Overcoming Participation Constraints*

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Abstract

This paper shows that linking a sufficiently large number of independent but unrelated social decisions can achieve approximate efficiency. A Groves mechanism amended with a veto game implements an efficient outcome with probability arbitrarily close to one, and satisfies interim participation, incentive and resource constraints.

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1 Introduction

Collective decision making is often a process in which many issues are resolved jointly. An example is the WTO trade agreement of 1994, which took eight years to negotiate (the 1986-1994 “Uruguay Round”), and is a staggeringly complex agreement, loaded with special provisions for various goods and services. Another example is the Transportation Act of 2005, including more than 3000 earmarks for specific “pet projects” such as new bridges, bike-paths, ramps, parking lots, landscaping enhancements among specific highways etc. Moreover, government institutions at all levels tend to fulfill many and arguable rather unrelated functions. For example, at the local government level, libraries, museums, public parks, public schools, police and fire protection and many other goods and services are provided jointly to all tax-paying residents of the community.

The traditional public finance view is that each problem needs a separate remedy. Highway congestion and provision of a local library are thought of as two distinct problems that need two separate solutions. In general, we understand that income effects and complementarities could rationalize a joint treatment, but exactly how is unclear, so this possibility is usually ignored.

In contrast, the pros and cons of linking tariff concessions with other issues, such as labor standards, environmental issues, investment liberalization, and human rights codes, have been discussed rather extensively in the recent international trade literature, but no consensus has emerged. Some, like Bagwell and Staiger [4] argue that giving individual countries more sovereignty over “domestic issues” (e.g., to make it harder to link certain issues) would improve bargaining outcomes. Others, such as Copeland and Taylor [13], and Hortsmann et al. [20], and Maskus [24] argue that linking may be beneficial, whereas yet others, such as Conconi and Perotti [12], argue that it depends on details. By contrast, spending bills with many earmarked projects are overwhelmingly interpreted as the result of log-rolling, and associated with inefficient pork-barrel spending (See Battaglini and Coate [6] and the references therein).

Another well-known example of linking is Armstrong [2], who considers a multiproduct monopoly problem. He demonstrates that by charging a fixed fee for the right to purchase any good at marginal cost, the monopolist can almost fully extract the surplus if there is a large number of goods. The inefficiency associated with monopoly pricing is thereby virtually eliminated by linking sales of many goods, with all gains going to the seller. In the context of (excludable) public goods, several papers have shown that bundling (the practice of selling several goods as a package) can be a useful instrument both for a profit maximizing monopolist (see Bakos and Brynjolfsson [5] and Geng et al [18]) and a welfare maximizing planner (see Bergstrom and Bergstrom [7] and Fang and
Norman). In the political economy literature, Casella [10] shows that a voting scheme that allows “vote storing” allows agents to concentrate their votes where preferences are more intense, which typically leads to ex ante welfare gains. Other examples include a risk-sharing agreement proposed by Townsend [29] and a mechanism for partnership dissolution proposed in McAfee [22].

The applications mentioned in the previous paragraph span several fields, but have one property in common: the underlying inefficiency is generated by informational asymmetries. One may therefore ask whether there is some general principle involved. Recently, Jackson and Sonnenschein [21] answered this question affirmatively. They fix an arbitrary underlying social choice problem, and show that first best efficiency is approximately attainable if a sufficiently large number of independent replicas of the underlying problem is available. The key idea is that their mechanism constrains agents to report preference profiles where the frequency of any particular “base-problem type” coincides with the true probability distribution. They show that, if the social choice function is efficient, then agents seek to be as “truthful as possible”. With a large number of independent replicas, the frequency distribution of announced preference types is close to the true probability distribution. As a result, all equilibria result in approximately efficient allocations with high probability when there are many replicas.

An important limitation of the analysis in Jackson and Sonnenschein [21] is that it they can only deal with identical replicas. In this paper, we address this shortcoming by restricting the analysis to problems with transferable utility. The transferable utility framework rules out some applications covered by Jackson and Sonnenschein [21], such as standard voting problems, but still covers a rich class of problems, including bilateral bargaining, partnership dissolution, multiproduct monopoly provision, public good problems, and many other common applications. The analytical advantage is that we are able to obtain results for arbitrary sequences of underlying choice problems. Hence, various public good problems, externalities, and bargaining problems may be linked into a “grand design problem”.

If common value aspects are ruled out (as we do), many classic implementation results apply. In particular, ex post efficiency is attainable if either interim participation or self-financing constraints are absent, since transfers making agents internalize their externalities exist (Groves [19], d’Aspremont and Gerard-Varet [16] and Arrow [3]). Hence, the restriction to transferable utility with no common values implies that the underlying source of the inefficiency comes from the participation constraints, as in Myerson and Satterthwaite [26], Cramton et al [14], McAfee [22], and Mailath and Postlewaite [23].
Our paper establishes two results. The first, fairly mechanical, result establishes that a standard Groves mechanism “almost works”, in the sense that violations of the participation constraints are unlikely. The second result, which is the main contribution of the paper, shows that a Groves mechanism amended with a veto game satisfies all constraints and generates approximate efficiency.

More precisely, our first result states that, in an economy with a sufficiently large number of unrelated decision problems (satisfying stochastic independence and some regularity conditions), there is an ex ante budget-balancing Groves mechanism in which participation constraints fail with vanishingly small probability. The key intuition is that, since Groves mechanisms are efficient, lump sum transfers can be constructed that balance the budget in expectation, and where all agents have a strict incentive to participate ex ante. But, the average interim utility (where the averaging is over the decision problems) converges in probability to the average ex ante expected utility. It follows that violations of participation constraints become rare as the number of decision problems grow large.

While rare, violations of the participation constraints occur with positive probability, regardless of how many problems are linked. We therefore ask whether a nearly efficient budget balancing mechanism that always fulfills the participation constraints exists. This analysis is more subtle. The main difficulty is to rule out unravelling of the participation constraints. Specifically, to always satisfy the participation constraints we must allow agents to opt out. But then, the incentives to opt into the mechanism depend on which other types will do so. Hence, even if few types have an incentive to opt out when assuming that all other agents will opt in, the types that opt out may upset the participation constraints for other types. In Section 6.4 we provide an explicit example where the participation constraints unravel, despite the probability of a violation in the underlying Groves mechanism converging to zero.

Our main result establishes that, under appropriate regularity conditions, a linking mechanism can be constructed so as to rule out unravelling of the participation constraints. Specifically, we consider a perturbation of the Groves mechanism, which satisfies all constraints – including the participation constraints – for all type realizations. If our regularity conditions are satisfied, this mechanism implements the ex post efficient outcome with probability arbitrarily close to one, provided that the number of unrelated decisions is sufficiently large.

\footnote{In order to accommodate for pure public goods problems, we therefore need to equip agents with veto power. In many more specific models, “opting out” could be considerably less extreme measures, such as excluding the particular individual from usage of some goods and services.}
The perturbed Groves mechanism has a sequential interpretation where in stage one, all agents decide whether to veto a particular Groves mechanism. If a veto is cast, a status quo outcome that gives all agents their reservation utility is implemented. If no agent vetoes the mechanism, truth-telling is a conditionally dominant strategy, and the efficient outcome is implemented. Interestingly, the use of a Groves mechanism (as opposed to an ex post budget balanced mechanism) is crucial for our argument, despite the fact that our result is about Bayesian implementability. The reason is that selection becomes irrelevant for behavior in the second stage, which in turn gives a tractable characterization of the veto game.

Besides showing the benefits from linking across different problems, our paper also contributes to the understanding of the fundamental source of the benefits from linking simply by using a more standard mechanism than Jackson and Sonnenschein. That is, the primary construction in Jackson and Sonnenschein [21] is a rationing scheme for messages, which is designed specifically to overcome the problem to obtain information about intensity of preferences.\footnote{One may indeed interpret Jackson and Sonnenschein [21]'s rationing mechanism as an ingenious way of creating transferrable utility out of a non-transferrable utility environment by adding constraints on the available announcements agents can make.} In transferable utility environments, the ultimate source of any efficiency is the interaction between participation, feasibility, and incentive constraints, and there is simply no need to ration any messages. Instead, a variation of a standard pivot mechanism is sufficient. Hence, Jackson and Sonnenschein’s [21] mechanism can be seen as a generalization of various rationing schemes, such as a vote storage mechanism proposed in Casella [10], a risk-sharing arrangement proposed by Townsend [29], which limits how often the risk-averse agent can claim a loss, and a compromising scheme proposed by Borgers and Postl [9]. In contrast, our mechanism may be thought of as a generalization of mechanisms discussed in Armstrong [2], Bergstrom and Bergstrom [7], and Fang and Norman [17].

2 Groves Mechanisms

Consider an environment with $n \geq 2$ agents and a set $D$ of possible social decisions. Each agent $i \in I \equiv \{1, 2, ..., n\}$ is privately informed about a preference parameter $\theta^i \in \Theta^i$ and has a quasi-linear von Neumann-Morgenstern utility function given by

$$v^i(d, \theta^i) - t^i, \quad (1)$$
where \( t^i \) is interpreted as a transfer from agent \( i \) in terms of a numeraire good. Implementing social decision \( d \in D \) costs \( C(d) \in \mathbb{R} \) units of the numeraire good. Let \( \Theta = \times_{i=1}^n \Theta^i \) denote the set of all possible type profiles, and denote a generic element of \( \Theta \) by \( \theta \). By appeal to the revelation principle we only consider direct mechanisms. A pure direct revelation mechanism (henceforth mechanism) is a pair \( (x,t) \), where \( x : \Theta \to D \) is the allocation rule and \( t : \Theta \to \mathbb{R}^n \) is the cost sharing rule, where \( t(\theta) = (t^1(\theta), t^2(\theta), \ldots, t^n(\theta)) \). We adopt the convention that a positive \( t^i(\theta) \) is a transfer from agent \( i \) when \( \theta \) is announced. A mechanism is called incentive compatible if truth-telling is a Bayesian Nash equilibrium in the game induced by \( (x,t) \). Dominant strategy incentive compatibility is explicitly referred to as such.

Since utility is transferable and the valuation function \( v^i \) is one with private values, many classical implementation results apply. In particular, absent either ex post budget balance or the combination of interim participation and ex ante budget balancing constraints, any efficient allocation rule can be implemented in dominant strategies by a Groves mechanism. For easy reference, we define this class of mechanisms explicitly.

\textbf{Definition 1} A mechanism \( (x,t) \) is a Groves mechanism if for each \( \theta \)

\[
x(\theta) \in \arg \max_{d \in D} \sum_{i=1}^n v^i(d, \theta^i) - C(d)
\]

and, where for each \( i \in I \),

\[
t^i(\theta) = C(x(\theta)) - \sum_{j \neq i} v^j(x(\theta), \theta^j) + \tau^i(\theta^{-i}),
\]

and \( \tau^i : \Theta^{-i} \to \mathbb{R} \) may be arbitrarily chosen.

\section{Problems with Many Independent Social Decisions}

In this section we describe an economy with many independent social decisions. Since our design problem consists of components that are design problems by themselves we refer to the social decision in an economy with a single decision as an issue. The term social decision is reserved for the fully linked problem.

