend on wages and here we borrow the human capital investment model and the information technology from Coate and Loury to close the model.

2 The Model

2.1 The Economic Environment

Investments in Human Capital

There are two firms and a continuum of workers with mass normalized to unity. Each worker belongs to one of two identifiable groups, B or W and we denote by λ^J the respective fraction in the population for $J = B, W$. Prior to entering the labor market each worker makes a binary human capital investment decision. A worker either invests in her human capital and becomes a *qualified* worker, or the worker does not invest. If a worker invests, she incurs cost c which is distributed over $[\underline{c}, \overline{c}] \subseteq R$ according to a continuous and strictly increasing cumulative distribution $G(c)$.

Workers are risk neutral with payoffs that are additively separable in income and the cost of investment and do not care directly about task assignments. That is, a worker with cost c who invests and get a wage w gets utility $w - c$, while a worker who does not invest get utility w.

Production Technology

[1] As in Phelps [17], Aigner and Cain [1], Cornell and Welch [6], Lundberg and Startz [10] and Oettinger [16]

[2] Examples include Arrow [4], Spence [18], Akerlof [2], Coate and Loury [5], Foster and Vohra [7]. Similar in spirit are models deriving unequal outcomes from search frictions (see Mailath *et al.* [11], and Arcidiacono [3]).

To generate output, firms need workers performing two tasks, a *complex task* and a *simple task.* Only qualified workers are able to perform the complex task while all workers are able to perform the simple task. The effective input of labor in the complex task, C, is thus taken to be the quantity of *qualified* workers employed in the complex task and the input of labor in the simple task, S, is the quantity of workers (of both types) employed in the task. Output is given by $y(C,S)$ where $y : R_+^2 \rightarrow R_+$ is a production function that satisfies the following assumptions:

A1 y is concave and strictly increasing in both arguments

A2 y is twice continuously differentiable in both arguments over R_{++}^2

A3 y satisfies constant returns to scale

Information Technology

Employers cannot observe qualifications, but do observe a signal $\theta \in [0,1]$, distributed according to density f_q if the worker is qualified and f_u otherwise. Both densities are bounded away from zero and, without further loss of generality, $f_q(\theta)/f_u(\theta)$ is increasing in θ . This monotone likelihood ratio property implies that the posterior probability that a worker from group J with signal θ is qualified given prior π^J,

$$p\left(\theta, \pi^J\right) \equiv \frac{\pi^J f_q(\theta)}{\pi^J f_q(\theta) + (1-\pi^J) f_u(\theta)}, \tag{1}$$

is increasing in θ. A high signal is thus good news about a worker. We denote the cumulative distributions by F_q and F_u and assume that a law of large numbers hold so that these are also the realized frequency distributions of signals for qualified and unqualified workers respectively.[3]

[3] Since workers are anonymous the well-known issues concerning laws of large numbers for continuum random variables (discussed by Judd [9], Feldman and Gilles [8] and others) may be avoided by a simple trick. Let H_q and H_u be the distributions of qualified and unqualified workers on $[\underline{c}, \overline{c}]$ and x be uniformly distributed on $[0,1]$. The test-score for a qualified agent c, $\theta_c(x)$, is then taken to be $\theta_c(x) = F_q^{-1}(H_q(c)+x)$ if $H_q(c)+x \leq 1$ and $\theta_c(x) = F_q^{-1}(H_q(c)+x-1)$ if $H_q(c)+x > 1$. A direct calculation shows that the individual and aggregate distributions coincide. While signals are correlated, this is not a problem because c is unobservable.

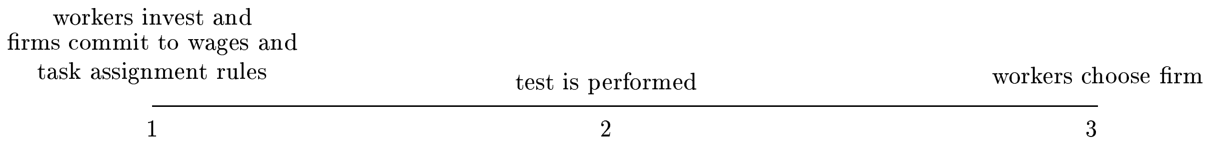

Figure 1: The timing of the model

2.2 The Game

The timing is described in Figure 1.[4] In the first stage of the game firms post wages and task assignment rules and workers simultaneously decide on human capital investments. Each worker decides whether to invest and an investment strategy profile in group J is a map $i^J : [\underline{c}, \overline{c}] \to [0,1]$, where $i^J(c)$ is the proportion of group J workers with cost c that invests.[5] The fraction of investors in group J is $\pi^J = \int i^J(c) dG(c)$, which, since c is unobservable and payoff irrelevant to the firm, contains all relevant information about the investment profile i^J for the firms.

Firms may condition wages and job assignments on θ. A strategy for firm i is to select some *wage schedule* $w_i^J : [0,1] \to R_+$ and a *task assignment rule* $t_i^J : [0,1] \to [0,1]$ for each group J, where $t_i^J(\theta)$ is interpreted as the fraction of workers with signal θ employed in the complex task.[6]

In the second stage nature assigns a signal θ to each worker in accordance with density f_q (f_u) for a worker who invested (did not invest). Workers then observe the posted wages and task assignment rules and decide where to work in the third and final stage.

Investment costs are sunk when workers are comparing wage offers. Hence it is a conditionally strictly dominated strategy not to accept the highest offer. After eliminating strictly dominated firm choice rules the *ex ante* payoff for a worker from group J with investment cost c can thus be written $E_q\left[\max\left\{w_1^J(\theta), w_2^J(\theta)\right\}\right] - c$ for a worker who invests and $E_u\left[\max\left\{w_1^J(\theta), w_2^J(\theta)\right\}\right]$ for a worker who does not, where E_q (E_u) is the expectation with respect to f_q (f_u).

[4]In earlier drafts we considered a more "natural" timing with wage posting carried out after the investment decisions and task asignments carried out after the clearance of the labor market. This makes the strategy sets more complicated, but equilibrium outcomes remain the same.

[5]The formulation assumes that workers with the same cost picks the same probability of investment. This could in equilibrium fail only for a single cost c, which is irrelevant since G has no atoms.

[6]The model would collapse if workers could be made residual claimants on "their contribution to output". Such model would produce a unique equilibrium which would be color-blind and efficient. Our exact specification is consistent with a world where firms can observe output, but not individual productivities. Whereas this is admittedly crude, what is *qualitatively* needed is that the "pre-market signal" θ matters for expected payoffs. We are currently exploring richer contracting environments where this is the case.

3 Color-Blind Equilibria

As a benchmark we first consider equilibria where both groups invest at the same rate and where firms ignore the payoff irrelevant group characteristic (equivalent to model without the group characteristic). We look for Nash equilibria that satisfy the additional requirement that workers choose firms in a sequentially rational manner after any history of play. Such equilibria are perfect Bayesian, but since beliefs are irrelevant for the optimal firm choice in the end of the game our requirement is really much weaker.

3.1 Equilibrium Job Assignments and Wages

Let $\pi \in [0,1]$ denote the fraction of investors (same for both groups) and imagine that job assignments are carried out by a planner who can choose any task assignment rule $t : [0,1] \to [0,1]$, but takes π as given. The reason for the introduction of a fictitious planner is that constrained efficiency in job assignments is necessary for equilibrium, so this is a convenient way of characterize equilibrium task assignments.

There is total of $\pi f_q(\theta)$ qualified workers and $(1-\pi) f_u(\theta)$ unqualified workers with signal θ. Hence, there are $t(\theta)\pi f_q(\theta)$ "units" of labor in the complex task and (since all workers are equally productive in the simple task) $[1-t(\theta)][\pi f_q(\theta) + (1-\pi) f_u(\theta)]$ "units" of labor in the simple task with signal θ, so the total inputs of labor in the two tasks are

$$\begin{aligned} C &= \int_0^1 t(\theta)\pi f_q(\theta)\,d\theta \\ S &= \int_0^1 [1-t(\theta)][\pi f_q(\theta) + (1-\pi) f_u(\theta)]d\theta. \end{aligned} \tag{2}$$

Since (1) is increasing in the signal it is without loss of generality to focus on rules $t(\cdot)$ with a threshold property, where workers with signals above the threshold are assigned to the complex task and workers with lower signals are assigned to the simple task. Given a threshold θ' the quantity of *qualified* workers with signals above the threshold is $\pi(1-F_q(\theta'))$ and the quantity of workers (qualified and unqualified) with signals below the threshold is $\pi F_q(\theta') + (1-\pi) F_u(\theta')$. Output is thus maximized with a threshold solving,

$$\max_{\theta' \in [0,1]} y\left(\pi[1-F_q(\theta')], \pi F_q(\theta') + [1-\pi]F_u(\theta')\right). \tag{3}$$

Let $\theta(\pi)$ be any solution to (3) and, with some abuse of notation, define

$$C(\pi) \equiv \pi[1 - F_q(\theta(\pi))] \qquad S(\pi) \equiv \pi F_q(\theta(\pi)) + (1-\pi) F_u(\theta(\pi)) \tag{4}$$

which are the effective factor inputs in the complex and simple task respectively given task assignments in accordance with a threshold rule with cutoff $\theta(\pi)$.

We call a strategy profile a *continuation equilibrium* if all equilibrium conditions except the requirement that investments are best responses to wages are satisfied. Our first result states that wages are given by expected marginal products and job assignments are constrained efficient in any continuation equilibrium.

Proposition 1 *Suppose that a fraction π of the workers invest and that $\theta(\pi)$ is a solution to (3). Then there exists a continuation equilibrium where both firms offer wages*

$$w(\theta;\pi) = \begin{cases} \frac{\partial y(C(\pi),S(\pi))}{\partial S} & \text{for } \theta < \theta(\pi) \\ p(\theta,\pi) \frac{\partial y(C(\pi),S(\pi))}{\partial C} & \text{for } \theta \geq \theta(\pi) \end{cases}. \tag{5}$$

and where a worker is assigned to the complex task if and only if $\theta \geq \theta(\pi)$. Moreover, in any continuation equilibrium where a fraction π of the workers invest the wage schedule posted by i, $w_i(\theta)$, must agree with (5) for almost all $\theta \in [0,1]$ for each firm i.

Proposition 1 implies:

Corollary 1 *Equilibrium wages are unique up to deviations on sets of signals with measure zero.*

This is obvious if $\theta(\pi)$ is uniquely defined. In case of multiple solutions to (3), constant returns to scale implies that the marginal products are the same evaluated at any solution.

