7 Signaling Games

7.1 What is Signaling?

The concept of "signaling" refers to strategic models where one or more informed agents take some observable actions *before* one or more uninformed agents make their strategic decisions. This leads to situations where the uninformed agent care about the actions taken by the informed agent not only because the actions affect payoffs directly, but also because the action taken say something about the type of the player. This in turn creates incentives to select actions to send the right signal about type.

7.2 The Basic Signaling Game

Consider the following setup:

- There are two players. We refer to player 1 as the *sender* and to player 2 as the *receiver.*

- Player 1 has private information about his type. We denote the type space by Θ and write θ for a generic element. The set of available actions for the sender is A_1, so a (pure) strategy is function s_1 with $s_1(\theta) \in A_1$ for every type θ. We let p denote the (prior) probability distribution over the senders type space.

- Player 2, the receiver, observes the action chosen by the sender and then take some action in A_2. A pure strategy for player is a function s_2, where $s_2(a_1) \in A_2$ for every $a_1 \in A_1$

- Utility functions,

$$\begin{aligned} &u_1(a_1, a_2; \theta) \text{ for the sender} \\ &u_2(a_1, a_2, \theta) \text{ for the receiver,} \end{aligned}$$

7.3 Equilibrium

Sometimes, it is sufficient to look at Nash equilibria. However, there are times where the set of Nash equilibria includes i) way to many possibilities; ii) some quite unreasonable "equilibria". The reason is that the Nash equilibrium concept has no restrictions on "off the equilibrium path play". We therefore want to do something in the spirit of backwards induction/subgame perfection, which is a bit trickier in games with incomplete information.

In essence, we want "equilibria" to satisfy two sorts of criteria:

1. *Sequential Rationality.* Whenever an agent is called to play, the agent does something optimal (to "refine away" Nash equilibria supported by play that is suboptimal off the path).

2. *Consistency of beliefs.* What is optimal often depends on what an agent believes about the opponent(s), and we want to rule out an agent thinking something that is contradicted by the equilibrium strategies (coordination assumption much in the same spirit as the "rational expectations" part of Nash equilibrium).

There are several equilibrium concepts that capture these two ideas to a smaller or larger extent. The definition below is not my favorite version, but we'll stick to it in this course because it is easier to define than the more appealing versions:

Definition 1 *A (pure) perfect Bayesian Equilibrium in a signaling game (of the form described above) is a strategy profile s^* and a system of beliefs μ such that*

1. $s_1^*(\theta)$ *solves* $\max_{a_1 \in A_1} u_1(a_1, s_2^*(a_1); \theta)$ *for all* $\theta \in \Theta$

2. $s_2^*(\theta)$ *solves* $\max_{a_2 \in A_1} \sum_{\theta \in \Theta} \mu(\theta|a_1) u_2(a_1, a_2, \theta)$ *for all* $a_1 \in A_1$

3. $\mu(\theta|a_1)$ *satisfies Bayes rule whenever applicable (i.e., whenever you can avoid dividing by zero)*