\subsection{Single-Issue Economies}

Formally, let \( e_k \equiv \{(\Theta^i_k, F^i_k, v^i_k)_{i=1}^n, D_k, C_k\} \) denote an economy with a single issue \( k \). Here, \( \Theta^i_k \) denotes the issue-\( k \) type-space for agent \( i \), and a generic element is denoted \( \theta^i_k \). We assume that
types are stochastically independent across agents, implying that the prior distribution over $\Theta^i_k$ can be represented by a cumulative distribution $F^i_k$, which also, regardless of the realized types for the other agents, is the perceived probability distribution over $\Theta^i_k$ for all other agents as well as for a fictitious planner. Preferences over different ways to resolve issue $k$ are described by the valuation function $v^i_k: \Theta^i_k \times D_k \to \mathbb{R}$, where $D_K$ is the set of possible alternatives for issue $k$. The von Neumann-Morgernstern utility for a type-$\theta^i_k$ agent $i$ in economy $e_k$ is given by

$$v^i_k (\theta^i_k, d_k) - t^i$$

(4)

for each $d_k \in D_k$, where $t^i$ is a transfer of the numeraire good. Finally, $C_k: D_k \to \mathbb{R}$ describes the cost of implementing each alternative $d_k \in D_k$ in terms of the numeraire good.

### 3.2 Many-Issue Economies

Let $E_K$ denote an economy consisting of issues $1, ..., K$. A social decision corresponds with a resolution of each issue and a transfer scheme, and we let $D_K = \times^K_{k=1} D_k$ denote the set of social decisions in economy $E_K$. A type realization for individual $i$ is a vector of realized types in single-issue economies $e_1, ..., e_K$, and we denote the type space for agent $i$ by $\Theta^i (K) = \times^K_{k=1} \Theta^i_k$. Adopting standard conventions, we write $\Theta (K) \equiv \times^n_{i=1} \Theta^i (K)$ for the space of type profiles, and $\Theta^{-i} (K) \equiv \times_{j \neq i} \Theta^j (K)$ for the space of possible type realizations among all agents except for $i$. To conserve space we suppress $K$ when it cannot cause confusion, and write $\theta^i$ (respectively $\theta$ and $\theta^{-i}$) for a generic element of $\Theta^i (K)$ (respectively, $\Theta (K)$ and $\Theta^{-i} (K)$). A mechanism in economy $E_K$ is a pair $\langle x_K, t_K \rangle$, where $x_K: \Theta (K) \to D_K$ is the allocation rule and $t_K \equiv (t^1_K, ..., t^n_K): \Theta (K) \to \mathbb{R}^n$ is the transfer rule.

For each $d \in D_K$, the cost of implementing $d$ is the sum of the costs of implementing its components,

$$C_K (d) \equiv \sum^K_{k=1} C_k (d_k).$$

(5)

The von Neumann-Morgernstern utility function for agent $i$ of type $\theta^i \in \Theta^i (K)$ is on the form in (1), with $v^i$ being replaced by the valuation function $V^i_K: \Theta^i (K) \times D_K \to \mathbb{R}$ defined as the sum of the single issue valuation functions

$$V^i_K (d, \theta^i) = \sum^K_{k=1} v^i_k (d_k, \theta^i_k).$$

(6)
Next, we assume that the issue-specific types of each agent are independently distributed across issues. That is, for each \( \theta^i \in \Theta^i (K) \), the cumulative distribution function is given by

\[
F^i_k (\theta^i) = \sum_{k=1}^{K} F^i_k (\theta^i)
\]

(7)

To sum up, the primitives of an economy \( E_K \) in terms of the components \( \{e_k\}_{k=1}^{K} \) are

\[
E_K = \left\{ \left( \Theta^i (K) = \sum_{k=1}^{K} \Theta^i_k, V^i_K = \sum_{k=1}^{K} v^i_k, F^i_K = \sum_{i=1}^{n} F^i_k \right) \right\}, D_K = \sum_{k=1}^{K} D_k, C_K = \sum_{k=1}^{K} C_k \}
\]

(8)

### 3.3 Reservation Utilities

Usually, participation constraints are imposed by assuming that there is some type independent reservation utility, which then can be normalized to zero. However, there are economic applications, such as the lemons problem in Akerlof [1] and the partnership-dissolution problem considered by Cramton et al [14], McAfee [22] and others, where type dependent reservation is the most natural way to capture the relevant economics. In order to accommodate such possibilities, we make no particular assumptions on how reservation utilities vary with type, except that they are derived from issue-specific reservation utilities.

We assume that, for each issue \( k \), there is a “status quo outcome” \( d^0_k \in D_k \) such that \( C_k (d^0_k) = 0 \). We let \( r^i_k (\theta^i_k) \equiv v^i_k (d^0_k, \theta^i_k) \) denote agent \( i \)'s reservation utility in the economy with the single issue \( k \). For each \( \theta^i \in \Theta^i (K) \), the reservation utility in economy \( E_K \), denoted \( R^i_K (\theta^i) \), is taken to be the utility for agent \( i \) in case the status quo outcome is implemented for each issue \( 1, ..., K \), that is

\[
R^i_K (\theta^i) = \sum_{k=1}^{K} r^i_k (\theta^i_k).
\]

(9)

### 3.4 Regularity

Let \( \theta_k = (\theta^1_k, ..., \theta^n_k) \in \times_{i=1}^{n} \Theta^i_k \) denote an issue-\( k \) type profile. In an economy with single issue \( k \), denote by \( x^* K (\theta_k) \) an efficient social decision rule, and by \( s_k (\theta_k) \) the associated maximized value of surplus. That is,

\[
x^* K (\theta_k) = \arg \max_{d_k \in D_k} \sum_{i=1}^{n} v^i_k (\theta^i_k, d_k) - C_k (d_k)
\]

(10)

\[
s_k (\theta_k) = \max_{d_k \in D_k} \sum_{i=1}^{n} v^i_k (\theta^i_k, d_k) - C_k (d_k),
\]
where we assume that appropriate continuity and compactness assumptions are satisfied, so that the optimization problem in (10) is well defined. It follows by definition that \( s_k(\theta_k) \geq \sum_{i=1}^{n} r^i_k(\theta^i_k) \) for every \( \theta_k \in \times_{i=1}^{n} \Theta^i_k \). However, what we need for our results is a bit stronger, namely that the expected maximized surplus from issue \( k \) is uniformly bounded away from the sum of issue-\( k \) reservation utilities. Moreover, we also require that the social surplus associated with each issue be bounded from above and below. Together, we refer to these restrictions as regularity conditions. Formally,

**Definition 2** We say that \( \{e_k\}_{k=1}^{\infty} \) is a regular sequence of single-issue economies if

**R1 [Non-vanishing Gains from Trade].** There exists some \( \delta > 0 \) such that

\[
E[s_k(\theta_k)] \geq E\left[ \sum_{i=1}^{n} r^i_k(\theta^i_k) \right] + \delta, \text{ for all } k;
\]

**R2 [Uniform Bounds on Surplus].** There exist some \( a < \bar{a} \) such that \( a < s_k(\theta_k) < \bar{a} \) for all \( \theta_k \in \times_{i=1}^{n} \Theta \) and for all \( k \);

**R3 [Uniform Bounds on Reservation Utilities].** There exist some \( b < \bar{b} \) such that \( b < r^i_k(\theta^i_k) < \bar{b} \) for all \( i, \theta^i_k \in \Theta^i_k \), and all \( k \).

### 3.5 Some Remarks about the Environment

1. We kept the set of agents the same and finite for all problems. Since we can allow agents with trivial roles for any issue \( k \), the restriction can be rephrased as saying that the union of the set of agents over all issues is finite.

2. Throughout the model we made several separability assumptions. Payoffs and costs are additively separable across issues, and, for each agent, the list of issue-specific types are stochastically independent. Moreover, the type parameter \( \theta^i_k \) is assumed to affect how agent \( i \) evaluates issue \( k \), but no other issue. These separable assumptions are somewhat weaker than it appears. If any of these assumptions fail for a pair of issues, we can always bunch these two issues into a single issue. This is possible because there are no dimensionality restrictions on either the issue specific set of alternatives or type spaces. Hence, our results hinge on the existence of a sufficiently large number of truly independent and separate problems, but not on the absence of any non-separabilities.
3. Our model also assumes that, for each issue, agents’ types are stochastically independent. This assumption is maintained because, if types were correlated across agents, then it is usually possible to construct efficient mechanisms similar to that of Cremer and McLean [15] that respects both participation and resource constraints, without any role for linking the issues.

4. Our environment is general enough to incorporate both private goods, public goods and other types of externalities. For example, Armstrong’s [2] non-linear pricing problem with many products fits nicely in this framework. Consider each issue as corresponding to a product. The set of alternatives for issue $D_k \in \mathbb{R}^n$ will then correspond to a quantity choice for product $k$ for each consumer. Agent $i$’s valuation function $v^i_k$ depends on the quantity choice for $i$, i.e. $i$-th component of $d_k$ (see Section 5.1 for more discussion). Myerson and Satterswaite’s [26] bilateral bargaining problem also fits in as an issue $k$ in our setup. A bilateral bargaining problem can be represented by an action space $D_k = [0, 1]$, denoting the probability that agent 1 will obtain the good. Besides agents 1 and 2 – the two agents in the bargaining situation – all other agents’ preferences on this issue are simply independent of $d_k$. Mailath and Postlewaite’s [23] many-agent bargaining problem with public goods can similarly be incorporated as an issue in our setup.

4 The Implementation Problem

The goal of the mechanism designer is to implement an efficient allocation, subject to incentive compatibility, individual rationality and resource constraints. Mechanism $(x_K, t_K)$ is incentive compatible if

$$E_{-i} [V^i_K (x_K (\theta), \theta^i) - t^i_K (\theta)] \geq E_{-i} [V^i_K (x_K (\theta^i, \theta_{-i}), \theta^i) - t^i_K (\theta^i, \theta)]$$

$$\forall i \in I, \theta_i, \theta^i_i \in \Theta^i (K).$$

(11)

We impose participation constraints in the interim state as

$$E_{-i} [V^i_K (x_K (\theta), \theta^i) - t^i_K (\theta)] \geq R^i_K (\theta^i)$$

$$\forall i \in I, \theta_i \in \Theta^i (K),$$

(12)

where $R^i_K (\theta^i)$ is defined in (9). Finally, the resource constraint is imposed in the ex ante form,

$$E [C_K (x_K (\theta))] = E \left[ \sum_{i=1}^n t^i_K (\theta) \right].$$

(13)
A seemingly more stringent condition is to require that the resources balance for every type profile, that is
\[ C_K (x_K (\theta)) = \sum_{i=1}^{n} t_i^K (\theta) \quad \forall \theta \in \Theta (K). \] (14)
When types are correlated or when stronger notions of incentive compatibility are considered, results sometimes depend on the form of the resource constraint. However, for the setup in this paper (13) and (14) are equivalent in the sense that for any \( \langle x, t \rangle \) satisfying (11), (12) and (13) there is some transfer rule \( t' \) such that \( \langle x, t' \rangle \) satisfies (11), (12) and (14).\(^3\) Hence, among the solutions to our problem with the seemingly weaker constraint (13), there is at least one of these solutions that also satisfies (14).

5 The Groves Mechanism Almost Works

In this section we consider the performance of a traditional Groves mechanism, with a lump sum transfer set so as to guarantee that all agents would be willing to participate if the participation decision is made behind a veil of ignorance. The result, which is a rather direct application of Chebyshev’s inequality, is that this mechanism violates a participation constraint with vanishing probability as the number of issues goes out of bounds.