3.2 Equilibrium Human Capital Investments

The final equilibrium condition is that investments are best responses to the wages, implying that a worker invests if and only if the gain in expected earnings is higher than the cost c. We refer to the gain in earnings as *the incentive to invest* and for wages consistent with a continuation equilibrium where a fraction π invests we use direct substitution from (5) to write this as

$$\begin{aligned} I(\pi) &= \int_0^1 w(\theta;\pi) f_q(\theta)\, d\theta - \int_0^1 w(\theta;\pi) f_u(\theta)\, d\theta = \qquad (6) \\ &= \frac{\partial y(C(\pi),S(\pi))}{\partial S} [F_q(\theta(\pi)) - F_u(\theta(\pi))] + \frac{\partial y(C(\pi),S(\pi))}{\partial C} \int_{\theta(\pi)}^1 p(\theta,\pi) [f_q(\theta) - f_u(\theta)]\, d\theta. \end{aligned}$$

The fraction of workers that gain from investing is thus $G(I(\pi))$. In a Nash equilibrium, firms have rational expectations about the fraction of investors and investment behavior must be rational given wages, so the equilibria are fully characterized as the solutions to

$$\pi = G(I(\pi)). \tag{7}$$

Hence, the fraction of investors in any equilibrium is a solution to (7) and from any solution to (7) we can construct wage schedules, task assignments and investment rules consistent with equilibrium.[7]

4 Asymmetric Equilibria

We now allow groups to invest at different rates and now let $\pi = \left(\pi^B, \pi^W\right)$ denote the group-specific fractions of investors. The analogue of problem (3) is

$$\max_{\theta^B, \theta^W \in [0,1]^2} y\left(\sum_{J=B,W} \lambda^J \pi^J \left[1 - F_q(\theta^J)\right], \sum_{J=B,W} \lambda^J \left[\pi^J F_q(\theta^J) + (1-\pi^J) F_u(\theta^J)\right]\right), \tag{8}$$

and given a solution $(\theta^B(\pi), \theta^W(\pi))$ to this program we now let

$$\begin{aligned} C(\pi) &\equiv \sum_{J=B,W} \lambda^J \pi^J \left[1 - F_q(\theta^J(\pi))\right] \\ S(\pi) &\equiv \sum_{J=B,W} \lambda^J \left[\pi^J F_q(\theta^J(\pi)) + (1-\pi^J) F_u(\theta^J(\pi))\right], \end{aligned} \tag{9}$$

with a similar abuse of notation as in (4). The characterization of equilibrium wages and task assignments is the obvious generalization of Proposition 1 and since the proof proceeds step by step as that in the model with a single group we have omitted the proof.

Proposition 2 *Suppose that a fractions $\pi = (\pi^B, \pi^W)$ of the workers invest and that $(\theta^B(\pi), \theta^W(\pi))$ solves (8). Then there exists a continuation equilibrium where both firms offer wages*

$$w^J(\theta; \pi) = \begin{cases} \frac{\partial y(C(\pi), S(\pi))}{\partial S} & \text{for } \theta < \theta^J(\pi) \\ p\left(\theta, \pi^J\right) \frac{\partial y(C(\pi), S(\pi))}{\partial C} & \text{for } \theta \geq \theta^J(\pi) \end{cases} \quad \text{for } J = B, W, \tag{10}$$

and assign a worker with characteristics (J, θ) to the complex task if and only if $\theta \geq \theta^J(\pi)$. Moreover, in any continuation equilibrium where fractions $\pi = (\pi^B, \pi^W)$ of the workers invest the wage schedule posted by i for group J, $w_i^J(\theta)$, agrees with (10) for almost all $\theta \in [0,1]$ for each firm i.

[7] That is, if π solves (7) there is an equilibrium $\left\{i, \langle w_i, t_i \rangle_{i=1,2}\right\}$ where $i(c) = e_q$ for all $c < G^{-1}(\pi)$ and $i(c) = e_u$ for all $c > G^{-1}(\pi)$, $w_i(\theta) = w(\theta)$ all θ and $t_i(\theta)$ is a rule with threshold $\theta(\pi)$.

The incentive to invest given any investment behavior π is

$$I^J(\pi) = \frac{\partial y(C(\pi), S(\pi))}{\partial S}\left[F_q(\theta^J(\pi)) - F_u(\theta^J(\pi))\right] + \frac{\partial y(C(\pi), S(\pi))}{\partial C}\int_{\theta^J(\pi)}^{1} p(\theta, \pi^J)\left[f_q(\theta) - f_u(\theta)\right]d\theta \tag{11}$$

and the fraction of agents in group j who invest is still given by the fraction with investment cost lower than the benefits, so the system of equations that characterize the equilibria is

$$\pi^J = G(I^J(\pi)) \text{ for } J = B, W. \tag{12}$$

We say that an equilibrium is *discriminatory* whenever $\pi^B \neq \pi^W$, which is consistent with the standard definition of economic discrimination in terms of average wage differentials (Aigner and Cain [1]). Inspection of (10) shows that if $\pi^B < \pi^W$, then the wage is higher for $W-$workers than for $B-$workers for each signal (strictly higher for $\theta > \theta^J(\pi)$). Moreover, more workers in group W have high signals, so in our stylized model the group with the lower fraction of investors is also the group with the lower average wage.

4.1 Sufficient Conditions for Unique Optimal Task Assignments

It is notationally convenient to rule out multiple solutions to (8). There is no substantial cost of doing this since multiplicity of solutions to (8) generates nothing qualitatively different (due to Corollary 1). Sufficient conditions are:

Lemma 1 *Suppose that either 1) y is quasi-concave and strictly increasing in both arguments and $\frac{f_q(\theta)}{f_u(\theta)}$ is strictly increasing in θ, or, 2) y is strictly quasi-concave and strictly increasing in both arguments. Then, there is a unique $(\theta^B(\pi), \theta^W(\pi)) \in [0,1]^2$ that solves (8) for any $\pi >> 0$.*

To understand this it is useful to restate the problem (8) as

$$\max_{C^J, S^J} y(\lambda^B C^B + \lambda^W C^W, \lambda^B S^B + \lambda^W S^W) \tag{13}$$

$$\text{subject to } S^J \leq \pi - C^J + (1-\pi) F_u\left(F_q^{-1}\left(\frac{\pi - C^J}{\pi}\right)\right) \text{ for } J = B, W.$$

The monotone likelihood ratio assumption implies that the right hand side of the constraint is a concave function of C (strictly concave with the strict monotone likelihood assumption), so the two cases can be illustrated as in Figure 2.

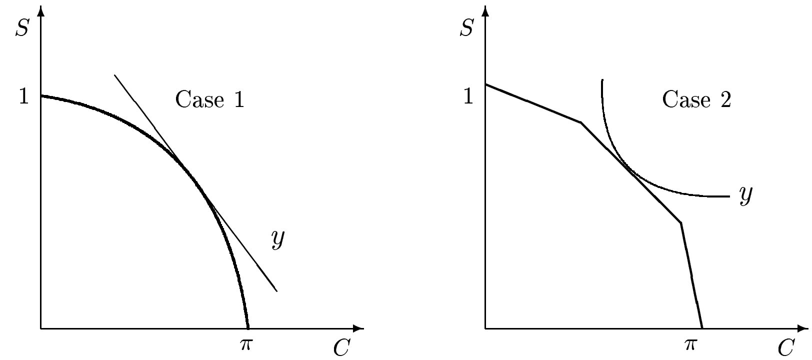

Figure 2: Sufficient conditions for unique optimal task assignments

4.2 The Linear Model

It is useful to first consider the special case with a linear production function, $y(C,S) = \alpha C + \beta S$, for some $\alpha > \beta > 0$. To avoid dealing with correspondences we assume that $\frac{f_q(\theta)}{f_u(\theta)}$ is strictly increasing in θ, so that Lemma 1 applies. The task assignment problem (8) then simplifies to

$$\sum_{J=B,W} \lambda^J \max_{\theta^J \in [0,1]} \left[\alpha \pi^J (1 - F_q(\theta^J)) + \beta(\pi^J F_q(\theta^J) + (1-\pi^J) F_u(\theta^J))\right] \tag{14}$$

That is, the task assignment problem can be solved separately for each group. To stress the separability we write $\widehat{\theta}^J(\pi^J)$ for the unique optimal threshold in group J, which is given by

$$\widehat{\theta}^J(\pi^J) = \begin{cases} 1 & \text{if } \alpha p\left(1,\pi^J\right) \leq \beta \\ 0 & \text{if } \alpha p\left(0,\pi^J\right) \geq \beta \\ \text{the unique solution to } \alpha p\left(\theta,\pi^J\right) = \beta & \text{if } \alpha p\left(0,\pi^J\right) < \beta < \alpha p\left(1,\pi^J\right) \end{cases}. \tag{15}$$

Equilibrium wages are thus $w^J(\theta;\pi^J) = \beta$ for $\theta \leq \widehat{\theta}^J(\pi^J)$ and $w^J(\theta;\pi^J) = \alpha p\left(\theta,\pi^J\right)$ for $\theta > \widehat{\theta}^J(\pi^J)$ and equilibria are fully described as a pair (π^B,π^W) such that

$$\pi^J = G\left(\widehat{I}^J(\pi^J)\right) \qquad \text{for } J = B,W \text{ , where} \tag{16}$$

$$\widehat{I}^J(\pi^J) = \beta\left[F_q(\widehat{\theta}^J(\pi^J)) - F_u(\widehat{\theta}^J(\pi^J))\right] + \alpha \int_{\widehat{\theta}^J(\pi^J)}^1 p\left(\theta,\pi^J\right)(f_q(\theta) - f_u(\theta))d\theta \tag{17}$$

Observe that $\widehat{I}^J(\pi^J)$ is a composition of continuous functions, which means that existence of equilibria is immediate.

There may be a unique solution to (16), in which case groups must be treated identically in any equilibrium. The more interesting possibility is that (16) may have multiple solutions, in which case

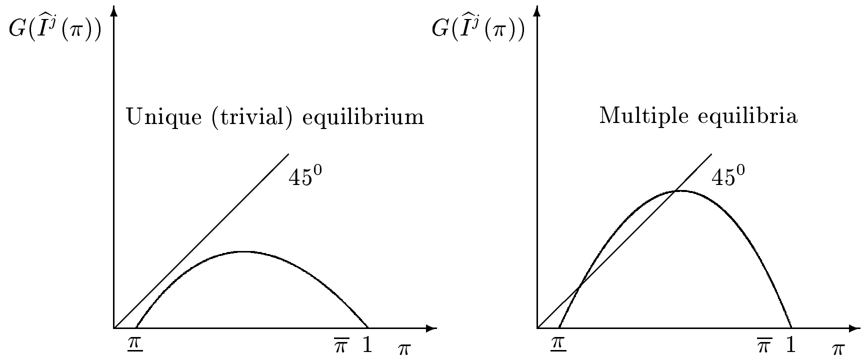

Figure 3: Illustration of the equilibrium fixed point problem in the linear case

there are equilibria with discrimination. This is illustrated in Figure 3. One shows from (15) that there exists $\underline{\pi} > 0$ such that all workers are assigned to the simple task if $\pi^J \leq \underline{\pi}$. Symmetrically, there is some $\overline{\pi} < 1$ such that all workers are assigned to the complex task. If $\pi^J \leq \underline{\pi}$ it follows that $\widehat{I}^J(\pi^J) = 0$, so if $G(0) \leq \underline{\pi}$ it follows that there is a trivial equilibrium where no workers invest (the picture is drawn for $G(0) = 0$). Observe that the incentive to invest is still strictly positive in the (non-empty) range $(\overline{\pi}, 1)$ since the posterior is increasing in the signal, thus generating wage inequality even with all agents allocated to the complex task.

The curve $G(\widehat{I}^j(\pi))$ can be moved up and down by changes in G alone. That is, if G first order stochastically dominates G', the whole curve shifts up if G is replaced by G'. Hence, if costs are "sufficiently low" the curve intersects the diagonal line, implying that there then are at least 2 interior equilibria in addition to the trivial equilibrium.[8]

If there are multiple solutions to (16) each group can have a fraction of investors corresponding to any of these solutions. These equilibria are Pareto rankable.

Proposition 3 *Let π^* be the largest solution to (16). Then $\left(\pi^B, \pi^W\right) = (\pi^*, \pi^*)$ Pareto dominates all other equilibria of the model.*

[8]There can be more than 2 interior equilibria. The easiest way to see that is to consider the (non-generic) case with a continuum of equilibria. Given any f_q, f_u, α, and β we can construct a distribution G that supports a continuum of equilibria as follows. Take an interval $(a, b) \subset [\underline{c}, \overline{c}]$ where B is strictly increasing (such range must exist since B is continuous and strictly positive at any $\underline{\pi} < \pi < 1$). Define $\widetilde{B} : (a, b) \rightarrow R$ as $\widetilde{B}(\pi) = B(\pi)$ for any $\pi \in (a, b)$ and let G be a function satisfying $G^{-1}(\pi) = \widetilde{B}(\pi)$ for any $\pi \in (a, b)$, which immediately implies that any $\pi \in (a, b)$ is an equilibrium. It should be intuitive from this that we may construct (more robust) examples where there are k equilibria for any integer k.