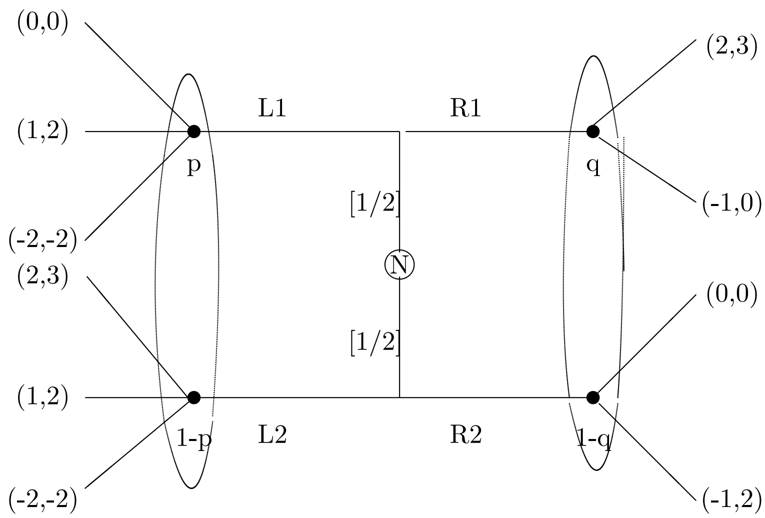

Figure 1: A Simple Signaling Game

7.4 Example

- $(R_1L_2, aT), (p,q) = (0,1)$ is Perfect Bayesian

  Bayes rule

$$\begin{aligned} p &= \Pr[t_1|L] = \frac{\Pr[L|t_1]\Pr[t_1]}{\Pr[L|t_1]\Pr[t_1] + \Pr[L|t_2]\Pr[t_2]} \\ &= \frac{0\frac{1}{2}}{0\frac{1}{2} + 1\frac{1}{2}} = 0 \\ q &= \Pr[t_1|R] = \frac{\Pr[R|t_1]\Pr[t_1]}{\Pr[R|t_1]\Pr[t_1] + \Pr[R|t_2]\Pr[t_2]} \\ &= \frac{1\frac{1}{2}}{1\frac{1}{2} + 0\frac{1}{2}} = 1 \end{aligned}$$

- CHECK! $(L_1L_2, bB); (p,q) = (\frac{1}{2}, q)$ is a PBE if $q \geq \frac{2}{5}$ (free to specify beliefs at unreached information sets, but must be done so that they support equilibrium strategies).

- CHECK! (R_1R_2, cT) is a Nash equilibrium, but not a PBE.

8 Job Market Signaling and Commitment

8.1 A Simple Version of the Static Spence Model

Consider the following formulation of the job market signalling model due to Spence;

- Types are given by $\alpha \in \{1, 2\}$
- μ_0 is the probability that $\alpha = 1$
- A Worker may choose (i.e. commit to) any education of length $t \geq 0$
- The utility is given by

$$u(w, t, \alpha) = w - \frac{t}{\alpha}$$

- Game: 1) Worker choose education. 2) Firms compete Bertrand for workers and get a profit

$$\pi = \alpha - w$$

  if it manages to attract the worker.

Let $\mu(t)$ denote the probability that firms' asses that $\alpha = 1$. In (a perfect Bayesian) equilibrium, the firms must both optimize given beliefs, implying that

$$w(t) = 2 - \mu(t)$$

8.1.1 Separating Equilibria

- Given that we restrict attention to equilibria where firms behave optimally after any $t \geq 0$ is follows that $t_1 = 0$ in any separating (perfect Bayesian) equilibrium: the reason is that if $t_1 > 0$ and the low productivity worker deviates to $t = 0$, then

$$\begin{aligned} \mu(0) &\in [0, 1] \Rightarrow \\ w(0) - 0 &\geq 1 > 1 - t_1 = w(t_1) - t_1. \end{aligned}$$

We conclude that the deviation is profitable. In words the logic is simply that the low type reveals to be the worst possible worker. Clearly, it cannot be worth anything to the low productivity worker to do so, so the worker must take the least costly action.

- By consistency of beliefs
$$\mu(t_2) = \frac{0 \times \mu_0}{0 + \mu_0 \times 1} = 0$$
so $w(t_2) = 2 - \mu(t_2) = 2$. In order for type 1 to be better off at $t_1 = 0$ than with t_2 it must be that,
$$\begin{aligned} w(t_1) &\geq w(t_2) - t_2 \\ 1 &\geq 2 - t_2 \Leftrightarrow t_2 \geq 1 \end{aligned}$$
and for type 2 to be better of with t_2 than with $t_1 = 0$ it must be that
$$\begin{aligned} w(t_2) - \frac{t_2}{2} &\geq w(0) \\ 2 - \frac{t_2}{2} &\geq 1 \Leftrightarrow t(2) \leq 2 \end{aligned}$$
Hence, if
$$1 \leq t_2 \leq 2$$
neither type has an incentive to pretend to be the other, and by considering beliefs;
$$\mu(t) = \begin{cases} 0 & \text{if } t = t_2 \\ 1 & \text{if } t \neq t_2 \end{cases}$$
it is immediate that neither type has a profitable deviation. In fact, it is sufficient to use beliefs
$$\text{or } \widetilde{\mu}(t) = \begin{cases} 0 & \text{if } t \geq t_2 \\ 1 & \text{if } t < t(2) \end{cases},$$
which you can verify by drawing a graph. $\widetilde{\mu}$ is “nicer” than μ because beliefs are monotonic in education.