Let
\[ x^*_K (\theta) \in \arg \max_{d \in D_K} \sum_{i=1}^{n} V^i_K (d, \theta^i) - C_K (d), \] (15)
\[ S_K (\theta) = \max_{d \in D_K} \sum_{i=1}^{n} V^i_K (d, \theta^i) - C_K (d) \] (16)
where \( V^i_K (d, \theta^i) \) is defined in (6) and \( C_K (d) \) in (5). Consider a Groves mechanism
\[ \langle x^*_K, t_{G,K} \equiv (t^1_{G,K}, \ldots, t^n_{G,K}) \rangle, \] (17)
where \( x^*_K \) is given by (15) and for each \( i \) the transfer \( t^i_{G,K} \) is given by
\[ t^i_{G,K} (\theta) = V^i_K (x^*_K (\theta), \theta^i) - S_K (\theta) \] (18)
\[ + \frac{(n-1)}{n} \mathbb{E} [S_K (\theta)] - \mathbb{E} \left[ R^i_K (\theta^i) \right] + \frac{1}{n} \mathbb{E} \left[ \sum_{j=1}^{n} R^j_K (\theta^j) \right] \]
\[ \text{lump sum transfer independent of } i \text{'s announcement} \]

\(^3\)See Börgers and Norman [8]. Applied to a Groves mechanism, this is the well-known AGV implementation result due to d’Aspremont and Gerard-Varet [16].
Truth-telling is a dominant strategy, since the mechanism is a Groves mechanism, implying that incentive constraints (11) are satisfied. Moreover, a routine calculation show that the resource constraint (13) holds. The only issue is thus the participation constraints (12).

Let $U^i_i(\theta^i)$ denote the interim expected utility for agent $i$ given mechanism $(x^*_K, t_{G,K})$, assuming that all other agents report truthfully. Substituting $x^*_K(\theta)$ and $t_{G,K}(\theta)$ into the payoff function for the agent and taking expectation the over $\theta^{-i}$ we may express this interim expected utility as

$$U^i_i(\theta^i) = E_{-i}[S_K(\theta)] - \frac{n-1}{n}E[S_K(\theta)] + E[R^i_K(\theta^i)] - \frac{1}{n}E\left[\sum_{j=1}^{n} R^j_K(\theta^j)\right]. \quad (19)$$

In this notation, the participation constraint in (12) is satisfied whenever $U^i_i(\theta^i) \geq R^i_K(\theta^i)$. We can now show:

**Proposition 1** Suppose that $\{E_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues (in the sense of Definition 2). Moreover, for each $K$, let $(x^*_K, t_{G,K})$ be the Groves mechanism as specified in (17). Then, for every $\varepsilon > 0$ there exists some finite $K^*(\varepsilon)$ such that

$$\Pr[U^i_i(\theta^i) - R^i_K(\theta^i) \geq 0] \geq 1 - \varepsilon \quad (20)$$

for every $i$ and every $K \geq K^*(\varepsilon)$.

The interpretation of the result is that the probability that all participation constraints in (12) are satisfied (i.e., $(1 - \varepsilon)^n$) can be made arbitrarily close to one when we link a sufficiently large number of independent social decisions using a standard Groves mechanism with appropriately chosen lump sum transfers.

For an intuitive understanding of the result, first note that the objective function in (15) is additively separable over the issues, so that

$$S_K(\theta) = \sum_{k=1}^{K} s_k(\theta_k), \quad (21)$$

where $s_k(\theta_k)$ is the maximized issue $k$ surplus defined in (10). Moreover, stochastic independence across issues implies that $S_K(\theta)/K$ converges in probability to its expectation as $K$ goes to infinity. This in turn implies that $E_{-i}[S_K(\theta)]/K$ is close to the unconditional expectation. Similarly, $R^i_K(\theta^i)/K$ is near the expected value for $K$ large, and, since the participation constraint can be
written as \( \frac{U_k^i(\theta)}{K} \geq \frac{R_k^i(\theta)}{K} \), the participation constraint is satisfied with high probability if

\[
\frac{1}{n} \left\{ \frac{E[S_K(\theta)]}{K} - \frac{E\left[\sum_{j=1}^{n} R_k^i(\theta^j)\right]}{K} \right\} > 0,
\]

(22)

which is guaranteed by the assumption that there is a uniform lower bound on the gains from trade over each issue \( k \).

**Proof.** Since \( S_K(\theta) = \sum_{k=1}^{K} s_k(\theta_k) \) and \( R_k^i(\theta^j) = \sum_{k=1}^{K} r_k^i(\theta_k^j) \), we may express (19) as

\[
U_K^i(\theta^i) = \sum_{k=1}^{K} \left\{ E_{-i}[s_k(\theta_k)] - \frac{n-1}{n} E[s_k(\theta_k)] + E[r_k^i(\theta_k^i)] - \frac{1}{n} E\left[\sum_{j=1}^{n} r_k^i(\theta_k^j)\right] \right\}
\]

(23)

Define \( \phi_k^i(\theta_k) \equiv E_{-i}[s_k(\theta_k)] - r_k^i(\theta_k^i) \). This allows us to express \( U_K^i(\theta^i) - R_k^i(\theta^j) \) as

\[
U_K^i(\theta^i) - R_k^i(\theta^j) = \sum_{k=1}^{K} \left\{ \phi_k^i(\theta_k^i) - E[\phi_k^i(\theta_k)] + \frac{1}{n} \left( E[s_k(\theta_k)] - E\left[\sum_{j=1}^{n} r_k^i(\theta_k^j)\right] \right) \right\}.
\]

(24)

By assumption all issues in economy \( E_K \) are regular, thus by (R1), we have:

\[
E_i[U_K^i(\theta^i) - R_k^i(\theta^j)] = \frac{1}{n} \sum_{k=1}^{K} \left\{ E[s_k(\theta_k)] - E\left[\sum_{j=1}^{n} r_k^i(\theta_k^j)\right] \right\} \geq \frac{K}{n} \delta.
\]

(25)

Moreover, since the issues are stochastically independent, \( \{\theta_k^i\}_{k=1}^{K} \) is a sequence of stochastically independent variables, thus \( \{\phi_k^i(\theta_k^i)\}_{k=1}^{K} \) is a sequence of independent variables. Finally, since all issues are regular, \( \phi_k^i(\theta_k^i) \) is bounded from above and below by (R2) and (R3). Hence, there exists \( \sigma^2 \) such that \( \text{Var}[\phi_k^i(\theta_k^i)] \leq \sigma^2 \) for every \( i \) and every \( k \), so

\[
\Pr[U_k^i(\theta^i) - R_k^i(\theta^j) < 0] \overset{(2\delta)}{=} \Pr\left[\sum_{k=1}^{K} \left( \phi_k^i(\theta_k^i) - E[\phi_k^i(\theta_k^i)] \right) < -\frac{1}{n} \sum_{k=1}^{K} \left( E[s_k(\theta_k)] - E\left[\sum_{j=1}^{n} r_k^i(\theta_k^j)\right] \right) \right]
\]

/By inequality in (25)/ \( \leq \Pr\left[\sum_{k=1}^{K} \left( \phi_k^i(\theta_k^i) - E[\phi_k^i(\theta_k^i)] \right) < -\frac{\delta K}{n} \right]
\]

\[ \leq \Pr\left[\left| \sum_{k=1}^{K} \left( \phi_k^i(\theta_k^i) - E[\phi_k^i(\theta_k^i)] \right) \right| > \frac{\delta K}{n} \right]
\]

/Chebyshev’s inequality/ \( \leq \left( \frac{1}{\delta K} \right)^2 \text{Var}\left[\sum_{k=1}^{K} \phi_k^i(\theta_k^i)\right] \leq \frac{n^2 \sigma^2}{\delta^2 K}
\]

(26)

Since \( \frac{n^2 \sigma^2}{\delta^2 K} \to 0 \) as \( K \to \infty \), we conclude that for every \( \varepsilon > 0 \), there exists some finite integer \( K^*_i(\varepsilon) \) such that (20) holds for agent \( i \). Let \( K^*_i(\varepsilon) = \max_{i \in I} K^*_i(\varepsilon) \) and the result follows.

While Proposition 1 is about “almost implementing” efficient allocations, the logic in the proof of Proposition 1 can be used to understand Armstrong’s [2] analysis of a multiproduct monopolist with \( K \) private goods. Armstrong assumes that each good \( k \) is produced at constant unit cost \( c_k \). Efficiency is trivially implementable in this environment by marginal cost pricing, but profit maximization will lead to inefficiencies due to informational rents for the consumers. His main result is that when \( K \) is sufficiently large, a cost-based two-part tariff can be almost profit maximizing in the sense that the monopolist extracts almost all consumer surplus.

Due to the constant unit cost assumption, each consumer can be treated separately, so there is no loss in assuming that there is a single consumer. We therefore drop the index \( i \) and write

\[
V_K(d, \theta) = \sum_{k=1}^{K} v_k(d_k, \theta_k)
\]

for the utility function of the consumer, where \( d_k \) now is to be interpreted as the quantity of good \( k \) consumed. It can be seen from (15) and (18) that the Groves mechanism in this environment reduces to marginal cost pricing together with a lump sum transfer where

\[
x^*_K(\theta) = \arg \max_{d \in \mathbb{R}_+^K} \left[ \sum_{k=1}^{K} v_k(d_k, \theta_k) - \sum_{k=1}^{K} c_k d_k \right]
\]

\[
t_{G,K}(\theta) = \sum_{k=1}^{K} c_k x^*_k(\theta) + T, \tag{27}
\]

where \( T \) is a lump sum transfer that does not depend on \( \theta \). Thus mechanism (17) reduces to a two-part tariff in this case, where the consumer pays a fixed fee of \( T \) for the right to purchase goods at marginal costs. While the budget-balanced Groves mechanism will imply \( T = 0 \) (obvious from (15) by setting \( n = 1 \)), the approximately profit maximizing two-part tariff is to set \( T = (1 - \varepsilon) \mathbb{E}[S_K(\theta)] \) where \( S_K(\theta) = \left[ \sum_{k=1}^{K} v_k(x^*_k(\theta), \theta_k) - \sum_{k=1}^{K} c_k x^*_k(\theta) \right] \) and \( \varepsilon > 0 \) can be made arbitrarily small. A straightforward application of law of large numbers implies that such a two-part tariff satisfies the consumer’s participation constraint almost always, and, since the problem can be solved separately for each consumer the mechanism becomes exactly incentive feasible if types that are not willing to pay \( T \) are allowed to opt out. This is exactly the construction Armstrong [2] uses to establish that the monopolist can extract almost the full surplus if there are many goods.
5.2 An Example

Simple examples where the conclusion in Proposition 1 fails to be true can be constructed by discarding \([R1]\), the assumption that the gains from trade are bounded away from zero. If this regularity condition is discarded one can simply consider an infinite sequence of an identical problem which generates inefficiency in the static context, and assume that payoffs are discounted in real time. If the discount factor is sufficiently small, only the first problem matters, so participation constraints are violated whenever participation constraints are violated in single issue economy 1.

In this section we will consider a somewhat more subtle counterexample, where \([R1]\) and \([R3]\) in Definition 2 hold, but where the uniform bound assumption \([R2]\) is violated. Besides showing the need for some assumption that makes sure that the variance of the maximized surplus stays bounded, this example is also a building block for the example in Section 6.4.

In essence, the example consists of a sequence of public goods problems, where the probability of implementing an outcome different from the status quo converges to zero, but where the associated surplus conditional on implementation goes to infinity at a rate which fixes the ex ante surplus to unity for each problem in the sequence.