Discrimination can thus be sustained in the linear model, but only as a pure coordination failure. The separability between groups implies that the "dominant group" would not be affected at all if the discriminated group could somehow re-coordinate on a better equilibrium. This property is shared by the model in Coate and Loury [5] and almost all equilibrium models that can rationalize unequal treatment of identical groups.[9]

Our view is that this separability is a weakness of the theory. Given the long-standing record of economic policies designed to exclude certain groups from high income professions, it seems that there simply must be some gains from such measures for those that the policies are intended to "protect". Put differently, a rather natural belief is that economic discrimination against blacks has something do with Jim Crow laws in the past, and it seems strange then to perform the analysis within a model where such laws would be irrational. Similarly, the idea of "reverse discrimination" as a remedy for past discrimination appears equally irrational, again suggesting that a richer model is needed.

5 Complementarities

We assume that, in addition to assumptions **A1-A3,** the production function y satisfies,

A4 y is strictly quasi-concave

A5 $\lim_{C\to 0} \frac{\partial y(C,S)}{\partial C} = \infty$ for any $S > 0$ and $\lim_{S\to 0} \frac{\partial y(C,S)}{\partial S} = \infty$ for any $C > 0$

A6 $y(0,S) = y(C,0) = 0$ for any $C, S > 0$

Strict quasi-concavity is the qualitatively important assumption, while **A5** and **A6** are for expositional simplicity. Existence of equilibria can be checked rather easily from the reduced form characterization in (11) and (12).

Proposition 4 *Suppose y satisfies assumptions A1-A6. Then there is always at least one symmetric equilibrium*

[9] The only exception we are aware of is that there are matching and search models where groups can not be analyzed in separation. See Mailath et al [11].

5.1 Cross-Group Effects on Incentives

Strict quasi-concavity implies (by Lemma 1) that there is a unique $(\theta^B(\pi), \theta^W(\pi))$ solving (8) whenever $(\pi^B, \pi^W) >> 0$. Moreover, since output is zero whenever all workers are assigned to the same task the factor ratio,

$$r(\pi) = r\left(\pi^B, \pi^W\right) = \frac{\sum_{J=B,W} \lambda^J \pi^J (1 - F_q(\theta^J(\pi))}{\sum_{J=B,W} \lambda^J \left[\pi^J F_q(\theta^J(\pi)) + (1 - \pi^J) F_u(\theta^J(\pi))\right]}, \tag{18}$$

is always well defined. In case of a fully interior solution to (8) the necessary and sufficient conditions for optimality may be expressed as

$$p\left(\theta^J(\pi), \pi^J\right) \frac{\partial y(r(\pi), 1)}{\partial C} = \frac{\partial y(r(\pi), 1)}{\partial S}. \tag{19}$$

The crucial observation to be made from (19) is that *the ratio of complex to simple labor is monotonically increasing in the fraction of investors in any group.* To see this, suppose to the contrary that the factor ratio goes down when π^J increases. To satisfy (19) it is then necessary for $\theta^J(\pi)$ to decrease in each group. This in turn would, as can be seen in (18), imply that the factor ratio increased, which is a contradiction.[10]

The monotonicity of the factor ratio in investments generates negative cross group effect on incentives.

Proposition 5 *Fix $\pi^J > 0$. Then $I^J(\pi^J, \pi^K)$ is decreasing in π^K over the whole unit interval and strictly decreasing for all π such that $\theta^J(\pi) < 1$ and $\theta^K(\pi) > 0$.*

Since these effects are central to the understanding of our model, we now provide a heuristic explanation. Rewrite the equilibrium wage schemes as

$$w^J(\theta; \pi) = \begin{cases} \frac{\partial y(r(\pi),1)}{\partial S} & \text{for } \theta < \theta^J(\pi) \\ p\left(\theta, \pi^J\right) \frac{\partial y(r(\pi),1)}{\partial C} & \text{for } \theta \geq \theta^J(\pi) \end{cases} \qquad \text{for } J = B, W. \tag{20}$$

Observe that $r(\pi)$ increases when π^W increases, which for group B implies lower wages for workers with high signals and higher wages for workers with low signals. It should then be clear from Figure 4 that this decreases incentives, since the distribution of signals for a qualified worker (F_q) first order stochastically dominates the distribution of signals for an unqualified worker.

[10]The proof of Proposition 5 also takes care of the possibility of corner solutions to (8).

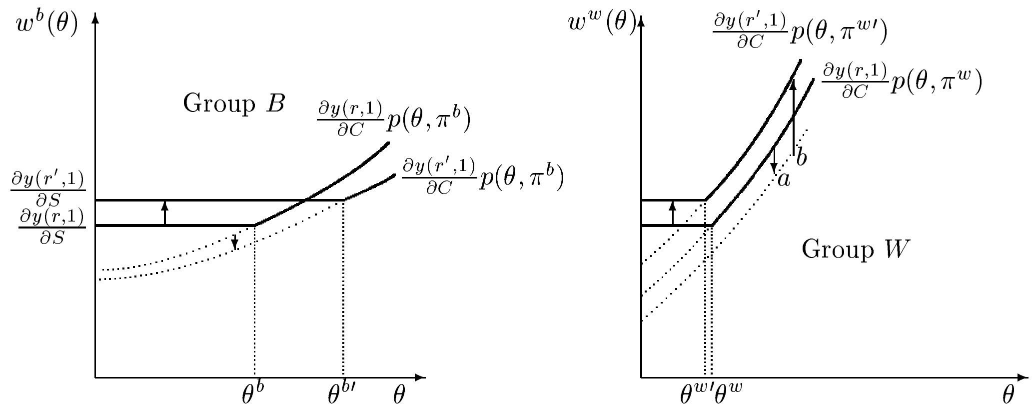

Figure 4: The effect of an increase in the fraction of investors in group W

This "as if externality" is driven by ordinary price effects, but the informational externality is still crucial. With observable investments, there would be a wage w_q for qualified workers and a wage w_u for unqualified workers. For the same reasons as in our model, an increase (in any group) of the fraction of investors would tend to decrease w_q and increase w_u in equilibrium. However, benefits to invest would be $w_q - w_u$ for both groups. A change in the proportion of qualified workers (in any group) thus affects incentives symmetrically. The informational asymmetry is therefore essential in our model since otherwise no differential treatment can occur in equilibrium.

5.2 When Will Asymmetric Equilibria Exist?

To construct an example where group B is discriminated against it suffices to find some $\pi = \left(\pi^B, \pi^W\right)$ where $\pi^B < \pi^W$ and $I^B(\pi) < I^W(\pi)$. Fixing all the other fundamentals of the model such an investment profile π always exists and once this is found any distribution function G such that $\pi^B = G\left(I^B(\pi)\right)$ and $\pi^W = G\left(I^W(\pi)\right)$ completes the example.

Parameters interact with each other in a rather complicated way, also in tightly parametrized versions of the model. It is therefore hard to come up with useful sufficient conditions for existence of asymmetric equilibria. The one general result that does provide some information is highly intuitive. To state the result, let G be some distribution function with $G(0) = 0$ and assume the single-group model with distribution G has a non-trivial equilibrium. Define the parametric sequence of distributions G_α, where $G_\alpha(c) = \alpha + (1-\alpha) G(c)$ for every c in the support of G.

Proposition 6 *Fix y, f_q, f_u, λ^B and λ^W. Then there exists $\overline{\alpha} > 0$ such that a discriminatory equilibrium exists in the model with cost distribution G_α for any $\alpha \leq \overline{\alpha}$.*

The intuition is straightforward: as long as there are not too many agents who get positive utility from the human capital investment it is possible to construct equilibria with all agents from one group in the simple task. This is in not possible in any symmetric equilibrium under the assumptions on page 12.

Of the other parameters in the model we suspect that discrimination is easier to sustain the larger is the size difference between groups. We have however not been able to prove this even for the parametric version of the model in Section 6. From the parametric example we know that the precision of the signal affects the likelihood of an asymmetric equilibrium non-monotonically. Very precise signals makes discrimination hard to obtain because prior information gets discounted a lot. If the signal is too uninformative, discrimination is hard to sustain because it gets hard to sustain incentives for any group. The "importance of the complex task" also matters. This is hard to formalize in general, but in Section 6 this is summarized in a single parameter and, not surprisingly, the effects are again non-monotonic (see Section 6.3).

5.3 Gains for the Dominant Group

The monotone spillover effects also create an incentive to discriminate. If a certain group could choose between a symmetric equilibrium and an equilibrium where the other group is discriminated against they would always choose to discriminate the other group. The result holds also if there are multiple symmetric equilibria given that the discriminatory equilibrium is wisely chosen, but is easiest to state under the assumption that the symmetric equilibrium is unique (in the next section we consider a parametrization where this is always the case):

Proposition 7 *Suppose that there is a unique symmetric equilibrium. Then, in any equilibrium with discrimination, the ex ante utility (before knowing the cost realization) in the group with the higher fraction of investors is higher than that in the symmetric equilibrium.*

While this may seem obvious the reader should note that this is never the case in a model where discrimination is a coordination failure, so the cross-group effects is what makes the model generate this rather natural prediction.

5.4 Gains and Losses from Specialization

To analyze the impact from groups specializing on society as a whole it is convenient to define the maximized value of output given any particular investment behavior as

$$Y(\pi) \equiv \max_{\theta^B, \theta^W} y\left(\sum_{J=B,W} \lambda^J \pi^J \left[1 - F_q(\theta^J)\right], \sum_{J=B,W} \lambda^J F_{\pi^J}\left(\theta^J\right) \right). \tag{21}$$

Social surplus in the economy is

$$Y(\pi) - \sum_{J=B,W} \lambda^J \int_{\underline{c}}^{G^{-1}(\pi^J)} cg(c)\, dc. \tag{22}$$

Let $h(\pi^W)$ be a function such that $\lambda^B h(\pi^W) + \lambda^W \pi^W = K$ holds for all π^W in some range for some constant $K > 0$. That is, $(h(\pi^W), \pi^W)$ defines a (linear) locus of group-specific fractions of investors such that the total investments in the economy is held constant.

Proposition 8 *Suppose that $\pi^B < \pi^W$. Then:*

1. $\frac{d}{d\pi^W} Y(h(\pi^W), \pi^W) > 0$
2. $\frac{d}{d\pi^W}\left(\lambda^B \int_{\underline{c}}^{G^{-1}(h(\pi^W))} cg(c)\, dc + \lambda^W \int_{\underline{c}}^{G^{-1}(\pi^W)} cg(c)\, dc\right) > 0.$

This result says that output increases with increased specialization and that aggregate investment costs increase with increased specialization. However, nothing guarantees that two *equilibria* have the same total quantity of investors. Proposition 8 is therefore only suggestive about welfare comparisons across equilibria.

Keeping this caveat in mind, the result that output is increasing in the degree of specialization has an intuitive explanation. Some workers are always assigned to the wrong job and this "mismatch" is reduced when groups specialize.[11] That aggregate investment costs increase should be obvious, since investments are transferred from lower to higher cost units.

[11] This is true also in the linear model, but then there is no particular reason for the *aggregate* quantity of investors to be at any particular level. With curvature in the technology, it is undesirable to vary factor inputs too much, which means that the trade-off summarized in Proposition 8 becomes relevant for efficiency.

6 A Parametric Example

For concreteness we will now illustrate the capabilities of the model with a parametric example. We let $y(C,S) = C^{\alpha}S^{1-\alpha}$ for some $\alpha \in (0,1)$ and assume that θ is drawn from $\{\theta_L, \theta_H\}$ in accordance with symmetric conditional probability distributions, where $\phi > \frac{1}{2}$ is the probability of drawing θ_H for a qualified worker, and $(1-\phi)$ is the probability of θ_H for an unqualified worker.