8.1.2 Pooling Equilibria

To support as large a set of education levels as pooling equilibria, suppose that $t_1 = t_2 = t^*$ is a pooling equilibrium and let beliefs be

$$\mu(t) = \begin{cases} \mu_0 & \text{if } t = t^* \\ 0 & \text{if } t \neq t^* \end{cases},$$

which obviously is consistent with Bayes rule where relevant. The best deviation for both types is then to $t = 0$ and this is not profitable for the low productivity/high cost of education type if

$$2 - \mu_0 - t^* \geq 1.$$

Clearly, the high productivity type has no incentive to deviate if the low productivity type has no incentive to deviate, so any

$$0 \leq t^* \leq 1 - \mu_0$$

can be supported as a pooling equilibrium.

8.2 The Role of Commitment

The insight with the job market signalling model is that it shows that the premium paid to workers with higher education doesn't necessarily mean that education increases productivity. It could, as is the case in the model, simply be that education allows more highly productive workers screen themselves out. However, if we think of education as something that happens in real time we may complain that once the low productivity worker has decided not to educate, then the firms know that it must be a high productivity worker if they observe strictly positive education. Hence, the firms should bid for the worker immediately after the low productivity worker has been screened out, which, it seems, would destroy any separating equilibrium.

There are models addressing this issue and, while there are some subtleties, the separation in the Spence model is rescued essentially by randomizations where there is a small probability ($\rightarrow 0$ as period length $\rightarrow 0$) that the low type pools with the high type.

9 Limit Pricing

Consider the following environment:

- There are two periods and a market for a homogenous good with inverse demand $p(q) = 1 - q$ in each period (i.e., a non-durable good)
- Two firms, labeled I (incumbent) and E (entrant)
- incumbent has marginal cost $c_I \in \{c_L, c_H\}$
- entrant has marginal cost c_E
- Incumbent (the sender) is a monopolist in the first period. Sets some q_I
- Entrant (the receiver) observes the realized q_I and decides whether or not to enter. For simplicity it is assumed that once the receiver has decided whether or not to enter the incomplete information magically gets resolved.
- Firms compete Cournot in second period if there is entry
- Incumbent acts a monopolist in second period if no entry
- Cost of entry given by K.

The monopoly and cournot profits are readily computed as in the following table (recall that the asymmetric information disappears after entry).

	Entrant enters and $c = c_L$	Entrant enters and $c = c_H$	Entrant stays out
Profit for incumbent c_L	$(1 - 2c_L + c_E)^2/9$		$(1 - c_L)^2/4$
Profit for incumbent c_H		$(1 - 2c_H + c_E)^2/9$	$(1 - c_H)^2/4$
Profit for entrant	$(1 - 2c_E + c_L)^2/9 - K$	$(1 - 2c_E + c_H)^2/9 - K$	0

YOU SHOULD CHECK THESE CALCULATIONS!

To make things interesting we assume that

$$(1 - 2c_E + c_H)^2/9 > K > (1 - 2c_E + c_L)^2/9,$$

meaning that *under symmetric information* the entrant would enter if and only if the incumbent would be a high cost firm.

9.0.1 Separating Equilibria

In a separating equilibrium, the entrant will (as would be the case with symmetric information) enter if and only if the incumbent has a high cost.

Let q_L and q_H denote the quantities chosen by the low and the high cost type respectively. Observe that a necessary condition is that

$$(1 - q_L - c_L)\, q_L + \frac{(1 - c_L)^2}{4} \geq (1 - q_H - c_L)\, q_H + \frac{(1 - 2c_L + c_E)^2}{9} \quad \text{(IC-L)}$$

$$(1 - q_H - c_H)\, q_H + \frac{(1 - 2c_H + c_E)^2}{9} \geq (1 - q_L - c_H)\, q_L + \frac{(1 - c_H)^2}{4} \quad \text{(IC-H)}$$

The first condition states that a low cost firm should not be tempted to select the quantity of a high cost firm, the second the opposite. Next we observe that;

- $q_H = \frac{(1-c_H)}{2}$ in any Perfect Bayesian equilibrium (not true in a Nash equilibrium).

  Reason: The worst case scenario for a high cost type is entry. There will be entry after q_H. Hence, if $q_H \neq \frac{(1-c_H)}{2}$ there would be a profitable deviation for the high cost firm.