Consider a sequence of economies where \(\Theta^i_k = \{l, h\}\) and \(D_k = \{0, 1\}\) for each \(k\). Let

\[
C_k(d_k) = \begin{cases} 
0 & \text{if } d_k = 0 \\
(k+1)^2 & \text{if } d_k = 0 
\end{cases}
\]

Let \(\alpha \in (0, \frac{1}{2})\) and assume that \(v^i_k(\theta^i_k, 0) = 0\) for \(\theta^i_k = l, h\), whereas

\[
v^i_k(\theta^i_k, 1) = \begin{cases} 
\frac{(k+1)^2}{2} - \left[\frac{1}{2} - \alpha\right] (k+1)^4 & \text{for } \theta^i_k = l \\
(k+1)^2 \left[1 + \left(\frac{1}{2} - \alpha\right) k (k+2)\right] & \text{for } \theta^i_k = h 
\end{cases}
\]

It is immediate that the reservation payoffs are given by \(r^i_k(\theta^i_k) = 0\) for \(\theta^i_k = l, h\) and that the efficient mechanism is \(x^*_k(ll) = 0\) and \(x^*_k(\theta_k)\) for \(\theta_k \in \{lh, hl, hh\}\). Some algebra shows that the associated maximized surplus is given by

\[
s_k(\theta_k) = \begin{cases} 
0 & \text{if } \theta_k = ll \\
\alpha (k+1)^2 & \text{if } \theta_k \in \{lh, hl\} \\
(k+1)^4 \left[1 - 2\alpha \frac{k(k+2)}{(k+1)^2}\right] & \text{if } \theta_k = hh 
\end{cases}
\]

From this point on, all the analysis can be done in terms of the maximized surplus function. The only reason to actually write down the primitives is that transferable utility and private values
impose some restrictions on the surplus function; not every \( s_k \) such that \( s_k (\theta_k) \geq r^1_k (\theta^1_k) + r^2_k (\theta^2_k) \) is consistent with surplus maximization. Finally, we assume that \( \Pr [\theta^j_k = h] = \frac{1}{(k+1)^2} \).

The unconditional expected value of the maximized surplus is thus

\[
E[s_k (\theta_k)] = 2 \frac{1}{(k+1)^2} \left[ 1 - \frac{1}{(k+1)^2} \right] \alpha (k+1)^2 + \left[ \frac{1}{(k+1)^2} \right]^2 (k+1)^4 \left[ 1 - 2\alpha \frac{k(k+2)}{(k+1)^2} \right]
\]

and

\[
E[s_k (\theta_k) | \theta^j_k = l] = \frac{1}{(k+1)^2} \alpha (k+1)^2 = \alpha
\]

Consider type \( l = (l, \ldots, l) \). The interim expected payoff in the Groves mechanism (reference) is then

\[
U^i (1) = \sum_{k=1}^{K} E[s_k (\theta_k) | \theta^j_k = l] - \frac{1}{2} \sum_{k=1}^{K} E[s_k (\theta_k)] = \alpha K - \frac{1}{2} K < 0,
\]

since \( \alpha < \frac{1}{2} \). It is easy to check that

\[
\Pr [\theta^j = l] = \prod_{k=1}^{K} \frac{k(k+2)}{(k+1)^2} = \frac{K+2}{2(K+1)} > \frac{1}{2},
\]

implying that the participation constraint is violated with a probability exceeding \( \frac{1}{2} \) for each player no matter how many problems are linked..

6 An Asymptotically Efficient Mechanism

Proposition 1 is suggestive, but it does not solve the implementation problem as stated in Section 4. Although the probability of a violation of participation constraints in (12) is small if \( K \) is large, the probability of a violation is strictly positive in general. Hence, the mechanism is not incentive feasible.

A natural remedy to this problem is to revert to the status quo outcome when a participation constraint is violated. Since failures are rare when \( K \) is large, this may appear to create an almost efficient incentive feasible mechanism. The problem with this idea, however, is that the agents that “opt out” change the interim expected payoff from participation for the agents that “opt in”.\(^4\) Hence, when some agents opt out of the mechanism, this may diminish the value of playing

\(^4\)This issue is not relevant for private goods problems such as Armstrong [2] where the efficient outcome for agent \( i \) does not depend on others’ types.
the Groves mechanism for the remaining agents. But, if types with interim payoffs below the
reservation utilities opt out, the best response for the remaining types may be that some additional
types should opt out due to the negative selection effect. The set of types that agree to play the
Groves mechanism must therefore be determined through a fixed point argument, and a priori it is
unclear how to rule out that the mechanism unravels, so that the only fixed point implements the
status quo with high probability. Section 6.4 constructs an explicit example where this occurs.

Our main result, Proposition 2, shows that the regularity conditions in Definition 2 are sufficient
to rule out such unraveling, so that an almost efficient mechanism satisfying constraints (11), (12),
and (13) exists if there is a sufficient number of independent issues. The main analytical difficulty
in establishing this result is the fixed point problem discussed above: constructing sequences of
direct revelation mechanisms generates a fixed point problem in the set of “non-vetoing types”,
which makes the mechanism difficult to describe. Instead, we construct an indirect mechanism with an
enlarged message space that includes an explicit choice as to whether to veto the Groves mechanism,
which makes the mechanism easy to describe. The fixed point problem is solved by combining a
constructive argument (Lemma 4, which establishes that a set of types have a strict incentive to
participate if everyone else in this large set participates.) with an existence result from Milgrom
and Weber [25] (which guarantees that all other types have a well-defined participation decision).

In Section 6.4 we provide an explicit example (violating the regularity conditions) where the
probability that a participation constraint is violated converges to zero as $K$ tends to infinity, but
where there nevertheless exists no equilibrium of the augmented Groves mechanism where any gains
from trade are realized. That is, despite the fact that the probability that a participation constraint
is violated is negligible, a veto will be cast for sure.

### 6.1 An Augmented Groves Mechanism

We add the non-type message $m^i \in \{0, 1\}$ to the message space, so that each agent reports a pair
($m^i, \theta^i$). If all agents report $m^i = 1$, then a Groves mechanism is implemented, whereas if any agent
chooses $m^i = 0$, the default outcome $d^0$ is implemented and all transfers are zero. The extra message
$m^i$ can thus be interpreted as a decision on whether or not to veto playing the Groves mechanism.
Notice here the difference with the construction used by Jackson and Sonnenschein [21] when dealing
with participation constraints. Our vetoes are cast before agents know the types of the other agents,
whereas they assume that the vetoing procedure is ex post, implying that their mechanism satisfies
ex post participation constraints. While ex post participation sometimes is a desirable feature, it
is very difficult to deal with in our model due to our (possibly) continuous social decisions and type spaces. In particular, conditional on no veto, our mechanism is an exact Groves mechanism, so incentive compatibility is a non-issue and an efficient outcome is implemented. In contrast, if agents look forward towards a veto game, then the dominance solvability of the revelation game breaks down, and agents need to make conjectures of the outcome in the ex post veto game. As a result, exact efficiency with high probability will be impossible to guarantee. We conjecture that a slightly weaker version of a result would continue to hold with ex post participation constraints, but a proof will necessarily involve several additional layers of approximations compared with the current proof.

We consider a sequence of mechanisms \( \{ \tilde{x}_K, \tilde{t}_K \}_{K=1}^{\infty} \), where for each \( K \)

\[
\tilde{x}_K (\theta, m) = \begin{cases} 
  x^*_K (\theta) & \text{if } m = (1, 1, ..., 1) \\
  d^0 = (d^0_1, ..., d^0_K) & \text{otherwise}
\end{cases}
\]

\[
\tilde{t}_K (\theta, m) = \begin{cases} 
  t^i_{G,K} (\theta) + \frac{\delta K}{4n} & \text{if } m = (1, 1, ..., 1) \\
  0 & \text{otherwise}
\end{cases}
\]

The term \( t^i_{G,K} (\theta) \) in the transfer scheme is given by the Groves transfer scheme in (18), and \( \delta \) is the lower bound on the per issue gains from trade defined in (R1).

While not explicitly written as such, (30) is equivalent with as a sequential mechanism, where in stage 1 the agents play a veto game. If any agent casts a veto, then the game is over. If, instead, all agents opt in, then the game proceeds into the second stage, where a Groves mechanism with almost the same transfers as in Section 5 is implemented. The only difference is that the term \( \frac{\delta K}{4n} \) has been added to the transfer scheme. If the transfer scheme is kept unchanged it is possible that the mechanism runs a budget deficit no matter how large is \( K \). In contrast, adding the term \( \frac{\delta K}{4n} \) to the “fixed fee of participation” guarantees a strict budget surplus when \( K \) is large, that is

\[
E [C_K (x_K (\theta))] < E \left[ \sum_{i=1}^{n} t_K^i (\theta) \right].
\]

Clearly, such a budget surplus is inefficient, but, since it can be rebated lump sum (which means that also to types that veto the mechanism get the rebate) without upsetting either participation or incentive constraints, this in turn guarantees existence of a transfer that balances the resource constraint (13) exactly (see Corollary 1 following Proposition 2).
6.2 The Approximate Efficiency Result

We now state our main result, which says that, if the underlying sequence of single-issue economies is regular, then mechanism (30) has equilibria that implement the efficient outcome with probability arbitrarily close to one, provided that sufficiently many issues are linked. Formally,

**Proposition 2** Suppose that \( \{E_K\}_{K=1}^{\infty} \) is sequence of economies consisting of stochastically independent regular issues (in the sense of Definition 2). Then, for every \( \varepsilon > 0 \) there exists some finite \( K^*(\varepsilon) \) such that, for every economy \( E_K \) with \( K \geq K^*(\varepsilon) \), there exists an equilibrium in the game induced by mechanism (30) where the efficient outcome is implemented with probability at least \( 1 - \varepsilon \), and where (31) is satisfied.

While not explicitly stated that way, it follows immediately from Proposition 2 that the solution to the implementation problem in Section 4 is approximately efficient if \( K \geq K^*(\varepsilon) \);

**Corollary 1** For every \( \varepsilon > 0 \) and \( K \geq K^*(\varepsilon) \) there exists a (direct revelation) mechanism that generates at least the same expected surplus as mechanism (30) and satisfies (11), (12) and (13)

**Proof.** By the revelation principle, there exists a direct revelation mechanism with an equilibrium that implements the same outcome as mechanism (30) for every \( K \). Hence, (11) is satisfied and (12) holds since each agent has an action \( (m_i^t = 0) \) that guarantees the reservation utility. Finally, since (31) is satisfied there exists a lump-sum rebate (where also types that veto the decision receive the rebate) such that (13) is satisfied, which leaves the incentives constraints (11) and participation constraints (12) unaffected. Hence, there is an incentive feasible mechanism that generates the same expected surplus as mechanism (30). The constrained efficient mechanisms is thus weakly better.