In an equilibrium with equal treatment of groups, let σ and γ denote the fraction of agents assigned to the complex task with signal θ_H and θ_L respectively, The associated inputs of labor are

$$\begin{aligned} C(\sigma,\gamma,\pi) &= \sigma\phi\pi + \gamma(1-\phi)\pi \\ S(\sigma,\gamma,\pi) &= (1-\sigma)\left[\phi\pi + (1-\phi)(1-\pi)\right] + (1-\gamma)\left[(1-\phi)\pi + \phi(1-\pi)\right], \end{aligned} \tag{23}$$

where π is the (common) fraction of investors. Optimal task assignments, denoted $(\sigma(\pi), \gamma(\pi))$, solve

$$\max_{\sigma,\gamma,\in[0,1]^2} C(\sigma,\gamma,\pi)^{\alpha} S(\sigma,\gamma,\pi)^{1-\alpha}, \tag{24}$$

and closed form solutions to (24) are easy to find from the first order conditions to the problem. This setup is equivalent to one with continuous signals with conditional distributions satisfying the monotone likelihood ratio property *weakly*, a case covered by Propositions 1 and 2. The unique continuation equilibrium wages are thus the expected marginal products, that is

$$w(\theta;\pi) = \max\left\{ p(\theta,\pi)\,\alpha \frac{S(\sigma(\pi),\gamma(\pi),\pi)}{C(\sigma(\pi),\gamma(\pi),\pi)}^{1-\alpha}, (1-\alpha)\frac{C(\sigma(\pi),\gamma(\pi),\pi)}{S(\sigma(\pi),\gamma(\pi),\pi)}^{\alpha} \right\}, \tag{25}$$

for $\theta \in \{\theta_H, \theta_L\}$. Hence, the closed form solution to (24) enables us to express the incentives to invest,

$$\begin{aligned} I(\pi) &= \underbrace{\phi w(\theta_H;\pi) + (1-\phi)w(\theta_L;\pi)}_{\text{Exp wage for qualified worker}} - \underbrace{((1-\phi)\phi w(\theta_H;\pi) + w(\theta_L;\pi))}_{\text{Exp wage for unqualified worker}} \\ &= (2\phi-1)\left(w(\theta_H;\pi) - w(\theta_L;\pi)\right). \end{aligned} \tag{26}$$

explicitly as a function of π and parameters of the model (which are suppressed in the notation above). The symmetric equilibria in the examples below are computed by solving the equation $\pi = G(I(\pi))$ numerically.

One advantage with this parametrization is that *symmetric* equilibria can be shown to be unique under some restrictions on the cost distribution G.

Proposition 9 *Suppose that G is concave and that $\underline{c} < 0$. Then there is a unique symmetric equilibrium π^* (which is non-trivial).*

This result is useful because it makes comparisons between situations with and without discrimination more straightforward. With multiplicity of symmetric equilibria we would have to either make set-wise comparisons or to select a plausible symmetric equilibrium according to some criterion. To avoid this, all examples below are parametrizations with unique equilibria.

Even with a unique symmetric equilibrium, there can be either multiple or no asymmetric equilibria at all. We have not shown it analytically, but it seems that the typical situation with groups of equal size is that *if* asymmetric equilibria exist, then there are two asymmetric equilibria with group W better off than group B and another two with the roles of the groups reversed. For simplicity, we therefore focus on equilibria which are "fully segregated" in the sense that all workers from one group are assigned to the simple task.

Focusing on asymmetric equilibria of this simple form also has the advantage that such equilibria can be computed much like the symmetric equilibrium. Assuming the equilibrium has all B workers in the simple task means that the equilibrium fraction of investors in W can be calculated from a trivial modification of (26), where the only difference is that an "exogenous" amount of simple labor λ^B is added to $S(\sigma, \gamma, \pi)$ in (23). In all numerical simulations we have done, the resulting equilibrium fraction π^W in the other group is unique. Finally, we must check that forcing the B workers into the simple task is consistent with equilibrium, that is that there are no incentives to hire any B worker in the complex task when $\pi^B = G(0)$.

6.1 Conflicts of Interest

We know from Proposition 7 that the group with the higher fraction of investors is always better off (ex ante) in an asymmetric equilibrium than in the unique symmetric equilibrium. It is natural to suspect that the "normal case" is that the discriminated group is worse off and have lower income than in the symmetric equilibrium. One such example, where B is the discriminated group, is displayed in Table 1.[12]. In this equilibrium, all workers of group B are employed in the simple task, which implies that they have no incentives to invest. The last 4 rows in the table show that the dominant group gains and the discriminated group loses relative to the symmetric equilibrium.

[12] Since group sizes are identical the roles of the two groups may be reversed.

This is not only true on average, but also conditional on investment. Every worker in group W is thus better off in the discriminatory equilibrium than in the symmetric equilibrium.[13]

$\phi = \frac{2}{3}, \alpha = \frac{1}{2}, \lambda^W = 1/2$ $c \sim U[-0.02, 0.18]$	Discriminatory Equilibrium		Symmetric
	Group B	Group W	Equilibrium
Equilibrium Investment	$\pi^B = 0.1$	$\pi^W = 0.548$	$\pi = .269$
Gross incentives to invest	$I^B(\pi) = 0$	$I^W(\pi) = 0.090$	$I^J(\pi) = 0.034$
Wages	$w^B(\theta_H, \pi) = .307$ $w^B(\theta_L, \pi) = .307$	$w^W(\theta_H, \pi) = .576$ $w^W(\theta_L, \pi) = .307$	$w(\theta_H, \pi) = .380$ $w(\theta_L, \pi) = .279$
Average expected welfare	.309	.416	.320
Expected welfare if invest	$.307 - c$	$.487 - c$	$.346 - c$
Exp. welfare if not invest	.307	.397	.313

Table 1: Symmetric and asymmetric equilibria, average welfare computed as expected wage less average investment costs

It is also possible that both groups gain from discrimination. The simplest example is if investment costs are so high that nobody invests in the symmetric equilibrium. One such case is when $c \sim U[.03, 0.23]$ and all other parameters are as in the example in Table 1. In this case there is a unique, symmetric equilibrium where investment is zero for both groups (which implies zero production). However there is also an asymmetric equilibrium where $(\pi^B, \pi^W) = (0, 0.39)$. Positive production implies that wages are strictly positive for all workers, so the asymmetric equilibrium is beneficial to both groups.

Less trivial examples where discrimination is beneficial despite positive production in the symmetric equilibrium can also be constructed. The general idea is that specialization increases output unless the total number of qualified workers falls (too much) and that the an increased wage in the simple task may make up for the reduced opportunities in the complex task for the discriminated group.

[13] In general, the effective ratio of complex to simple labor may either increase or decrease. If the ratio decreases, the wage in the simple task declines, so $W-$workers with high c (who don't invest) could in principle be made worse off.

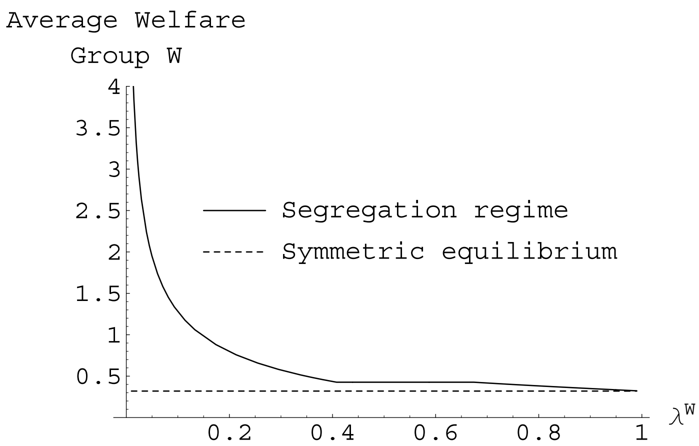

Figure 5: Average welfare in the dominant group as a function of group size

6.2 A Rationale for Institutionalized Discrimination

Since the dominant group gains from discrimination our model immediately suggests one rationale for apartheid or other discriminatory measures. We may think of such a policy as a *coordination device* assuring that the most preferred equilibrium for the group with political control is realized.

There is also a second, maybe more interesting, rationale for discriminatory policies in the model. While discrimination is always preferable for the dominant group it may simply not be sustainable in equilibrium. Hence, a law that forbids firms to assign workers from group B to the complex task may make workers from group W better off than in any "laissez faire" equilibrium.

Other parameters matter, but it is always hard to discriminate without intervention when the group with political control is relatively small. Suppose all B workers are in the simple task and assume that there are *some* workers in group B that are qualified ($\underline{c} < 0$). The basic idea for why discrimination gets harder to sustain the smaller is group W is simply that workers in the complex task will become more and more valuable as group W gets smaller. Eventually the marginal product in the complex task is so high that employers want to hire B workers to the complex task even if the likelihood that they are qualified is small. This in turn creates better incentives and, given a small enough λ^W, the segregation equilibrium unravels.[14]

[14]The critical group size can be big. With other parameters kept as in the Example of table 1 full segregation is no longer consistent with equilibrium of the *laissez faire* model if $\lambda^W < 0.212$, but if the cost distribution has more mass on lower costs a larger dominant group is needed.

While a decrease in λ^W makes a discriminatory equilibrium less likely the welfare gains from segregation increase as λ^W decreases. In Figure 5 we illustrate these gains for one example where we have kept all parameters except relative group size as in the example in Table 1 and varied λ^W. The figure plots the average payoff for a worker in group B as a function of λ^W under the assumption that workers from group B are all assigned to the simple task. Since this is incompatible with equilibrium if λ^W is small, *the force of law is needed for the dominant group to reap the benefits when they are the largest*. The example therefore suggests that segregated labor markets must be supported by apartheid-like legislation if the group in power is small. While this appeals to common sense, we are unaware of any other model of discrimination that generates similar results.

6.3 Technology and Incentives to Segregate

The parameter α also matters for the incentives to discriminate. With some stretching, one may think of an increase in α as "skill-biased technical change". It then seems natural to ask whether such technical change will increase or decrease the incentives to discriminate. Potentially, this could provide a technological explanation for Civil Rights legislation and the abolishment of slavery or apartheid. The reader should take notice that *some* aspects of the patterns displayed in this example are parameter-specific, even within this very restricted parametric class, so this section is only meant to be suggestive.

We have let the technology parameter α vary and kept all other parameters as in the example in Table 1. Figure 6 compares the average welfare of the dominant group in three different regimes. The dotted line correspond to the unique, symmetric equilibrium. The dashed line refers to the "segregation regime" in which all B workers are forced *by law* to be employed in the simple task. Finally, the continuous line in the figure plots the welfare of the dominant group in a regime where, in addition to being segregated, the B workers' earnings are expropriated by the W workers (we label this the "slavery" regime). The "segregation" and "slavery" regimes correspond to the same labor market outcomes and differ only in the distribution of property rights over wages.

Changes in α affect the possibility to support segregation in equilibrium in a straightforward manner. The higher is α the harder segregation is to sustain. In the example the threshold is approximately at $\alpha = 0.76$, and for lower α segregation is an equilibrium also without mandated segregation, whereas the force of law is necessary for higher values of α.