  The reason why this is not true in every Nash equilibrium is that the entrant in A Nash equilibrium may play something different than the Cournot duopoly equilibrium off the equilibrium path. For example, let q_L^*, q_H^* be the equilibrium outputs in the first period and consider a strategy where the entrant stays out if and only if $q = q_L^*$, where the entrant enters and plays the Cournot quantity if $q = q_H^*$, and where the entrant enters and plays $q_E = 1$ if $q \notin \{q_L^*, q_H^*\}$. This strategy would give the incumbent a profit=0 if $q \notin \{q_L^*, q_H^*\}$, implying that this construction can support a range of values for q_H^* based on the non-credible threat by the entrant to flood the market in case the incumbent plays the "wrong" first period quantity.

- Symmetrically, $\frac{(1-2c_L+c_E)^2}{9}$ is the worst possible second period payoff for the low cost type⇒if the low cost type would choose the static monopoly quantity in the first period, it must get a payoff of *at least*

$$\frac{(1 - c_L)^2}{4} + \frac{(1 - 2c_L + c_E)^2}{9} \geq (1 - q_H - c_L)\, q_H + \frac{(1 - 2c_L + c_E)^2}{9}. \quad (1)$$

- Taking these considerations together, this leads us to focus on the following necessary "incentive compatibility" constraints

$$\begin{aligned}
(1-q_L-c_L)\,q_L+\frac{(1-c_L)^2}{4} &\geq \frac{(1-c_L)^2}{4}+\frac{(1-2c_L+c_E)^2}{9} \\
\frac{(1-c_H)^2}{4}+\frac{(1-2c_H+c_E)^2}{9} &\geq (1-q_L-c_H)\,q_L+\frac{(1-c_H)^2}{4},
\end{aligned}$$

where we have;

1. replaced the right hand side in the IC constraint for L by using (1)
2. substituted $q_H=\frac{(1-c_H)}{2}$ into the IC constraint for H.

Let Δ_L and Δ_H be the value of keeping the entrant out for the low and high cost incumbent, that is

$$\begin{aligned}
\Delta_L &= \frac{(1-c_L)^2}{4}-\frac{(1-2c_L+c_E)^2}{9} \\
\Delta_H &= \frac{(1-c_H)^2}{4}-\frac{(1-2c_H+c_E)^2}{9},
\end{aligned}$$

One can check that Δ_L may be smaller or larger than Δ_H depending on c_E. The way to understand this is that as c_E increases, eventually one reaches a point where the duopoly profit equals the monopoly profit for the low cost firm. At this cost, the entrant will still produce a strictly positive quantity against the high cost firm, so the high cost firm still has a strict gain from the entrant staying out. On the other hand side, it is easy to check that for $c_E=c_L$, then $\Delta_L>\Delta_H$, so here the larger payoff under monopoly dominates.

We solve this as real economists:

Assumption $\Delta_L \geq \Delta_H$

Define,

$$\begin{aligned}
\Pi_L^M &= \frac{(1-c_L)^2}{4} \\
\Pi_H^M &= \frac{(1-c_H)^2}{4} \\
\Pi_L(q) &= (1-q-c_L)q \\
\Pi_H(q) &= (1-q-c_H)q,
\end{aligned}$$

so that the necessary incentive constraints may be written for short as

$$\Delta_H \leq \Pi_H^M - \Pi_H(q_L)$$
$$\Delta_L \geq \Pi_L^M - \Pi_L(q_L)$$

In Figure 2 the range of separating equilibrium quantities q_L is illustrated in terms of these two inequalities. To draw the picture, note that the "cost of eliminating entry" $\Pi_J^M - \Pi_J(q_L)$ has a minimum of zero at the static monopoly quantity, which is a higher quantity for the low cost firm. Using the fact that

$$\frac{d}{dq}\left(\Pi_L(q) - \Pi_H(q)\right) = (1 - 2q - c_L) - (1 - 2q - c_H) = c_H - c_L > 0,$$

one checks that $\Pi_L^M - \Pi_L(q)$ and $\Pi_H^M - \Pi_H(q)$ intersects at most once. It should be clear from the figure that $\Delta_L \geq \Delta_H$ is a sufficient (but not necessary) condition for the range of separating equilibria to be nonempty.