Intuitively, the explanation for Proposition 2 is that, conditional on not casting a veto, truth-telling is a dominant strategy. Moreover, when opting out, the reported type is irrelevant. Together, this implies that assuming truth-telling of the type message is without loss of generality, and the only issue is how often a veto will be cast. Note that using a Groves mechanism in the “second stage” is crucial for this argument. If instead, we would use a Bayesian mechanism, such as the AGV mechanism of d’Aspermont and Gerard-Varet [16], the second stage equilibrium would depend on the selection of agents who do not veto the mechanism, creating a rather complicated problem, where in the second stage the best we could hope for would be a nearly efficient outcome.
The crucial step in the proof (Lemma 4) asserts that if $K$ is sufficiently large there is a set of type profiles such that no type in this set will veto the Groves mechanism provided that all agents believe that no other type in this set will cast a veto. Moreover, the probability that the realized type profile belongs to this set approaches unity as $K$ goes out of bounds. This set of type profiles is a strict subset of the profiles with a strict incentive to participate in the original Groves mechanism. This is necessary, because the “admissions fee” has been raised by $\frac{\delta K}{2n}$, and because we need some room to assure that a vanishing probability of a veto does not upset the participation decision in this set. Concretely, the set consists of types that, playing the regular Groves mechanism, earn an interim expected payoff of at least $\frac{\delta K}{2n}$. Under the regularity assumptions in Definition 2 the probability that the type is drawn outside this set, while always larger than upsetting a participation constraint in the regular Groves mechanism, also goes to zero as $K \to \infty$. Moreover, under the belief that other agents in the set choose not to veto, the expected payoff from not vetoing exceeds the reservation payoff for $K$ large enough. Hence, while we cannot rule out that the mechanism unravels for a given $K$, we know that there is some large enough $K$ that guarantees only types outside the set will ever cast a veto. To complete the proof we must ensure that types outside the set have well-defined equilibrium strategies (otherwise we could have non-existence of equilibria), which is done by appealing to a theorem from Milgrom and Weber [25].

6.3 The Proof of Proposition 2

Define the set,

$$
\Theta^i (K) = \left\{ \theta^i \in \Theta^i (K) | U^j_K (\theta^i) - R^j_K (\theta^j) \geq \frac{\delta K}{2n} \right\},
$$

(32)

where the interim utility function $U^j_K (\theta^j)$ is defined in (19), using the Groves mechanism in the previous section. To interpret $\Theta^i (K)$, recall that $\delta K / n$ is a lower bound for the expected value for $U^j_K (\theta^j) - R^j_K (\theta^j)$, so $\Theta^i (K)$ is a set of agent $i$’s types for which her interim expected surplus is at least 1/2 of this lower bound. This set is useful because, just like the set of agents for which the participation constraints are fulfilled under the Groves mechanism, the probability that $\theta^i$ belongs to $\Theta^i (K)$ is close to one when $K$ is large. Moreover, since the types in $\Theta^i (K)$ have a per-issue ex ante expected payoff of at least $\delta / 2n$ there is room to consider small deviations from the Groves mechanism without upsetting their participation constraints for agents in the set $\Theta^i (K)$. 

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It is convenient to introduce the notation
\[
\begin{align*}
\overline{\Theta}(K) &= \left\{ \theta \in \Theta(K) \mid \theta^i \in \overline{\Theta}^i(K) \text{ for each } i \right\} \quad (33) \\
\overline{\Theta}^{-i}(K) &= \left\{ \theta^{-i} \in \overline{\Theta}^{-i}(K) \mid \theta^j \in \overline{\Theta}^j(K) \text{ for each } j \neq i \right\}. \quad (34)
\end{align*}
\]

Using roughly the same style of argument as in the proof of Proposition 1, we begin by establishing that for large \( K \), the probability that \( \theta^{-i} \) belongs to \( \overline{\Theta}^{-i}(K) \) is near one. This is fairly obvious, since the lump sum transfers from our Groves mechanism are constructed so that
\[
E \left[ U^i_K (\theta^i) - R^i_K (\theta^i) \right] = \frac{1}{n} \left\{ E \left[ S_K (\theta) \right] - E \left[ \sum_{j=1}^{n} R^j_K (\theta^j) \right] \right\} \geq \frac{\delta K}{n}, \quad (35)
\]
by the assumption that the per-issue gains from trade are bounded below by \( \delta \) in (R1).

**Lemma 1** Suppose that \( \{E_K\}_{K=1}^{\infty} \) is sequence of economies consisting of stochastically independent regular issues. Then, for every \( \varepsilon > 0 \) there exists finite \( K'(\varepsilon) \) such that \( \Pr \left[ \overline{\Theta}^{-i}(K) \right] \geq 1 - \varepsilon \) for all \( K \geq K'(\varepsilon) \).

Define \( E_{-i} \left[ S_K (\theta) \mid \Theta^{-i}(K) \right] \) as the interim expectation of the social surplus under an efficient social decision rule for agent \( i \) conditional on \( \theta^i \) and conditional on the types of the other agents being in the set \( \Theta^{-i}(K) \). Using the regularity assumptions (R2) and (R3) we show that:

**Lemma 2** Suppose that \( \{E_K\}_{K=1}^{\infty} \) is sequence of economies consisting of stochastically independent regular issues. Then, for every \( \varepsilon > 0 \) there exists finite \( K''(\varepsilon) \) such that for all \( K \geq K''(\varepsilon) \),
\[
\frac{E_{-i} \left[ S_K (\theta) \mid \Theta^{-i}(K) \right] - \varepsilon}{K} \geq \frac{E_{-i} \left[ S_K (\theta) \right] - \varepsilon}{K}, \quad \text{for all } \theta^i \in \Theta^i(K).
\]

Intuitively, the idea is that, since the probability that \( \theta^{-i} \) lies in \( \Theta^{-i}(K) \) can be made arbitrarily close to one by linking sufficiently many problems, the conditional expectation is almost exclusively taken over \( \Theta^{-i}(K) \).

Next, we start to characterize equilibrium play in the mechanism. Since the “second stage mechanism” is a Groves mechanism, truth-telling is dominant conditional on no veto being cast. Moreover, there is nothing to lose from telling the truth if a veto is cast. Hence, it is fairly obvious that there is nothing to gain from misrepresenting the type profile. In terms of the decision when to veto the Groves mechanism, it is then just a question of calculating the interim expected payoff from not vetoing, which is a computation where the agents need to consider selection effects, and
compare this with the reservation utility. While this is all conceptually rather clear, it is useful to formalize this in order to make the proof of Lemma 4 and Proposition 2 more transparent.

To state this formally we need to introduce notation for the veto rules. Later, we need to allow randomizations (for types outside of $\Theta^i (K)$) to ensure existence of equilibria, so let a veto rule in economy $\mathcal{E}^i$ be a map $\psi^i_{K} : \Theta^i (K) \rightarrow [0, 1]$, where $\psi^i_{K} (\theta^i)$ is interpreted as the probability that type $\theta^i$ chooses $m^i = 1$, a vote to play the Groves mechanism. Following standard conventions, let $\psi_{K} = (\psi^1_{K}, ..., \psi^n_{K})$ and $\psi^{-i}_{K} = (\psi^1_{K}, ..., \psi^{i-1}_{K}, \psi^{i+1}_{K}, ..., \psi^n_{K})$. Also, let $\hat{U}^i (m_i, \psi^{-i}_{K}, \theta^i)$ denote the interim expected payoff for type $\theta^i$ from announcement $(m^i, \theta^i)$ under the assumption that all other agents report type truthfully and follow an arbitrary veto rule $\psi^{-i}_{K} : \Theta^{-i} (K) \rightarrow [0, 1]^{n-1}$. For brevity, we let $\Psi^{-i}_{K} (\theta^{-i}) \equiv \prod_{j \neq i} \psi^j_{K} (\theta^j)$ and $F^{-i}_{K} (\theta^{-i}) = \prod_{j \neq i} F^j_{K} (\theta^j)$.

To formalize this in order to make the proof of Lemma 4 and Proposition 2 more transparent, we need to allow randomizations (for types outside of $\Theta^i (K)$) to ensure existence of equilibria, so let a veto game. It says that, if $K$ is large enough and if, for all $i$ and $j \neq i$, agent $i$ believes that no type in the set $\Theta^{-i} (K)$ will veto the mechanism, then all types $\theta^i$ in $\Theta^i (K)$ have a strict incentive not to veto the mechanism. This is true regardless of the behavior of types outside of the set $\Theta^{-i} (K)$.

---

\footnote{\textsuperscript{5}At the cost of more complicated notation, we could allow for non-truthful announcements. In the end, this extra “bite” is useless since there is no hope to get dominance in the veto rules.}
In other words, we cannot rule out complete unraveling of the mechanism for a given finite \( K \), but, eventually, incentives are dominated by best responding to \( \Theta^{-1}(K) \) and when \( K \) is sufficiently large this rationalizes voting to go ahead with the Groves mechanism for types in \( \Theta^j(K) \).

**Lemma 4** Suppose that \( \{\mathcal{E}_K\}_{K=1}^\infty \) is sequence of economies consisting of stochastically independent regular issues. Then, there exists finite \( K^* \) such that, for all \( K \geq K^* \), if \( \psi_K^j(\theta^j) = 1 \) for each type \( \theta^j \in \overline{\Theta}^j(K) \), \( j \neq i \), then it is a best response for agent \( i \) to set \( \psi_K^i(\theta^i) = 1 \) for each \( \theta^i \in \overline{\Theta}^i(K) \).

Recall that the whole exercise would be useless unless we can verify that the mechanism balances the budget. Again, we can only guarantee this if \( K \) is large enough. If \( K \) is too small the probability that the mechanism is vetoed is non-negligible, and it may be that the types that stay out are those that generate the most revenue for the mechanism designer. However, when \( K \) is large the probability of a veto is small and the extra lump sum revenue from the term \( \frac{\delta \hat{K}}{4n} \) in (30) eventually suffices for a budget surplus.

**Lemma 5** Suppose that \( \{\mathcal{E}_K\}_{K=1}^\infty \) is sequence of economies consisting of stochastically independent regular issues and that, for every \( K \), \( \psi_K^j(\theta^j) = 1 \) for each \( \theta^j \in \overline{\Theta}^j(K) \). Then, there exists finite \( K^{**} \) such that (31) is satisfied for all \( K \geq K^{**} \).

The idea is that \( \Pr[\overline{\Theta}(K)] \) converges in probability to unity and so that the mechanism runs a budget surplus that approaches \( \frac{\delta \hat{K}}{4} \) when \( K \) is large.

To complete the proof we now combine the five Lemmas above and argue that an equilibrium exists where the efficient outcome is implemented with high probability and the budget is balanced.

**Proof of Proposition 2.** Consider a (fictitious) game where \( m^i = 1 \) is the only available action for types in \( \overline{\Theta}^j(K) \) and where types in \( \Theta^j(K) \setminus \overline{\Theta}^j(K) \) may choose \( m^i \in \{0, 1\} \), and the interim expected payoffs are given by (36). The action space is finite for each player, so payoffs are equicontinuous in the sense of Milgrom and Weber [25] (see Proposition 1 in Milgrom and Weber [25]). Moreover, stochastic independence implies that the information structure is absolutely continuous (Proposition 3 in Milgrom and Weber [25]). Applying Theorem 1 in Milgrom and Weber the game has an equilibrium in distributional strategies. Since the action space is finite, we can represent this equilibrium as a behavioral strategy \( \psi_K^* : \Theta(K) \rightarrow [0,1]^n \), where by construction \( \psi_K^* (\theta^i) = 1 \) whenever \( \theta^i \in \overline{\Theta}^i(K) \). But, applying Lemma 4 it follows that that there exists \( K^* < \infty \) such that \( \psi_K^* \) is an equilibrium also when types in \( \overline{\Theta}^j(K) \) have the option to freely pick \( m^i \in \{0, 1\} \),

\[ ^6 \text{Trivially, there will always be equilibria where the Groves mechanism is vetoed with probability 1.} \]
which by use of Lemma 5 implies that (31) is satisfied for \( K \geq \max \{ K^*, K^{**} \} \). Moreover, for any \( \varepsilon > 0 \) Lemma 1 assures that we may pick \( K' (\varepsilon) \) such that such that \( \Pr \left[ \Theta^i (K) \right] \geq 1 - \varepsilon \) for each \( K > K' (\varepsilon) \). Finally, Lemma 3 guarantees that truth-telling is optimal provided that each agent announce \( m^i \) in accordance with \( \psi^i_{K^*} \). The result follows by letting \( K^* (\varepsilon) = \max \{ K^*, K^{**}, K' (\varepsilon) \} \).

\[ \text{6.4 An Example where the Participation Constraints Unravel} \]

At the abstract level, our proof uses the uniform bound assumption [R2] directly in Lemma 4 in order to conclude that if a veto is cast with a negligible probability, then the impact on the interim expected payoff is negligible. In Proposition 1 on the other hand, it is only used to bound the variances. Nevertheless, there may be other proofs, so the reader may rightfully question whether unravelling of participation constraints can ever be an issue if the probability of the participation constraints in the original Groves mechanism converges to zero. We have therefore constructed an explicit example where this occurs.