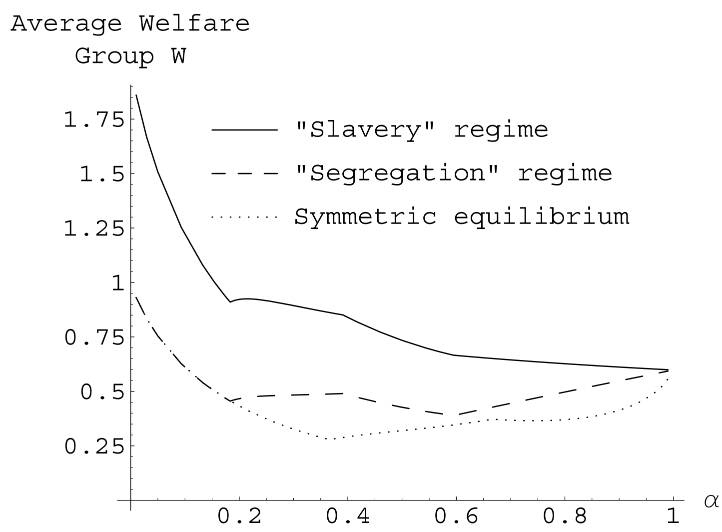

Figure 6: Welfare of the dominant group under three different regimes as a function of α

The effects on *incentives* to segregate are more subtle. When α is small, most workers are employed in the simple task in either regime. In particular, a fraction of W workers with high signals are employed in the simple task in both regimes, implying that aggregate inputs are identical in all regimes.[15] There are therefore no gains from segregation for W workers, whereas slavery is still advantageous because they expropriate group B.

When α is high, forcing the segregated group into the simple task has negligible effects on output and productivity for workers in the complex task. Segregation then becomes like a scaled down version of the symmetric equilibrium, where the labor of the segregated group is almost totally wasted. The gains for W workers are small, and the losses for society as a whole are large. Since there is little to steal from B workers, slavery is also unprofitable.

The big gains from segregation occur for intermediate values of α, when segregating group B into the simple task has significant effects on the productivity of W workers. However, as is evident from Figure 6, the payoff difference between segregation and the color-blind equilibrium is not necessarily single-peaked. There are two peaks in the payoff difference in the example, which has

[15] Labor market outcomes coincide for small values of α because a fraction of W workers with signal θ_H are employed in the simple task. There are therefore no incentives to invest and the equilibrium investment is $\pi^W = G(0)$ both with and without segregation. The factor ratio is the same in all regimes, there is no effect on the marginal productivity in the complex task, and therefore no incentive to segregate.

to do qualitative changes in the type of equilibrium as α increases. Some W workers are in the simple task if α is low. This implies that the wage for W workers with signal θ_L decreases as α increases because of a decline the marginal productivity in the complex task. Eventually, all W workers are assigned to the complex task, and at this point (which is around $\alpha = 0.6$) the wage of the low signal workers start to increase in α (see Figure 7).

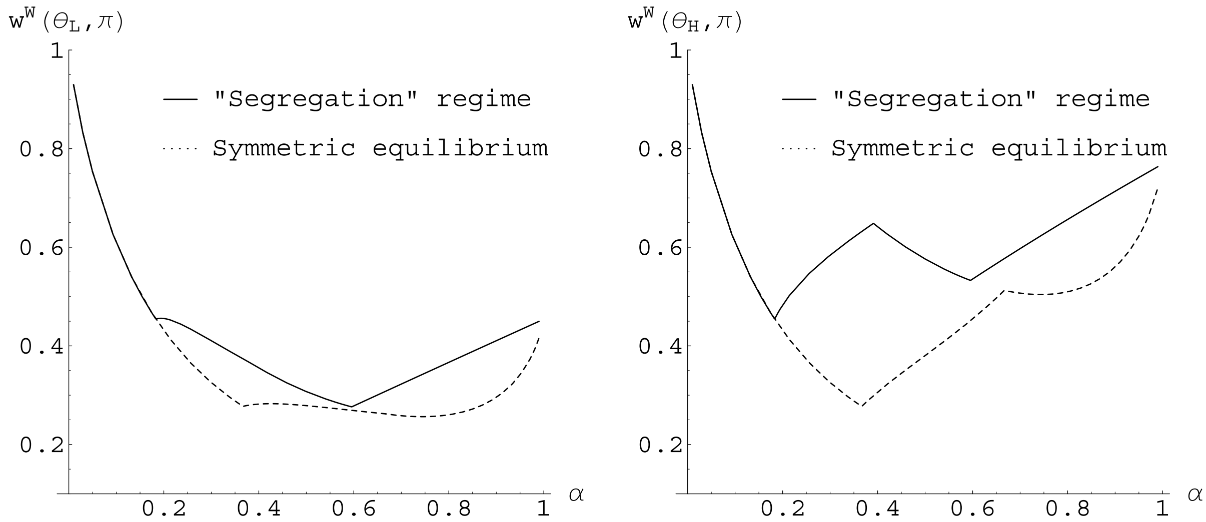

Figure 7: Wage under two different regimes as a function of α

Harder to understand is the behavior of the wage for W workers with signal θ_H. After being increasing in α for a while, it begins to decrease at $\alpha = 0.4$. Again, this corresponds with a qualitative change in the equilibrium. The peak at $\alpha = 0.4$ in Figure 7 corresponds with a change from only signal θ_H workers being assigned to the complex task to a situation where a fraction of the θ_L workers are also assigned to the complex task. At this point the factor ratio starts to increase at a fast enough rate so that incentives are affected negatively and π^W begins to decrease in α, up to the point where the economy runs out of W workers to reassign to the complex task (at $\alpha = 0.6$, where the high signal wage starts to increase again).

The discussion above suggests that the two local maxima with respect to the payoff difference between the segregation regime and the symmetric equilibrium may be an artefact of the parametrization. In spite of the many different effects that are active, it is possible that a richer set of signals would create a more systematic pattern, but this would be computationally more demanding and we have not tried this yet.

In sum, what seems robust is that incentives to segregate are small for very low and very high α. In an intermediate range the gains are substantial, but behaves non-systematically both within and across examples.

7 Summary and Concluding Remarks

The main contribution of this paper is to develop a model of statistical discrimination with true interaction effects between groups. Incentives to acquire human capital are affected not only by investment behavior in the own group, but also by human capital investments in the other group. This "cross-group effect" makes it possible for discrimination to arise also when there is a unique symmetric equilibrium. In our parametric examples it is in fact the case that if investments in one group is taken as given there is a unique fraction of investors in the other group. Hence, equality can only be obtained if both groups re-coordinate simultaneously. The dominant group is better off in equilibria with discrimination, which we view as an appealing property since the model then can rationalize why active measures are taken to institutionalize discrimination.

The interaction between groups in our model arises because of curvature in the production technology, but there alternative mechanisms. One possibility (see Moro and Norman [13]) is to replace the curvature in technology with curvature in preferences over different goods, which have different intensities in qualified labor. Such a setup generates similar cross-group effects.

Our second contribution is that the model is a full-fledged general equilibrium model. Given our focus, a general equilibrium approach is necessary: meaningful welfare analysis of (economy-wide) economic policies requires (economy-wide) feasibility constraints for the policy maker to respect. However, we think that there are compelling reasons to avoid partial equilibrium models also for strictly positive analysis. Since discrimination is considered a social problem one is often interested in what would happen if different (economy-wide) policies are introduced. In a partial equilibrium framework one therefore worries that feedback on wages or other important margins are ignored[16].

[16] An instructive example is the differences between the analysis of affirmative action in Coate and Loury [5] and Moro and Norman [12], which is an application of the model considered in this paper. Coate and Loury's argument that affirmative action may lead to "patronization" has little support in our model, where wages adjust. On the other hand, we find that affirmative action may harm the intended beneficiaries, which is impossible in Coate and Loury's model, where wages are fixed.

We find it interesting to note that the relative size of the discriminated group matters in an intuitive way in our parametric examples. One the one hand side, the larger the discriminated group is, the higher are the benefits from discrimination for the other group. On the other hand, the larger the discriminated group is the harder it is to sustain discrimination as an equilibrium. The model can therefore rationalize why a segregated labor market must be supported by coercive measures when the group in power is small.

A Appendix: Proofs

To conserve space we use $f_\pi(\theta)$ as shorthand notation for $\pi f_q(\theta) + (1-\pi) f_u(\theta)$ and $F_\pi(\theta)$ for $\pi F_q(\theta) + (1-\pi) F_u(\theta)$. Limits of integration are suppressed when integrating over the whole interval $[0,1]$ and no confusion can arise. In the interest of brevity we have also omitted the proofs of some of the more intuitive intermediate steps. These are available in Moro and Norman [14].

A.1 Proof of Proposition 1

Proof of Proposition 1 (sufficiency). Given $\pi \in (0,1]$, let $\theta(\pi)$ solve the task assignment problem (3), $t : [0,1] \to [0,1]$ be the threshold rule with cutoff $\theta(\pi)$, and $(C(\pi), S(\pi)) = (\pi F_q(\theta), F_\pi(\theta))$ be the associated (aggregate) factor inputs. Suppose that each firm posts the wage schedule $w : [0,1] \to R$ given by $w(\theta;\pi)$ in (5). Moreover, suppose (which is consistent with equilibrium) that *all* workers with signal realizations on the measurable set $\Theta_1 \subset [0,1]$ break ties in favor of firm 1, while all workers on $\Theta_2 = \Theta \backslash [0,1]$ break ties in favor of firm 2, where Θ_1 and Θ_2 are such that there exists $0 \le \rho \le 1$ such that,

$$\begin{aligned} C_1(\pi) &= \int_{\theta \in \Theta \cap [\theta(\pi),1]} \pi f_q(\theta)\, d\theta = \rho C(\pi) \\ S_1(\pi) &= \int_{\theta \in \Theta_1 \cap [0,\theta(\pi))} f_\pi(\theta)\, d\theta = \rho S(\pi). \end{aligned} \tag{A1}$$

This implies that $C_2(\pi) = (1-\rho) C(\pi)$ and $S_2(\pi) = (1-\rho) S(\pi)$. Given any ρ there is a multitude of sets Θ_1 and Θ_2 satisfying (A1) and the profit for firm 1 is (firm 2 is symmetric)

$$\begin{aligned} \Pi_1 &= y(C_1(\pi), S_1(\pi)) - \int_{\theta \in \Theta_1} w(\theta;\pi) f_\pi(\theta)\, d\theta \quad \text{/from (5)/} \\ &= y(C_1(\pi), S_1(\pi)) - \int_{\theta \in \Theta_1 \cap [\theta(\pi),1]} \underbrace{\frac{\pi f_q(\theta)}{f_\pi(\theta)}}_{p(\theta,\pi)} \frac{\partial y(C(\pi), S(\pi))}{\partial C} f_\pi(\theta)\, d\theta \end{aligned} \tag{A2}$$

$$-\int_{\theta\in\Theta_1\cap[0,\theta(\pi))}\frac{\partial y\left(C\left(\pi\right),S\left(\pi\right)\right)}{\partial S}f_\pi\left(\theta\right)d\theta\quad\text{/from (A1)/}$$
$$=\quad y\left(C_1\left(\pi\right),S_1\left(\pi\right)\right)-\frac{\partial y\left(C\left(\pi\right),S\left(\pi\right)\right)}{\partial C}\rho C\left(\pi\right)-\frac{\partial y\left(C\left(\pi\right),S\left(\pi\right)\right)}{\partial S}\rho S\left(\pi\right)=0,$$

where the last equality uses homogeneity of degree zero of the partials of y and Euler's theorem. Suppose one firm deviates to $(w',t')\neq(w,t)$. Let C' and S' denote the implied factor inputs and let $a(\theta)\in[0,1]$ denote the fraction of workers with signal θ that accepts a job at the deviating firm (tie-breaking rules are restricted so that a is integrable and independent of the investment, which is fine since we are arguing that the candidate wages and task assignments are supportable as an equilibrium). Since $w'(\theta;\pi)\geq w(\theta;\pi)$ for all θ such that $a(\theta)>0$ the profit for the deviating firm, Π_i', satisfies

$$\Pi_i'\leq y\left(C',S'\right)-\int w\left(\theta,\pi\right)a\left(\theta\right)f_\pi\left(\theta\right)d\theta,\tag{A3}$$

where (here the assumption that ties are broken the same way by qualified and unqualified workers is used)

$$C'=\int t'\left(\theta\right)\pi f_q\left(\theta\right)a\left(\theta\right)d\theta\text{ and }S'=\int(1-t'\left(\theta\right))a\left(\theta\right)f_\pi\left(\theta\right)d\theta\tag{A4}$$