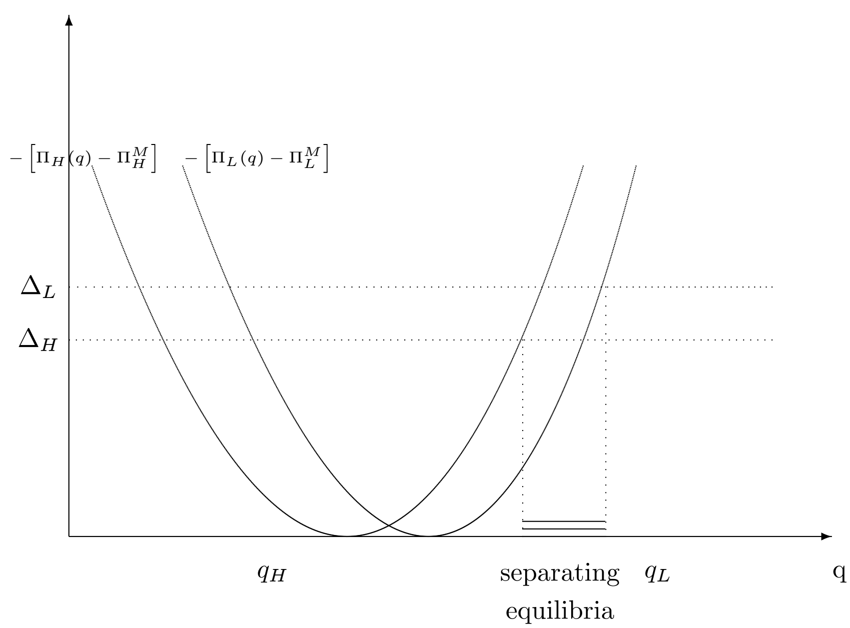

Figure 2: A Continuum of Separating Equilibria in the 2-type Limit Pricing Model

Finally, to support a quantity q_L in the range indicated in the figure as a separating equilibrium (where the high cost firm chooses its static monopoly output in the first period) in the simplest possible way we let the beliefs off the equilibrium path be "as pessimistic as possible". That is, if the entrant thinks that any quantity other than q_L must be evidence that the firm is a high cost firm, so that

$$\mu\left(c = c_L | q\right) = \begin{cases} 1 & \text{if } q = q_L \\ 0 & \text{otherwise} \end{cases},$$

then *any deviation q for the low cost firm gives a payoff*

$$\left(1 - q - c_L\right) q_H + \frac{\left(1 - 2c_L + c_E\right)^2}{9} \leq \frac{\left(1 - c_L\right)^2}{4} + \frac{\left(1 - 2c_L + c_E\right)^2}{9}.$$

so deviating to the monopoly output is the best deviation under these beliefs. Moreover, from the point of view of the high cost firm any $q = q_L$ gives a profit

$$\left(1 - q - c_H\right) q + \frac{\left(1 - 2c_L + c_E\right)^2}{9},$$

under these beliefs, which obviously is maximized for $q = q_H$. In words, if the only quantity that deters entry is q_L, then the best first period quantity for the high cost firm must be either q_L or the static monopoly quantity.

9.0.2 Remarks

1. The analysis above shows that it is a possibility that a relatively efficient incumbent firm engages in "limit pricing" (higher quantity=lower price) to signal to potential entrants that they'll be so strong competitors that they better keep out of the market

2. Note that in the equilibrium constructed nobody is fooled. However, an efficient firm would be mistaken for a not so efficient firm should they deviate.

3. The example also shows that limit pricing (sometimes considered "predatory behavior") may actually benefit consumers (although one would have to consider more carefully exactly what the alternative is id one would think of this as a serious policy-related exercise) by lowering the price.

4. There is in general also a continuum of pooling equilibria (as well as semi-pooling). This huge multiplicity of equilibria is typical for (simple) signaling games and has led to a monstrous literature on equilibrium selection. I personally think that refinements beyond standard sequential rationality arguments is a suspect idea and that we have to live with the multiplicity. I may talk a little bit about this refinements literature, mainly to illustrate what I think is wrong with the whole idea. However, to some extent the multiplicity is an artefact of the very simplicity of the model (just a few types) and at least when we look for pure separating equilibria, this is mitigated by a richer (continuous) type space.