Let \( n = 2 \) and assume that, for each \( k \) and \( i = 1, 2 \), the single issue \( k \) type space is given by \( \Theta^i = \{ l, m, h \} \). Assume that the set of alternative resolutions of issue \( k \) are \( D_k = \{ d^0_k, d^m_k, d^h_k \} \), where \( d^0_k \) will be the “status quo” outcome and \( d^m_k \) and \( d^h_k \) two alternative ways to resolve issue \( k \). Let \( \varepsilon \in (0, \frac{1}{16}) \) and assume that the issue \( k \) valuation functions are,

\[
v^i_k (d^0_k, \theta^i_k) = 0 \quad \text{for all } \theta^i_k
\]

\[
v^i_k (d^m_k, \theta^i_k) = \begin{cases} 
\frac{1}{4} \left( \frac{k+1}{k} \right)^2 & \text{if } \theta^i_k = l \\
\frac{k(k+1)^2}{2} - 4\varepsilon k (k+1)(k+2) - \varepsilon (k+1) & \text{if } \theta^i_k = m \\
- \left[ \frac{(k+1)^2}{4k} + 3 (k+1)^3 \right] & \text{if } \theta^i_k = h
\end{cases}
\]

\[
v^i_k (d^h_k, \theta^i_k) = \begin{cases} 
0 & \text{if } \theta^i_k = l \\
- \left[ \frac{k(k+1)^2}{2} - 4\varepsilon k (k+1)(k+2) - \varepsilon (k+1) \right] & \text{if } \theta^i_k = m \\
\frac{(k+1)^2}{4k} + 3 (k+1)^3 & \text{if } \theta^i_k = h
\end{cases}
\]

and let the cost function be

\[
C_k (d_k) = \begin{cases} 
0 & \text{if } d_k = d^0_k \\
\frac{k(k+1)^2}{2} - 4\varepsilon k (k+1)(k+2) - 2\varepsilon (k+1) & \text{if } d_k = d^m_k \\
2 (k+1)^3 & \text{if } d_k = d^h_k
\end{cases}
\]

One way to interpret the example is that the society may provide one of two possible public goods, but where there are two high valuation agents who mutually dislike the public good preferred by the
other agent\textsuperscript{7} The reader verify that the efficient social decision rule in the single issue \(k\) economy is
\[
x_k^* (\theta_k) = \begin{cases} 
\theta_k & \text{if} \ \theta_k \in \{ll, mh, hm\} \\
\theta_k & \text{if} \ \theta_k \in \{lm, ml, mm\} \\
\theta_k & \text{if} \ \theta_k \in \{lh, hl, hh\}
\end{cases}
\]
and that the associated maximized issue \(k\) surplus is
\[
s_k (\theta_k) = \begin{cases} 
0 & \text{if} \ \theta_k \in \{ll, mh, hm\} \\
\frac{1}{4} \frac{(k+1)^2}{k} + \varepsilon (k + 1) & \text{if} \ \theta_k \in \{lm, ml\} \\
\frac{k(k+1)^2}{2} - 4\varepsilon k (k + 1) (k + 2) & \text{if} \ \theta_k = mm \\
\left[ \frac{4(k+1)-1}{4k} \right] (k+1)^2 & \text{if} \ \theta_k \in \{lh, hl\} \\
(4 (k + 1) - \frac{k}{2}) (k + 1)^2 & \text{if} \ \theta_k = hh
\end{cases}
\]
Let the probability distribution over \(\Theta_k^i\) be given by
\[
(Pr [\theta^i = l], Pr [\theta^i = m], Pr [\theta^i = h]) = \left( \frac{k(k+2)}{(k+1)^2}, \frac{1}{2(k+1)^2}, \frac{1}{2(k+1)^2} \right)
\]
Calculating the relevant expected values of the maximized issue \(k\) surplus we have that
\[
E[s_k (\theta_k) | l] = \frac{k + 1}{2} + \varepsilon \frac{1}{2(k + 1)}
\]
\[
E[s_k (\theta_k) | m] = \frac{k + 1}{2} - \varepsilon k - \varepsilon k \frac{1}{(k+1)}
\]
\[
E[s_k (\theta_k) | h] = k + 1.
\]
First, consider the interim expected payoff of type \(l = (l, \ldots, l)\). Applying (19), the general expression for the interim expected payoff in the Groves mechanism with transfer function given by (18) we have that the interim expected payoff in the linking Groves mechanism is
\[
U (l) = \sum_{k=1}^{K} \left[ \frac{k + 1}{2} + \varepsilon \frac{1}{2(k + 1)} - \frac{1}{2} \sum_{k=1}^{K} k \right]
\]
\[
= \varepsilon \frac{1}{2} \sum_{k=1}^{K} \frac{1}{k + 1} = \varepsilon \frac{K+1}{2} \sum_{k=2}^{K+1} \frac{1}{k} = \frac{\varepsilon}{2} \sum_{k=1}^{K+1} \frac{1}{k+1} - 1 \approx \frac{\varepsilon}{2} [\ln (K+1) - 1].
\]
Next, consider a type with the first \(K^*\) realizations being \(m\) and the remaining \(K - K^*\) being \(l\).

\textsuperscript{7}The fact that \(v^i_k (a^h_k, l) = 0\) is only for numerical convenience. We could make \(v^i_k (a^h_k, l)\) a positive number by appropriately adjusting the cost and the valuation for type \(h\).
Such a type gets an interim expected payoff of

\[
U \left( m, \ldots, m, l, \ldots, l \right)_{K^* \text{ terms } K-K^* \text{ terms}} = - \sum_{k=1}^{K^*} \left[ \frac{\varepsilon k - \varepsilon k}{(k + 1)} \right] + \sum_{k=K^*}^{K} \frac{\varepsilon}{2(k+1)}
\]

\[
= \varepsilon \left[ \frac{1}{2} \sum_{k=K^*}^{K} \frac{1}{(k + 1)} - \sum_{k=1}^{K^*} \left[ k - \frac{k}{(k + 1)} \right] \right] \equiv H(K^*, K)
\]

Define \( K^* (K) \) as the largest number such that \( U(m, \ldots, m, l, \ldots, l) > 0 \), which is uniquely defined since \( H \) is strictly decreasing in \( K^* \). Moreover, \( K^* (K) \) goes (slowly) to infinity as \( K \) goes to infinity, which is the crucial part of the construction of the example. It follows that if \( \theta^i \) is such that \( U(\theta^i) < 0 \), then there exists \( k > K^* (K) \) such that \( \theta_k^i = m \). But, the probability of that is

\[
\frac{1}{2} \left[ 1 - \Pr \left( \left( \theta_{K^* (K)}^i, \ldots, \theta_K^i \right) = (l, \ldots, l) \right) \right]
\]

\[
= \frac{1}{2} \left[ 1 - \prod_{k=K^* (K)}^{K} \frac{k(k + 2)}{(k + 1)^2} \right] = \frac{1}{2} \left[ 1 - \frac{K^* (K)(K + 2)}{(K^* (K) + 1)(K + 1)} \right]
\]

\[
= \frac{1}{2} \left( \frac{K - K^* (K)}{(K^* (K) + 1)(K + 1)} \right) \to 0 \text{ as } K \to \infty
\]

It follows that the Groves mechanism with transfers (18) is almost incentive feasible in the example. While the sequence is not regular in the sense of Definition 2, the conclusion in Proposition 1 is still true.

However, consider type \( \theta^i \) where all coordinates are \( l \) except for \( \theta_k^{i*} = m \). Denote this type \( \theta^i = (l|\theta_k^i = m) \) and note that

\[
U \left( (l|\theta_k^i = m) \right) = \sum_{k=1}^{K} \left[ \frac{\varepsilon}{2(k + 1)} \right] - \frac{\varepsilon}{2(k^* + 1)} - \varepsilon k^* - \frac{\varepsilon k^*}{(k + 1)}
\]

\[
< \frac{\varepsilon}{2} \left[ \ln K - 1 - \frac{1}{k^* + 1} - k^* - \frac{2k^*}{k^* + 1} \right]
\]

\[
= \frac{\varepsilon}{2} \left[ \ln K - 1 - \left[ \frac{k^* + 2k^* + 1}{k^* + 1} \right] \right] = \frac{\varepsilon}{2} \left[ \ln K - 2 - k^* \right]
\]

Define \( k^* (K) \equiv \ln K - 2 + 1 \). Since \( \frac{k^* (K)}{K} = \frac{\ln K - 2 + 1}{K} \to 0 \) as \( K \to \infty \) and since the issue \( k \) interim expected utility is always higher when \( \theta_k^i = l \) than when \( \theta_k^i = m \) it follows that any type \( \theta^i \) such that i) \( \theta_k^i \neq h \) for all \( k \) and ii) \( \theta_k^i = m \) for some \( k > k^* (K) \) will have a strict incentive to veto the Groves mechanism with transfers (18). Because the transfers in mechanism (30) differs from those
in the original Groves mechanism by a positive lump sum transfer for each agent, all these types have an incentive to veto also mechanism (30). For brevity, define the set of such types by

$$\Theta_V = \{ \theta^i | \theta_k^i \in \{l, h\} \text{ for all } k, \theta_k^i = m \text{ for some } k \}$$

Consider any \( k > k^* (K) \) and observe that, by using the same calculation as in (29), we have that

$$\Pr \left[ \theta^i_{k^*} = (l, \ldots, l | \theta_k^i = m) > \Pr [\theta^i = l] = \frac{K + 2}{2(K + 1)} > \frac{1}{2} \right.$$  

$$\Rightarrow \Pr [\theta^i \notin \Theta_V | \theta_k^i = m] < \frac{1}{2}$$

Hence, conditional on a veto from all \( \theta^i \in \Theta_V \), the single issue \( k \) interim expected payoff for type \( l \) for every issue \( k > k^* (K) \) is

$$E \left[ s_k (\theta_k) | l, \text{ veto by } \theta^i \in \Theta_V \right]$$  

$$= \frac{1}{2(k + 1)^2} \left[ \Pr [\theta^i \notin \Theta_V | \theta_k^i = m] \right] \left[ \frac{1}{4} \left( \frac{k + 1)^2}{k} + \varepsilon (k + 1) \right] + \left[ \frac{4k(k + 1) - 1}{4k} \right] (k + 1)^2 \right]$$  

$$< \frac{1}{2} \frac{1}{2(k + 1)^2} \left[ \frac{1}{4} \left( \frac{k + 1)^2}{k} + \varepsilon (k + 1) \right] + \frac{1}{2} (k + 1)^2 \left[ \frac{4k(k + 1) - 1}{4k} \right] (k + 1)^2 \right]$$  

$$= \frac{\varepsilon}{2(k + 1)^2} + \frac{(k + 1)^2}{2} - \frac{1}{2} \frac{1}{2(k + 1)^2} \left[ \frac{1}{4} \left( \frac{k + 1)^2}{k} + \varepsilon (k + 1) \right] \right.$$

$$= \frac{k + 1}{2} + \frac{\varepsilon}{2(k + 1)} - \frac{1}{16k} - \frac{\varepsilon}{2(k + 1)} = \frac{k + 1}{2} - \frac{1}{16k}$$