Moreover $w(\theta,\pi)=\max\{p(\theta,\pi)y_1(C(\pi),S(\pi)),y_2(C(\pi),S(\pi))\}$, so

$$\begin{aligned}&\int w\left(\theta,\pi\right)a\left(\theta\right)f_\pi\left(\theta\right)d\theta \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(A5)}\\
=\;&\int t'\left(\theta\right)w\left(\theta,\pi\right)a\left(\theta\right)f_\pi\left(\theta\right)d\theta+\int\left(1-t'\left(\theta\right)\right)w\left(\theta,\pi\right)a\left(\theta\right)f_\pi\left(\theta\right)d\theta\\
\geq\;&y_1\left(C\left(\pi\right),S\left(\pi\right)\right)\int t'\left(\theta\right)\pi f_q\left(\theta\right)a\left(\theta\right)d\theta+y_2\left(C\left(\pi\right),S\left(\pi\right)\right)\int(1-t'\left(\theta\right))a\left(\theta\right)f_\pi\left(\theta\right)d\theta\\
=\;&y_1\left(C\left(\pi\right),S\left(\pi\right)\right)C'+y_2\left(C\left(\pi\right),S\left(\pi\right)\right)S'\end{aligned}$$

implying that $\Pi_i'\leq y(C',S')-y_1(C(\pi),S(\pi))C'+y_2(C(\pi),S(\pi))S'\leq 0$ (by concavity and constant returns). ■

To prove necessity of the conditions in Proposition 1 we proceed by proving a sequence of intermediate results:

Lemma A1 *Each firm earns a zero profit in any equilibrium*

Proof. The proof, which is omitted, is based on the same style of reasoning as in the usual Bertrand competition model, but some work has to be done to make sure that the deviant firm attracts a distribution of workers such that efficiency in production is possible after the deviation (see Moro and Norman [14] for the formal argument). ■

Lemma A2 $w_1(\theta) = w_2(\theta)$ *for almost all* $\theta \in [0,1]$ *in any equilibrium*

Proof. The basic idea of the proof (available in Moro and Norman [14]) is that if wages differ over a non-negligible set it is possible to attract the same workers at a lower cost. Since a deviation may trigger a change in the tie-breaking rules by the workers some work must be done to assure that productive efficiency does not decline. This is done by a deviation that attracts all workers. ∎

Lemma A3 $y(C_1, S_1) + y(C_2, S_1) = y(C(\pi), S(\pi))$

Proof. By feasibility, $y(C_1, S_1) + y(C_2, S_1) \leq y(C(\pi), S(\pi))$, so assume for contradiction that $y(C(\pi), S(\pi)) - y(C_1, S_1) - y(C_2, S_1) = \delta > 0$. Suppose firm 1 offers $w_1'(\theta) = w_2(\theta) + \epsilon$ for all θ and assigns all workers in accordance with a solution to (3). The implied profit is

$$\begin{aligned} \Pi_1'(\epsilon) &= y(C(\pi), S(\pi)) - \int w_2(\theta) f_\pi(\theta) d\theta - \epsilon \\ &> y(C_1, S_1) + y(C_2, S_2) - \int w_2(\theta) f_\pi(\theta) d\theta - \epsilon \end{aligned} \tag{A6}$$

But (Lemma A2) $w_1(\theta) = w_2(\theta)$ almost everywhere, so $\int_\theta w_2(\theta) f_\pi(\theta) d\theta$ is the sum of wages paid out by firms 1 and 2 before the deviation. By zero profits (Lemma A1) this implies that $\Pi_1'(\epsilon) = \delta - \epsilon$, so for ε small enough the deviation is profitable. ∎

Lemma A4 *Suppose* $\langle w_1, w_2 \rangle$ *is a pair of equilibrium wage schedules and let* $\theta(\pi)$ *be the unique solution to (3). Then there is a pair* (k_s, k_c) *such that 1)* $w_i(\theta) = k_s$ *for* $i = 1, 2$ *and for almost all* $\theta < \theta(\pi)$, *2)* $w_i(\theta) = p(\theta, \pi) k_c$ *for* $i = 1, 2$ *and for almost all* $\theta \geq \theta(\pi)$.

Proof. The two parts have almost identical proofs, so we prove only part 2), which may appear as less obvious. Let $w(\theta) = \max(w_1(\theta), w_2(\theta))$ and $(C(\pi), S(\pi))$ be the factor inputs corresponding to a solution to (3). For contradiction, suppose there is a set $A \subset [\theta(\pi), 1]$, where $m = \int_A \pi f_q(\theta) d\theta > 0$, and some $\delta > 0$ such that for all $\theta \in A$,

$$\begin{aligned} \frac{w(\theta)}{p(\theta,\pi)} &\leq \frac{1}{1 - F_q(\theta(\pi))} \int_{\theta(\pi)}^1 \frac{w(\theta)}{p(\theta,\pi)} f_q(\theta) - \delta = \frac{1}{\pi(1 - F_q(\theta(\pi)))} \int_{\theta(\pi)}^1 w(\theta) f_\pi(\theta) - \delta \\ &= \frac{1}{C(\pi)} \int_{\theta(\pi)}^1 w(\theta) f_\pi(\theta) - \delta \ . \end{aligned} \tag{A7}$$

By continuity, there exists a set $B \in [0, \theta(\pi))$ such that $\int_B f_\pi(\theta) d\theta = \frac{S(\pi)}{C(\pi)} m$ and

$$w(\theta) \leq \frac{1}{F_\pi(\theta(\pi))} \int_{\theta(\pi)}^1 w(\theta) f_\pi(\theta) = \frac{1}{S(\pi)} \int_0^{\theta(\pi)} w(\theta) f_\pi(\theta) \tag{A8}$$

for every $\theta \in B$. Consider a deviation by firm i, where it offers $w_i'(\theta) = w(\theta) + \epsilon$ to workers with $\theta \in A \cup B$ and $w_i'(\theta) = 0$ for all other θ, and assigns workers from A to the complex task and workers from B to the simple task. The profit from this deviation is

$$\begin{aligned}
\Pi' &= y\left(\int_{\theta\in A} \pi f_q(\theta)\, d\theta, \int_{\theta \in B} f_\pi(\theta)\, d\theta\right) - \int_{\theta\in A\cup B} (w(\theta)+\epsilon) f_\pi(\theta)\, d\theta \quad /\text{(A7) \& (A8)}/ \qquad \text{(A9)} \\
&\geq y(C(\pi), S(\pi)) \frac{m}{C(\pi)} - \left(\frac{1}{C(\pi)}\int_{\theta(\pi)}^1 w(\theta) f_\pi(\theta) - \delta\right) \int_{\theta \in A} \underbrace{p(\theta,\pi) f_\pi(\theta)}_{=\pi f_q(\theta)} d\theta \\
&\quad - \int_{\theta\in B} f_\pi(\theta)\, d\theta \left[\frac{1}{S(\pi)} \int_0^{\theta(\pi)} w(\theta) f_\pi(\theta)\right] - \epsilon \int_{\theta\in A\cup B} f_\pi(\theta)\, d\theta \Bigg/ \begin{array}{c} \int_A \pi f_q(\theta)\, d\theta = m \\ \int_B f_\pi(\theta)\, d\theta = \frac{S(\pi)}{C(\pi)} m \end{array} \Bigg/ \\
&= y(C(\pi), S(\pi)) \frac{m}{C(\pi)} - \left(\int_{\theta(\pi)}^1 w(\theta) f_\pi(\theta) - \delta\right) \frac{m}{C(\pi)} \\
&\quad - \frac{S(\pi)}{C(\pi)} m \left[\frac{1}{S(\pi)} \int_0^{\theta(\pi)} w(\theta) f_\pi(\theta)\right] - \epsilon \int_{\theta\in A\cup B} f_\pi(\theta)\, d\theta \\
&= \frac{m}{C(\pi)} \underbrace{\left(y(C(\pi), S(\pi)) - \int_{\theta\in[0,1]} w(\theta) f_\pi(\theta)\right)}_{=0 \text{ by Lemmas A1 and A3}} + \delta \frac{m}{C(\pi)} - \epsilon \int_{\theta \in \Theta' \cup \Theta''} f_\pi(\theta)\, d\theta
\end{aligned}$$

Hence, $\Pi' \geq \delta \frac{m}{C(\pi)} - \epsilon \int_{\theta\in A\cup B} f_\pi(\theta)\, d\theta > 0$ for ε small enough, which together with Lemma A2 establishes part 2) of the claim. The proof of the other half is symmetric. Removing the δ from (A7) and inserting a δ in the inequality in (A8) and again constructing A and B such that the factor ratio is as in a solution to (3) (i.e., satisfying the second condition in (A8)), the rest of the argument is unaltered. ■

Proof of Proposition 1 (necessity). It remains to be shown that $k_s = y_2(C(\pi), S(\pi))$ and $k_c = y_1(C(\pi), S(\pi))$. Firm would make positive profits if $k_s < y_2(C(\pi), S(\pi))$ and $k_c < y_1(C(\pi), S(\pi))$ and negative profits if the inequalities go the other way. Hence, we need only consider the cases where the inequalities go opposite directions. The arguments are symmetric and we only consider the case with $k_s > y_2(C(\pi), S(\pi))$ and $k_c < y_1(C(\pi), S(\pi))$. If $\theta(\pi) = 0, (1)$ each firm makes a positive profits (loss), so the only case to consider is when $\theta(\pi)$ is interior. A necessary condition for optimality for problem (3) is that $y_1(C(\pi), S(\pi))\, p(\theta(\pi), \pi) = y_2(C(\pi), S(\pi))$. Hence, there must be an interval $(\theta(\pi), \theta^*)$ where $w_i(\theta) = p(\theta, \pi) k_c < k_s$ for all $\theta \in (\theta(\pi), \theta^*)$.

Consider the deviation

$$w_i'(\theta) = \begin{cases} w(\theta) + \epsilon & \text{for } \theta \in (\theta(\pi), \theta^*) \\ 0 & \text{otherwise} \end{cases} \quad t_i'(\theta) = \begin{cases} 0 & \text{for } \theta \in (\theta(\pi), \theta') \\ 1 & \text{for } \theta \in [\theta', \theta^*) \end{cases}, . \tag{A10}$$

where θ' is set so that the factor ratio is as in the solution to (3) ($\frac{\int_{\theta'}^{\theta^*} \pi f_q(\theta) d\theta}{\int_{\theta(\pi)}^{\theta'} f_\pi(\theta) d\theta} = \frac{C(\pi)}{S(\pi)}$). The profit is

$$\begin{aligned} \Pi' &= y\left(\int_{\theta'}^{\theta^*} \pi f_q(\theta)\, d\theta, \int_{\theta(\pi)}^{\theta'} f_\pi(\theta)\, d\theta\right) - \int_{\theta(\pi)}^{\theta'} w(\theta) f_\pi(\theta)\, d\theta - \int_{\theta'}^{\theta^*} w(\theta) f_\pi(\theta)\, d\theta \qquad \text{(A11)} \\ &> \frac{y(C(\pi), S(\pi)) \int_{\theta'}^{\theta^*} \pi f_q(\theta)\, d\theta}{C(\pi)} \\ &\quad - y_2(C(\pi), S(\pi)) \int_{\theta'}^{\theta^*} f_\pi(\theta)\, d\theta - y_1(C(\pi), S(\pi)) \int_{\theta'}^{\theta^*} \pi f_q(\theta)\, d\theta \\ &= \int_{\theta'}^{\theta^*} \pi f_q(\theta)\, d\theta \left[y(C(\pi), S(\pi)) - y_2(C(\pi), S(\pi))\ S(\pi) - y_1(C(\pi), S(\pi))\, C(\pi)\right] = 0, \end{aligned}$$

which completes the proof of Proposition 1. ■

A.2 Proof of Proposition 3.

Proof. Suppose π' and π^* solve (16) and let $\pi' < \pi^*$. A calculation shows that $\int w^J(\theta; \pi^*) f_q(\theta) > \int w^J(\theta; \pi') f_q(\theta)$ and $\int w^J(\theta; \pi^*) f_u(\theta) > \int w^J(\theta; \pi') f_u(\theta)$, so workers with unchanged investment choices are strictly better off in the higher equilibrium. For workers with costs so that they choose to invest in the π^*-equilibrium, but not in the π'-equilibrium we have that

$$\begin{aligned} \int w^J(\theta; \pi^*) (f_q(\theta) - f_u(\theta))\, d\theta &\geq c > \int w^J(\theta; \pi') (f_q(\theta) - f_u(\theta))\, d\theta \qquad \text{(A12)} \\ \Rightarrow \underbrace{\int w^J(\theta; \pi^*) f_q(\theta)\, d\theta - c}_{\text{payoff for agent } c \text{ in } \pi^* \text{ eq}} &> \int w^J(\theta; \pi') f_q(\theta)\, d\theta + \int \underbrace{\left[w^J(\theta; \pi^*) - w^J(\theta; \pi')\right]}_{>0} f_u(\theta)\, d\theta \\ &> \int w^J(\theta; \pi') f_q(\theta)\, d\theta > \underbrace{\int w^J(\theta; \pi') f_u(\theta)\, d\theta}_{\text{payoff for agent } c \text{ in } \pi' \text{ eq}} \end{aligned}$$

so these workers are also strictly better off when $\pi = \pi^*$. ■

A.3 Proof of Proposition 4.

The proof, which is relatively routine, establishes that $I(\pi)$ in (6) is continuous in π. Most work goes into establishing continuity at $\pi = 0$ and $\pi = 1$. See Moro and Norman [14].

A.4 Proof of Proposition 5

Lemma A5 *$r(\pi)$ is increasing in both arguments and strictly increasing in π^J for each π such that $\theta^J(\pi) > 0$.*

The proof, which is omitted (available in Moro and Norman [14]), uses the Kuhn-Tucker conditions to (8). The strategy is to assume that π^J increases and $r(\pi)$ decreases, which by use of the Kuhn-Tucker conditions implies that $\theta^J(\pi)$ must decrease for both groups. Using the definition of the equilibrium factor ratio, (18), this implies that $r(\pi)$ increases, a contradiction. ■

Lemma A6 *$\theta(\pi)$ is continuously differentiable over the range where both thresholds are interior.*

Proof. The proof is a direct application of the implicit function theorem and omitted. ■

Proof of Proposition 5. To complete the proof for the weak version of the result, note that if $\pi^W < \pi^{W\prime}$ then Lemma A5 implies that $r(\pi) \leq r(\pi')$ and the difference in the gross benefits of invest for group B is

$$\Delta I^B = I^B(\pi) - I^B(\pi') = \int (w^B(\theta,\pi) - w^B(\theta,\pi'))(f_q(\theta) - f_u(\theta))d\theta \tag{A13}$$

The wage in the simple task increases and the wage in the complex task decreases for group B so $w^B(\theta,\pi) - w^B(\theta,\pi')$ is an increasing function of θ, which since F_q first order stochastically dominates F_u gives the result. For the strict part, in the case of a fully interior solution differentiate $I^B(\pi)$ (defined in (11) with respect to π^W to get

$$\begin{aligned} \frac{d}{d\pi^W} I^B(\pi) &= \underbrace{\frac{\partial^2 y(r(\pi),1)}{\partial C \partial S}}_{>0} \underbrace{(F_q(\theta^B(\pi)) - F_u(\theta^B(\pi)))}_{<0 \text{ by MLRP}} \underbrace{\frac{\partial r(\pi)}{\partial \pi^W}}_{>0} \\ &+ \underbrace{\frac{\partial^2 y(r(\pi),1)}{\partial C^2}}_{<0} \int_{\theta^B(\pi)}^{1} \underbrace{p(\theta,\pi^B)(f_q(\theta) - f_u(\theta))d\theta}_{>0 \text{ by MLRP}} \underbrace{\frac{\partial r(\pi)}{\partial \pi^W}}_{>0} < 0,. \end{aligned} \tag{A14}$$

where $\frac{\partial^2 y(r(\pi),1)}{\partial C \partial S} > 0$ and $\frac{\partial^2 y(r(\pi),1)}{\partial C^2} < 0$ follows from strict quasi-concavity and $\frac{\partial r(\pi)}{\partial \pi^W} > 0$ by Lemma A5. We leave to the reader to verify that if $\theta^W(\pi) = 1$, we still have that $\frac{\partial r(\pi)}{\partial \pi^W} > 0$ if $\frac{\partial r(\pi)}{\partial \pi^W}$ is suitably reinterpreted as a right-hand derivative. Hence, (A14) applies for this case as well. ■

A.5 Proof of Proposition 6

Proof. Suppose $\pi^B = \alpha$ and let $\pi^W(\alpha)$ be the largest solution to $\pi^W = G_\alpha\left(I^W\left(\pi^W, \pi^B = 0\right)\right)$. By assumption there exists some π^* such that $\pi^* = G\left(I^W(\pi^*, \pi^*)\right)$ and since $G_\alpha(c) > G(c)$ for every c in the interior of the support and since (Proposition 5) $I^W\left(\pi^W, \pi^B\right)$ is decreasing in π^B it follows that $\pi^* = G\left(I^W(\pi^*, \pi^*)\right) < G\left(I^W(\pi^*, 0)\right)$. This implies that $\pi^W(\alpha) > \pi^*$for every $\alpha > 0$. For $\left(\pi^B, \pi^W\right) = \left(\alpha, \pi^W(\alpha)\right)$ to be consistent with equilibrium it is sufficient that $\theta^B\left(\alpha, \pi^W(\alpha)\right) = 1$ (because $I^W\left(\pi^W, \pi^B\right) = I^W\left(\pi^W, 0\right)$ for all π^B such that $\theta^B\left(\alpha, \pi^W\right) = 1$ this implies that $\pi^W(\alpha)$ is a best response as well), which requires that

$$p(1, \alpha)\, y_1\left(r\left(a, \pi^W(\alpha)\right), 1\right) \leq y_2\left(r\left(a, \pi^W(\alpha)\right), 1\right). \tag{A15}$$

But $r\left(a, \pi^W(\alpha)\right) \geq \lambda^W r^*$, where r^* is the factor ratio in the symmetric equilibrium π^* under distribution G. Hence, (A15) is satisfied for α small enough, which establishes the claim. ■

A.6 Proof of Proposition 7

Lemma A7 *Given any $\pi \in [0,1]$ and $\pi^B \in [0,1]$, the average income in group W is always higher when investments are given (π^B, π) than when both groups have a fraction π of qualified workers.*

Proof. Consider a fictitious economy with one B agent and one W agent, where agents can either sell complex and simple labor at the market at prices $w_c = \frac{\partial y(*)}{\partial C}$ and $w_s = \frac{\partial y(*)}{\partial S}$ per effective unit, and where the $*$-argument is shorthand for evaluating the marginal products at equilibrium factor inputs given fractions of investors (π^B, π^W). Also assume that the fictitious representative J agent has available the technology y. The problem for a utility maximizing $J-$agent in this economy would then be to solve

$$\max_{C^J, S^J, \alpha^J, \beta^J} y\left(\alpha^J C^J, \beta^J S^J\right) + w_c\left(1 - \alpha^J\right) C^J + w_s\left(1 - \beta^J\right) S^J \tag{A16}$$
$$\text{subj to. } S^J \leq H\left(C^J\right) = \pi^J - C^J + \left(1 - \pi^J\right) F_u\left(F_q^{-1}\left(\frac{\pi^J - C^J}{\pi^J}\right)\right)$$

From (A16) one shows one solution is to set $\alpha^J = \beta^J = 0$ and provide exactly the effective factor inputs as in the equilibrium (all other solutions are equivalent in terms of total effective factor inputs. The indeterminacy that comes in is that there are positive α^J and β^J that can equalize marginal products in "domestic" production with market wages). Since "autarky" (the symmetric equilibrium) is a feasible solution to (A16) for any (π^B, π^W) we conclude that group W is weakly better off when investments are (π^B, π) than if investments are (π, π) .for any $\pi^B \in [0, 1]$. ■

Lemma A8 *If π^* is the symmetric equilirbium and (π^{B*}, π^{W*}) is an equilibrium where $\pi^{B*} < \pi^{W*}$, then $\pi^{B*} < \pi^* < \pi^{W*}$.*

Proof. If π^* is the only symmetric equilibrium, then $\pi > G\left(I^J(\pi, \pi)\right)$ for all $\pi > \pi^*$ and $J = B, W$ (otherwise there would be at least one additional equilibrium). If $\pi^* \leq \pi^{B*} < \pi^{W*}$, then Proposition 5 implies that $I^B\left(\pi^{B*}, \pi^{W*}\right) < I^B\left(\pi^{B*}, \pi^{B*}\right)$ (the solution at (π^{B*}, π^{B*}) in a neighborhood is necessarily interior so that the inequality is strict). But, since $\pi^{B*} \geq G\left(I^B\left(\pi^{B*}, \pi^{B*}\right)\right)$ $> G\left(I^B\left(\pi^{B*}, \pi^{W*}\right)\right)$ this contradicts the assumption that (π^{B*}, π^{W*}) is an equilibrium, so we conclude that $\pi^{B*} < \pi^*$. The proof of $\pi^* < \pi^{W*}$ is similar and left to the reader. ■

Proof of Proposition 7. Output increases if investments change from (π^*, π^*) to $\left(\pi^{W*}, \pi^{W*}\right)$, and since groups are treated symmetrically and all output is paid back to workers the average wage also increases. By Lemma A7, the average wage in group W is further increased when investments change from $\left(\pi^{W*}, \pi^{W*}\right)$ to (π^{B*}, π^{W*}), which taken together with the first change means that the average wage in group W is higher in (π^{B*}, π^{W*}) than in (π^*, π^*). Finally, all agents in group W may choose (i.e., it is a feasible option) to invest exactly as in the symmetric equilibrium (π^*, π^*) , in which case higher surplus follows immediately from the higher average wage. Since this is feasible, it must also be that investing at the higher rate must be weakly better for all agents who change their behavior across equilibria, so the average utility in group W must be higher in the equilibrium with discrimination. ■

A.7 Proof of Proposition 8

Proof (part 1), By direct differentiation of (21)

$$\frac{d}{d\pi^W} Y(h(\pi^W), \pi^W) = \frac{\partial Y(h(\pi^W), \pi^W)}{\partial \pi^B} \frac{dh(\pi^W)}{d\pi^W} + \frac{\partial Y(h(\pi^W), \pi^W)}{\partial \pi^W}. \tag{A17}$$

Appealing to the envelope theorem, the partial derivative of Y with respect to π^J is

$$\frac{\partial Y(\pi^B, \pi^W)}{\partial \pi^J} = \frac{\partial y}{\partial C} \lambda^J (1 - F_q(\theta^J)) + \frac{\partial y}{\partial S} \lambda^J (F_q(\theta^J) - F_u(\theta^J)), \tag{A18}$$

where the arguments have been omitted for brevity. Since $dh(\pi^W)/d\pi^W = -\lambda^W/\lambda^B$ (A17) and (A18) can be combined to yield

$$\frac{d}{d\pi^W}Y(h(\pi^W),\pi^W) = \lambda^W\left(\left(\frac{\partial y}{\partial C}-\frac{\partial y}{\partial S}\right)\left(F_q(\theta^B)-F_q(\theta^W)\right)+\frac{\partial y}{\partial S}\left(F_u(\theta^B)-F_u(\theta^W)\right)\right) \quad \text{(A19)}$$