Hence

$$U \left( 1 | \text{ veto by } \theta^i \in \Theta_V \right)$$  

$$< \frac{\varepsilon}{2} \sum_{k=1}^{\ln (k^* (K) + 1)} \frac{1}{k} - \frac{1}{16} \sum_{k=1}^{\ln (k^* (K) + 1)} \frac{1}{k} = \frac{1}{16} \sum_{k=1}^{\ln (k^* (K) + 1)} \frac{1}{k} - \frac{1}{16} \sum_{k=1}^{\ln (k^* (K) + 1)} \frac{1}{k}$$  

$$\approx \frac{1}{16} \{8\varepsilon + 1 \ln ((k^* (K) + 1)) - \ln K \} = \frac{1}{16} \{8\varepsilon + 1 \ln ((k^* (K) + 1)) - k^* (K) + 1 \} < 0$$

for any \( \varepsilon > 0 \) provided that \( K \) is large enough. Hence, type \((l, \ldots, l)\) will veto the mechanism provided that \( \theta^i \in \Theta_V \) veto the mechanism. Consequently, any type \( \theta^i \) such that \( \theta_k^i \in \{l, m\} \) for all \( k \) must veto the mechanism. Hence, the probability of a veto is at least \( \frac{3}{4} \) when \( K \) is large.

Further recursions can be used to show that the probability of a veto must converge to unity, but we omit these derivations since the important fact has already been established. In the example, the conclusion of Proposition 1 holds true (despite a violation of the regularity assumption), but the participation constraints unravel under mechanism (30).
7 Discussion

7.1 Linking in Trade Negotiations

International trade negotiations is in some ways a perfect application for our result, provided that one believes that informational asymmetries regarding the relative valuation between, say, a given tariff reduction and improved intellectual property rights is an important consideration. The self-financing constraint (13) then simply says that there is no outside source of resources, and the participation constraint (12) is just national sovereignty, which seems to be an appropriate assumption.

The obvious limitation of our model is that many issues in a trade treaty are fundamentally interrelated. For example, technical barriers of trade become an important issue only after tariffs have been reduced to a level which is below the unilaterally most preferred tariff, providing a direct rationale for linking tariff negotiations with rules governing non-tariff barriers of trade. While this is indeed a shortcoming, we do not think that our result is irrelevant for thinking about real world trade agreements. Instead, we think that our result highlights a fundamental force that works in favor of highly multidimensional agreements that is absent in the previous literature on the topic, which deals exclusively with complete information environments. At present we don’t know how generalizable the approximate efficiency result is to more realistic models, but the basic point that one participation constraint is easier to satisfy than many suggests that there should be a role for linking more generally. Our logic then suggests that it may be a mistake to negotiate a treaty for reducing greenhouse gas emissions in separation, since linking with other issues, such as trade concessions, would make a veto more costly.

Unlike our paper, papers in the international trade literature on issue linking in trade negotiations, such as Bagwell and Staiger [4], Horstmann et al [20], and Spagnolo, rely on a given bargaining protocol. The pros and cons of linking in these papers are therefore largely derived from strategic effects. Since such effects usually depend on details of the bargaining protocol, it is not that surprising that results are mixed. In contrast, the mechanism design methodology used in this paper has the advantage of ruling out results that are driven by particularities in a non-cooperative bargaining game.
7.2 Governments as Linking Mechanisms

Real world government institutions usually fulfill many and arguably quite unrelated functions, and the results in this paper can be seen as a simple theory explaining this, as far as we know, previously unexplained fact. The existing theory of public finance has identified many potential sources of market failure, and interpreted these as a rationale for “government intervention.” However, the existing theory only explains that it may be beneficial to set up some institution to deal with each particular market failure, but does not provide a foundation for why a single government institution should be responsible for dealing with all these problem. For example, we understand that there are externalities involved in garbage collection, that public parks would be under-provided by voluntary provisions, and that for-profit policing may be a bad idea, but it seems hard to argue that there are technological reasons for why these services should be provided jointly as part of a local government bundle, as they tend to be. Our results suggest a possible explanation; linking all these seemingly unrelated social decisions via a single government institution helps achieve efficiency by alleviating citizens’ participation constraints.

The reader may complain that our mechanism is an unrealistic description of what real world governments do. In particular, it may be argued that a single citizen usually doesn’t have veto power at the local government level, which is the level of government where participation constraints seem the most realistic. We agree, but we also note that most goods and services provided at the local government level are such that use exclusion is possible. Veto power is needed for our formal result in order to accommodate pure public goods problems. However, if it is possible to realize the reservation utility for an agent by excluding the agent from usage, then the veto game may be replaced by a game where agents independently choose between opting in and opting out, and where the efficient outcome conditional on whatever agents are opting in is implemented in the second stage.

7.3 Excludable versus Non-Excludable Public Goods

The distinction between vetoing and use exclusions is the most transparent in the contexts of an economy with $K$ public goods. Since mechanism (30) rests on veto power, it does not distinguish between excludable and non-excludable public goods. Either all agents opt in and consume all the public goods that are produced, or a veto is cast, in which case none of the public goods is provided. Since the probability of a veto goes to zero as $K$ goes out of bounds, use exclusions are not needed for asymptotic efficiency. This “asymptotic irrelevance of exclusions” depends crucially on the fact
that we keep the number of agents $n$ fixed. In contrast, Norman [27] and Fang and Norman [17] demonstrate that the exclusion instrument is a crucial feature of the constrained optimal mechanism when $K$ is fixed (to 1 and 2 respectively) and $n$ is large.

Our proof doesn't apply to sequences where both $K$ and $n$ tend to infinity. We conjecture that in the case of non-excludable public goods, asymptotic efficiency is impossible if $K$ and $n$ go out of bounds at the same rate. On the other hand, if goods are excludable, there is no need to equip agents with veto power; reservation utilities can be attained by the “milder” exclusion instrument. As a result, asymptotic efficiency is attainable in this case (regardless of the asymptotic behavior of $K/n$). Indeed, since the efficient provision for good $k$ converges either to “always provide” or “never provide” as $n$ tends to infinity approximate efficiency can be implemented with a mechanism where the provision decisions are done ex ante and a fixed price is charged for access to all public goods, much in the same spirit as Armstrong’s [2] two-part tariff scheme.

References


A Appendix

A.1 Proof of Lemma 3

Proof. Assuming that all other agents announce truthfully and that $m = (1, ..., 1) = 1$ is announced, the ex post payoff for agent $i$ of type $\theta^i$ from announcement $\hat{\theta}^i$ is

$$u^i (\hat{\theta}^i, 1, \theta) = V_K \left( x^i_K \left( \hat{\theta}^i, \theta^{-i} \right), \theta^i \right) + \sum_{j \neq i} V_K^j \left( x^j_K \left( \hat{\theta}^i, \theta^{-i} \right), \theta^j \right) + T_K^i,$$

where $T_K^i$ is defined in (37). By construction, $x^i_K (\cdot)$ is maximized at $\hat{\theta}^i = \theta^i$, resulting in ex post payoff $u^i (\theta^i, 1, \theta) = S_K (\theta) - T_K^i$. For $m \neq 1$ we have that $u^i (\hat{\theta}, m, \theta) = R_K^i (\theta)$ for any $\hat{\theta}^i$. Taking expectations over $\theta^{-i}$ gives the result. ■
A.2 Proof of Lemma 1

Proof. Using the expression for $U^j_K (\theta^j) - R^j_K (\theta^j)$ in equation (24), we can proceed just like in that proof, with the only difference being that now the calculation is for a bound on the probability that the payoff is less than half of the lower bound on the expected value of the payoff. That is,

$$1 - \Pr \left[ \overline{\Theta}^j (K) \right] = \Pr \left[ U^j_K (\theta^j) - R^j_K (\theta^j) < \frac{\delta K}{2n} \right]$$

$$= \Pr \left[ \sum_{k=1}^K \left( \phi_k^j (\theta_k^j) - \frac{E[s_k (\theta_k)] - E \left[ \sum_{j=1}^n r_k^j (\theta^j) \right]}{n} \right) < \frac{\delta K}{2n} \right]$$

/By inequality in (25)/ $\leq \Pr \left[ \sum_{k=1}^K \left( \phi_k^j (\theta_k^j) - \frac{E[s_k (\theta_k)] - E \left[ \sum_{j=1}^n r_k^j (\theta^j) \right]}{n} \right) < -\frac{\delta K}{2n} \right]$ 

$\leq \Pr \left[ \sum_{k=1}^K \left( \phi_k^j (\theta_k^j) - \frac{E[s_k (\theta_k)] - E \left[ \sum_{j=1}^n r_k^j (\theta^j) \right]}{n} \right) > -\frac{\delta K}{2n} \right]$ 

/Chebyshev's inequality/ $\leq \left( \frac{1}{2n} \right)^2 \text{Var} \left[ \sum_{k=1}^K \phi_k^j (\theta_k^j) \right] \leq \frac{4n^2 \sigma^2}{\delta^2 K}$

Since $\frac{n^2 \sigma^2}{\delta^2 K} \to 0$ as $K \to \infty$, we conclude that, for every $\varepsilon > 0$, there exists some finite integer $K' (\varepsilon)$ such that for every $K \geq K' (\varepsilon)$,

$$1 - \Pr \left[ \overline{\Theta}^j (K, \delta) \right] \leq 1 - (1 - \varepsilon)^{\frac{1}{n+1}}.$$ 

Hence, by letting $K' (\varepsilon) = \max_{j \neq i} K' (\varepsilon)$ and using stochastic independence we have that

$$\Pr \left[ \overline{\Theta}^{-i} (K) \right] = \times_{j \neq i} \Pr \left[ \overline{\Theta}^j (K, \delta) \right] \geq 1 - \varepsilon$$

for every $K \geq K' (\varepsilon)$.