The Kuhn-Tucker conditions for an optimal solution to (8) may be written

$$\begin{aligned} -p\left(\theta^J(\pi),\pi^J\right)\frac{\partial y(r(\pi),1)}{\partial C}+\frac{\partial y(r(\pi),1)}{\partial S}+\gamma^J-\kappa^J &= 0 \qquad \text{for } J=B,W, \qquad \text{(A20)} \\ \gamma^J\theta^J(\pi) = 0, \quad \kappa^J\left(1-\theta^J(\pi)\right) = 0, \quad \gamma^J \geq 0, \quad \kappa^J &\geq 0. \end{aligned}$$

Conditions (A20) imply that if $\theta^B \leq \theta^W$,then $p\left(\theta^B,\pi^B\right) < p\left(\theta^W,\pi^W\right)$ and

$$0 < \frac{\partial y}{\partial C}\left(p\left(\theta^W,\pi^W\right)-p\left(\theta^B,\pi^B\right)\right) = \gamma^W-\gamma^B-\eta^W+\eta^B \quad \text{(A21)}$$

For (A21) to hold, either γ^W or η^B is strictly positive. But, if $\gamma^W > 0$, then $\theta^W = 0 \Rightarrow \theta^B = 0$ (since $\theta^B \leq \theta^W$). By the Inada conditions this implies that $\frac{\partial y}{\partial S} = \infty$ and $\frac{\partial y}{\partial C}p\left(\theta^J,\pi^J\right) = 0$, violating (A20). Next, $\eta^B > 0 \Rightarrow \theta^B = 1 \Rightarrow \theta^W = 1$ (since $\theta^B \leq \theta^{\mathrm{W}}$). By the Inada conditions this implies that $\frac{\partial y}{\partial S} = 0$ and $\frac{\partial y}{\partial C}p\left(\theta^J,\pi^J\right) = \infty$, again violating (A20). We conclude that $\theta^B > \theta^W \Rightarrow F_q(\theta^B) - F_q(\theta^W) > 0$ and $F_u(\theta^B) - F_u(\theta^W) > 0$. Finally, we observe that this also implies that $\theta^W < 1 \Rightarrow \eta^J = 0 \Rightarrow \frac{\partial y}{\partial C}p\left(\theta^J,\pi^J\right) \geq \frac{\partial y}{\partial S}$, which since $p\left(\theta^J,\pi^J\right) < 1$ guarantees that $\frac{\partial y_1}{\partial C} > \frac{\partial y_1}{\partial S}$. Thus, all terms in (A19) are strictly positive, establishing the first part.

Proof (part 2). Again using the chain rule and that $dh(\pi^W)/d\pi^W = -\lambda^W/\lambda^B$

$$\begin{aligned} &\frac{d}{d\pi^W}\left(\lambda^B\int_{\underline{c}}^{G^{-1}\left(h(\pi^W)\right)}cg(c)dc+\lambda^W\int_{\underline{c}}^{G^{-1}\left(\pi^W\right)}cg(c)\,dc\right). \qquad \text{(A22)} \\ = \;& -\lambda^W G^{-1}\left(h(\pi^W)\right)g\left(G^{-1}\left(h(\pi^W)\right)\right)\frac{dG^{-1}\left(h(\pi^W)\right)}{d\pi^B}\frac{dh\left(\pi^W\right)}{d\pi^W} \\ & +\lambda^W G^{-1}\left(\pi^W\right)g\left(G^{-1}\left(\pi^W\right)\right)\frac{dG^{-1}\left(\pi^W\right)}{d\pi^W} = \lambda^W\left(G^{-1}\left(\pi^W\right)-G^{-1}\left(\pi^B\right)\right) > 0, \end{aligned}$$

where the last equality follows since $g\left(G^{-1}\left(\pi^J\right)\right)\frac{dG^{-1}\left(\pi^J\right)}{d\pi^J} = 1$ by the inverse function theorem and the inequality follows since G^{-1} is strictly increasing. ■

A.8 Proof of Proposition 9.

Proof. First consider the case with $\alpha \leq \phi$ in which case the optimal solution to the task assignment problem (24) is to set $(\sigma(\pi),\gamma(\pi)) = (1,0)$ when $0 \leq \pi \leq \frac{\alpha+\phi-1}{2\phi-1}$. The associated equilibrium wages are

$$w(\theta_H;\pi) = \frac{\phi\pi}{\phi\pi+(1-\phi)(1-\pi)}\alpha\left(\frac{(1-\phi)\pi+\phi(1-\pi)}{\phi\pi}\right)^{1-\alpha} \tag{A23}$$
$$w(\theta_L;\pi) = (1-\alpha)\left(\frac{\phi\pi}{(1-\phi)\pi+\phi(1-\pi)}\right)^{\alpha}$$

When $\pi > \frac{\alpha+\phi-1}{2\phi-1}$ a fraction $\sigma(\pi) > 0$ of the workers with signal θ_H are in the simple task, so in this case there $w(\theta_H;\pi) = w(\theta_L;\pi)$ and there are no incentives to invest. Substituting (A23) into the expression for incentives to invest in (26) and simplifying the result we may write these incentives concisely as

$$I(\pi) = \max\left\{k\left(\frac{\phi\pi}{\phi-k\pi}\right)^{\alpha}\left(\frac{\alpha}{k\pi+(1-\phi)}-1\right),0\right\} \quad \text{for } k\equiv 2\phi-1, \tag{A24}$$

where the reason for the max-operator is that $\frac{\alpha}{k\pi+(1-p)}-1<0$ when $\pi > \frac{\alpha+p-1}{2p-1}$, which is the range where $\sigma(\pi)>0$ and incentives consequently are equal to zero. Define

$$J(\pi) = \left(\frac{\pi}{\phi-k\pi}\right)^{\alpha}\left(\frac{\alpha}{k\pi+(1-\phi)}-1\right), \tag{A25}$$

so that $I(\pi)$ in (A24) is given by $I(\pi) = k\phi^{\alpha}J(\pi)$ whenever $I(\pi)>0$.

STEP 1: G UNIFORM: If G is uniform $G(I(\pi)) = QJ(\pi)+R$ for any $I(\pi)\in[\underline{c},\overline{c}]$, where $Q=\frac{1}{[\underline{c},\overline{c}]}k\phi^{\alpha}$ and $R=-\frac{\underline{c}}{[\underline{c},\overline{c}]}>0$ ($\underline{c}<0$ by assumption). By a direct calculation we have that

$$J'(\pi) = J(\pi)\alpha\left(\frac{\phi}{\pi(\phi-k\pi)}-\frac{k}{(k\pi+1-\phi)(\alpha-k\pi-(1-\phi.))}\right). \tag{A26}$$

A sufficient condition for uniqueness is that $\frac{d}{d\pi}G(I(\pi^*))<1$ in any equilibrium π^* (since $\underline{c}<0\Rightarrow G(I(0))$ is above the diagonal). We drop the $*-$superscript for equilibria and note that an equilibrium point satisfies $\pi = G(I(\pi)) = QJ(\pi)+R$, so

$$\frac{d}{d\pi}G(I(\pi)) = QJ'(\pi) = QJ(\pi)\alpha\left(\frac{\phi}{\pi(\phi-k\pi)}-\frac{k}{(k\pi+1-\phi)^2(\alpha-k\pi-(1-\phi.))}\right) \tag{A27}$$
$$= (\pi-R)\alpha\left(\frac{\phi}{\pi(\phi-k\pi)}-\frac{k}{(k\pi+1-\phi)(\alpha-k\pi-(1-\phi.))}\right)$$
$$< \pi\alpha\left(\frac{\phi}{\pi(\phi-k\pi)}-\frac{k}{(k\pi+1-\phi)(\alpha-k\pi-(1-\phi.))}\right),$$

where the equality is from evaluating the derivative at an equilibrium and the inequality follows since $R > 0$. The expression on the third line of (A27) is increasing in α and $\alpha \leq \phi$, so

$$\begin{aligned} \frac{d}{d\pi} G(I(\pi)) &< \pi\alpha \left(\frac{\phi}{\pi(\phi - k\pi)} - \frac{k}{(k\pi + 1 - \phi)(\alpha - k\pi - (1 - \phi.))} \right) \\ &\leq \pi\phi \left(\frac{\phi}{\pi(\phi - k\pi)} - \frac{k}{(k\pi + 1 - \phi)(\phi - k\pi - (1 - \phi.))} \right) \\ &= \frac{\phi^2}{\phi - k\pi} - \frac{\pi\phi}{(k\pi + 1 - \phi)(1 - \pi)}. \end{aligned} \tag{A28}$$

Now, $k\pi + 1 - \phi < k + (1 - \phi) = \phi \Rightarrow \frac{\phi}{k\pi + 1 - \phi} > 1$, implying that

$$\frac{d}{d\pi} G(I(\pi)) < \frac{\phi^2}{\phi - k\pi} - \frac{\pi}{(1 - \pi)} = \frac{\phi^2}{\phi - (2\phi - 1)\pi} - \frac{\pi}{(1 - \pi)}. \tag{A29}$$

Differentiating and simplifying we find that $\frac{d}{d\phi}\left(\frac{\phi^2}{\phi - (2\phi - 1)\pi}\right) > 0$ and since $\phi \in \left[\frac{1}{2}, 1\right]$ the derivative $\frac{d}{d\pi} G(I(\pi))$ is bounded by the right hand side of (A29) evaluated at $\phi = 1$. That is, $\frac{d}{d\pi} G(I(\pi)) < \frac{1}{1 - \pi} - \frac{\pi}{(1 - \pi)} = 1$, which establishes that equilibria must be unique when G is uniform.

STEP 2: G CONCAVE: For a general concave distribution we note that for every $c \in [\underline{c}, \overline{c}]$, the mean value theorem implies that there exists $c^* \in [\underline{c}, c]$ such that $G(c) - G(0) = G(c) = G'(c^*)\, c$, hence the equilibrium must satisfy $\pi = G(I(\pi)) = G'(c^*)\, I(\pi)$ for some $c^* \leq I(\pi)$. Concavity implies that $G'(I(\pi)) \leq G'(c^*)$, so

$$\frac{d}{d\pi} G(I(\pi)) = G'(I(\pi))\, I'(\pi) \leq G'(c^*)\, I'(\pi) = G'(c^*)\, k\phi^\alpha J'(\pi) = QJ'(\pi) \tag{A30}$$

for $Q = G'(c^*)\, k\phi^\alpha$. At this point it is just to proceed as with a uniform distribution (with $R = 0$).

Finally, the other case with $\alpha > \phi$ can be handed in a similar way. Tedious, but straightforward algebra shows that the incentives in this case are given by

$$I(\pi) = M \left(\frac{\phi}{k\pi + (1 - \phi)} - 1 \right) \left(\frac{\pi}{\phi - k\pi} \right)^\alpha \tag{A31}$$

where $M = \frac{k\alpha^\alpha (1 - \alpha)^{1 - \alpha}}{(1 - \phi)^{1 - \alpha}}$. Compared with the previous case there are only two changes. The first is that the constant has changed, which doesn't matter since the constant disappears when evaluating the derivative at an equilibrium. The second change is that the term $\frac{\phi}{k\pi + (1 - \phi)}$ replaces $\frac{\alpha}{k\pi + (1 - \phi)}$, which does affect the remaining calculations. Details available in Moro and Norman [14]. ■

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