A.3 Proof of Lemma 2

Proof. By regularity assumption (R2), there is a uniform bound $\bar{a} > 0$ so that $s_k (\theta_k) \leq \bar{a}$ for every $\theta_k$, which in turn implies that $S_K (\theta) / K \leq \bar{a}$ for every $K$ and $\theta \in \Theta (K)$. Thus

$$\frac{E_i \left[ S_K (\theta) \right]}{K} = \Pr \left[ \overline{\Theta}^{-i} (K) \right] \frac{E_{-i} \left[ S_K (\theta) \right]}{K} \frac{\overline{\Theta}^{-i} (K)}{K}$$

$$+ \left( 1 - \Pr \left[ \overline{\Theta}^{-i} (K) \right] \right) \frac{E_{-i} \left[ S_K (\theta) \right] \theta^{-i} \notin \overline{\Theta}^{-i} (K)}{K}$$

$$\leq \frac{E_{-i} \left[ S_K (\theta) \right]}{K} + \left( 1 - \Pr \left[ \overline{\Theta}^{-i} (K) \right] \right) \bar{a}$$
Fix $\varepsilon > 0$. By Lemma 1 there exists $K' (\frac{\varepsilon}{a})$ such that $\Pr \left[ \Theta^{-i} (K) \right] \geq 1 - \frac{\varepsilon}{a}$ for every $K \geq K' (\frac{\varepsilon}{a})$, implying that

$$\frac{E_{\theta-i} [S_K (\theta) | \Theta^{-i} (K)]}{K} \geq \frac{E_{\theta-i} [S_K (\theta)]}{K} - [1 - \left( 1 - \frac{\varepsilon}{a} \right)] \bar{a} = \frac{E_{\theta-i} [S_K (\theta)]}{K} - \varepsilon,$$

which gives the result for $K'' (\varepsilon) = K' (\frac{\varepsilon}{a})$.

\[ \text{A.4 Proof of Lemma 4} \]

\text{Proof.} Assuming that $\psi^j_i (\theta^j) = 1$ for all $j \neq i$ and all $\theta^j \in \Theta^j (K)$, the interim expected payoff for agent $i$ from setting $m^i = 1$ as defined in (36) can be written as

$$\tilde{U}^i (1, \psi^{-i}, \theta^i) = \int_{\theta^{-i}} \left( [S_K (\theta) + T_k] \Psi^j_i (\theta^{-i}) + R^j_i (\theta^i) \left[ 1 - \Psi^{-i}_K (\theta^{-i}) \right] \right) dF^i_K (\theta^{-i})$$

$$= \Pr \left[ \Theta^{-i} (K) \right] \left( E_{\theta-i} [S_K (\theta) | \Theta^{-i} (K)] + T_k \right) + \int_{\theta^{-i} \in \Theta^{-i} (K), \Theta^{-i} (K)} \left( [S_K (\theta) + T_{k'}] \Psi^j_i (\theta^{-i}) + R^j_i (\theta^i) \left[ 1 - \Psi^{-i}_K (\theta^{-i}) \right] \right) dF^i_K (\theta^{-i}).$$

Since $s_k (\theta_k)$ is uniformly bounded above by $\bar{a}$ (by R2) and $r^i_k (\theta^i_k)$ is uniformly bounded below by $\underline{b}$ (by R3), we can bound $T^i_k$ as defined in (37) as follows:

$$T^i_k \equiv - \frac{n - 1}{n} E \left[ S_K (\theta) \right] + E \left[ R^i_K (\theta^i) \right] - \frac{1}{n} E \left[ \sum_{j=1}^n R^j_i (\theta^j) \right] - \frac{\delta K}{4n}$$

$$= \left\{ \begin{array}{l}
S_K (\theta) \geq \sum_{j=1}^n R^j_i (\theta^j) \\
\end{array} \right\} \geq - E \left[ S_K (\theta) \right] + E \left[ R^i_K (\theta^i) \right] - \frac{\delta K}{4n} \geq \left( b - \bar{a} - \frac{\delta}{4n} \right) K.$$

Moreover $s_k (\theta_k)$ is uniformly bounded below by $a$ (by R2), so $S_K (\theta) \geq aK$. We conclude that

$$S_K (\theta) + T^i_k \geq \left( a + b - \bar{a} - \frac{\delta}{4n} \right) K$$

$$R^i_K (\theta^i) \geq bK > \left( a + b - \bar{a} - \frac{\delta}{4n} \right) K,$$

where the strict inequality follows since $a < \bar{a}$. Hence,

$$\int_{\theta^{-i} \in \Theta^{-i} (K), \Theta^{-i} (K)} \left( [S_K (\theta) + T_{k'}] \Psi^j_i (\theta^{-i}) + R^j_i (\theta^i) \left[ 1 - \Psi^{-i}_K (\theta^{-i}) \right] \right) dF^i_K (\theta^{-i})$$

$$\geq K \left( a + b - \bar{a} - \frac{\delta}{4n} \right) \left( 1 - \Pr \left[ \Theta^{-i} (K) \right] \right) \left( a + b - \bar{a} - \frac{\delta}{4n} \right) K.$$

Combining (A1) and (A2) we obtain

$$\tilde{U}^i (1, \psi^{-i}, \theta^i) \geq \Pr \left[ \Theta^{-i} (K) \right] \left( E_{\theta-i} [S_K (\theta) | \Theta^{-i} (K)] + T_k \right) + \left( 1 - \Pr \left[ \Theta^{-i} (K) \right] \right) K \left( a + b - \bar{a} - \frac{\delta}{4n} \right).$$
By Lemma 2, there exists $K_1$ so that for each $K \geq K_1$, 
\[
\frac{\mathbb{E}_i \left[ S_K (\theta) | \Theta^{-i} (K) \right]}{K} \geq \frac{\mathbb{E}_i \left[ S_K (\theta) \right]}{K} - \frac{\delta}{8n},
\]
where $\delta > 0$ is the uniform bound of the difference between the maximized expected surplus and the sum of the participation utilities (see Definition 2). Hence,
\[
\frac{\tilde{U}_i (1, \psi^{-i}, \theta^i)}{K} \geq \Pr \left[ \Theta^{-i} (K) \right] \left( \frac{\mathbb{E}_i S_K (\theta)}{K} + T_K^i - \frac{\delta}{8n} \right)
\]
\[
+ \left( 1 - \Pr \left[ \Theta^{-i} (K) \right] \right) \left( a + b - \bar{a} - \frac{\delta}{4n} \right),
\]
for every $K \geq K_1$. By (19), we know that $\mathbb{E}_i \left[ S_K (\theta) \right] + T_K^i + \frac{\delta}{4n}$ is the interim expected payoff for type $\theta^i$ in the Groves mechanism. Since $\theta^i \in \Theta^{-i} (K)$ it follows from definition (32) that $\mathbb{E}_i \left[ S_K (\theta) \right] + T_K^i + \frac{\delta}{4n} \geq R_K^i (\theta^i) + \frac{\delta K}{2n}$. Hence
\[
\frac{\tilde{U}_i (1, \psi^{-i}, \theta^i)}{K} \geq \Pr \left[ \Theta^{-i} (K) \right] \left( R_K^i (\theta^i) + \frac{\delta K}{4n} - \frac{\delta}{8n} \right)
\]
\[
+ \left( 1 - \Pr \left[ \Theta^{-i} (K) \right] \right) \left( a + b - \bar{a} \right),
\]
or
\[
\frac{\tilde{U}_i (1, \psi^{-i}, \theta^i) - R_K^i (\theta^i)}{K} \geq \Pr \left[ \Theta^{-i} (K) \right] \left( \frac{\delta}{8n} \right) + \left( 1 - \Pr \left[ \Theta^{-i} (K) \right] \right) \left( a + b - \bar{a} \right),
\]
The right hand side converges to $\delta/(8n)$ as $\Pr \left[ \Theta^{-i} (K) \right]$ approaches one, so there exists $\varepsilon_2 > 0$ such that $\tilde{U}_i (1, \psi^{-i}, \theta^i) - R_K^i (\theta^i) \geq 0$ if $\Pr \left[ \Theta^{-i} (K) \right] \geq 1 - \varepsilon_2$. Lemma 1 assures that there exists $K_2$ such that $\Pr \left[ \Theta^{-i} (K) \right] \geq 1 - \varepsilon_2$ for every $K \geq K_2$. For $K^* = \max \{ K_1, K_2 \}$ it therefore follows that $\tilde{U}_i (1, \psi^{-i}, \theta^i) - R_K^i (\theta^i) \geq 0$.

**A.5 Proof of Lemma 5**

Let $\Psi_K (\theta) \equiv \prod_{i=1}^n \psi_i^j (\theta^i)$ denote the probability that $m = (1, ..., 1)$ given type profile $\theta \in \Theta (K)$. The expected budget tax revenues can then be expressed as
\[
\mathbb{E} \Psi_K (\theta) \left[ \sum_{i=1}^n \hat{U}_i (\theta, 1) \right]
\]
\[
= \mathbb{E} \Psi_K (\theta) \left[ \sum_{i=1}^n V_K^i \left( x_K (\theta), \theta^i \right) - S_K (\theta) + \frac{n-1}{n} \mathbb{E} S_K (\theta) - \mathbb{E} \left[ R_K^i (\theta^i) \right] + \frac{1}{n} \mathbb{E} \left[ \sum_{j=1}^n R_K^j (\theta^j) \right] + \mathbb{E} \left[ \sum_{j=1}^n R_K^j (\theta^j) \right] + \frac{\delta K}{4n} \right]
\]
\[
= \mathbb{E} \Psi_K (\theta) \left[ (n-1) (\mathbb{E} S_K (\theta) - S_K (\theta)) + \frac{\delta K}{4} + C_K (\hat{x} (\theta, m)) \right]
\]

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By assumption, $\Psi_K(\theta) = 1$ when $\theta \in \Theta(K)$, so the budget surplus/deficit satisfies

$$
\mathbb{E}[\Psi_K(\theta) \left( \sum_{i=1}^{n} \hat{Y}_{K}(\theta, 1) - C_{K}(\hat{x}(\theta, m)) \right)]
$$

$$
= (n-1) \int_{\theta \in \Theta(K)} \left[ (\mathbb{E}[S_{K}(\theta)] - S_{K}(\theta)) + \frac{\delta K}{4} \right] dF_{K}(\theta)
$$

$$
+ (n-1) \int_{\theta \notin \Theta(K)} \Psi_{K}(\theta) \left[ (\mathbb{E}[S_{K}(\theta)] - S_{K}(\theta)) + \frac{\delta K}{4} \right] dF_{K}(\theta),
$$

But, $\mathbb{E}[S_{K}(\theta)] = \int_{\theta \in \Theta(K)} S_{K}(\theta) dF(\theta) + \int_{\theta \notin \Theta(K)} S_{K}(\theta) dF(\theta)$, so we may rearrange the expression above as

$$
\mathbb{E}[\Psi_K(\theta) \left( \sum_{i=1}^{n} \hat{Y}_{K}(\theta, 1) - C_{K}(\hat{x}(\theta, m)) \right)]
$$

$$
= \frac{1}{n-1} \mathbb{E}[S_{K}(\theta)] + \frac{\delta K}{4n} - \left[ 1 - \Pr[\Theta(K)] \right] \mathbb{E}[S_{K}(\theta)]
$$

$$
+ \int_{\theta \notin \Theta(K)} \Psi_{K}(\theta) \left( \mathbb{E}[S_{K}(\theta)] + \frac{\delta K}{4n} \right) dF_{K}(\theta)
$$

$$
> \frac{\delta K}{4n} + (1 - \Pr[\Theta(K)]) (\underline{a} - \bar{a}) K,
$$

where the inequality follows since $\mathbb{E}[S_{K}(\theta)] + \frac{\delta K}{4} > \underline{a}K$, $S_{K}(\theta) > \underline{a}K$, and $\mathbb{E}[S_{K}(\theta)] < \bar{a}K$ by assumption [R2]. Since $\Pr[\Theta(K)] \to 1$ as $K \to \infty$ it follows that there exists some finite $K^{**}$ such that the expected surplus is positive for any $K \geq K^{**}$.

\[\blacksquare\]