Multiple Regression

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Part I

References

- Wooldridge (2013) chapter 3
- Stock and Watson (2009) chapter 6 (parts of 7,8,9, & 18 also relevant)
- Angrist and Pischke (2014) chapter 2
- Bierens (2010)
- Baltagi (2002) chapter 4
- Linton (2017) Part III (more advanced)

Section 1

Introduction

Why we need a multiple regression model

- There are many factors affecting the outcome variable y.
- If we want to estimate the marginal effect of one of the factors (regressors), we need to control for other factors.
- Suppose that we are interested in the effect of x_1 on y , but y is affected by both x_1 and x_2:

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \epsilon_i \tag{1}$$

Why we need a multiple regression model

- Assume:

MLR.1 (linear model)

MLR.2 (independence) $\{(x_{1,i}, x_{2,i}, y_i)\}_{i=1}^n$ is independent random sample

MLR.3 (rank condition) no multicollinearity

MLR.4 (exogeneity) $\mathrm{E}[\epsilon|X] = 0$

- Suppose we regress y only against x_1 :

$$\hat{\beta}_1^s = \frac{\sum_{i=1}^n (x_{1,i} - \bar{x}_1) y_i}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2}$$

- What is $\mathrm{E}[\hat{\beta}_1^s]$?

Why we need a multiple regression model

- What is $\mathrm{E}[\hat{\beta}_1^s]$?

$$\begin{aligned}
\hat{\beta}_1^s =& \frac{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)(\beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \epsilon_i)}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2} \\
=& \beta_1 + \beta_2 \frac{\sum_{i=1}^n (x_{1,i} - \bar{x}_1) x_{2,i}}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2} + \frac{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)\epsilon_i}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2} \\
\mathrm{E}[\hat{\beta}_1^s | x\text{'s}] =& \beta_1 + \beta_2 \frac{\sum_{i=1}^n (x_{1,i} - \bar{x}_1) x_{2,i}}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2} \\
=& \beta_1 + \beta_2 \frac{\widehat{\mathrm{Cov}}(x_1, x_2)}{\widehat{\mathrm{Var}}(x_1)}
\end{aligned}$$

- $\hat{\beta}_1^s$ biased unless $\widehat{\mathrm{Cov}(x_1, x_2)} = 0$ or $\beta_2 = 0$

Omitted variable bias

- When true model is "long regression"

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \epsilon_i$$

  but we only estimate the "short regression",

$$y_i = \beta_0 + \beta_1^s x_{1,i} + \underbrace{v_i}_{=\beta_2 x_{2,i} + \epsilon_i}$$

  then

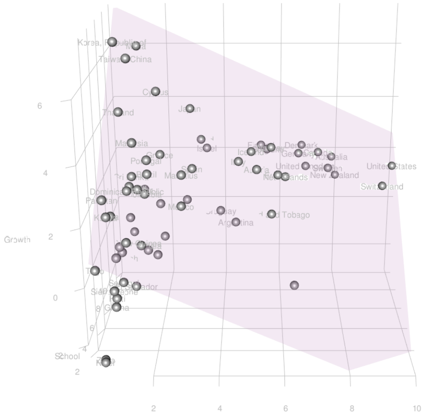
$$\mathrm{E}[\hat{\beta}_1^s] = \beta_1 + \beta_2 \frac{\widehat{\mathrm{Cov}}(x_1, x_2)}{\widehat{\mathrm{Var}}(x_1)}$$

- If x_1 and x_2 related, we can no longer say that $\mathrm{E}[v_i | x_{1,i}] = 0$
- When x_1 changes, x_2 changes as well, which contaminates estimation of the effect of x_1 on y
- As a result, the short regression estimate, $\hat{\beta}_1^s$, is biased

Section 2

Examples

Example: growth, GDP, and schooling



[0] Code

Example: growth, GDP, and schooling

	Model 1	Model 2	Model 3
(Intercept)	1.796***	0.958*	0.895*
	(0.378)	(0.418)	(0.389)
rgdp60	0.047		−0.485**
	(0.095)		(0.146)
yearsschool		0.247**	0.640***
		(0.089)	(0.144)
R^2	0.004	0.110	0.244
Adj. R^2	-0.012	0.095	0.219
Num. obs.	65	65	65
RMSE	1.908	1.804	1.676

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table: Growth and GDP and education in 1960

Test scores and student teacher ratios

- Data from Stock and Watson (2009)
- School average test score and student teacher ratios
- Code

Test scores and student teacher ratios

	1	2	3	4
β_0	698.93^{***}	739.45^{***}	638.73^{***}	640.32^{***}
	(9.47)	(60.72)	(7.45)	(5.77)
$\frac{student}{teacher}$	-2.28^{***}	-6.45	-0.65^{*}	-0.07
	(0.48)	(6.19)	(0.35)	(0.28)
$\left(\frac{student}{teacher}\right)^2$		0.11		
		(0.16)		
Income			1.84^{***}	1.49^{***}
			(0.09)	(0.07)
English learners				-0.49^{***}
				(0.03)
Num. obs.	420	420	420	420

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Chandra et al. (2008)

"Does Watching Sex on Television Predict Teen Pregnancy?"

- Long regression:

$$preg = \beta_0 + \beta_1 sexTV + \beta_2 totalTV + \epsilon$$

  - $preg$ = teen pregnancy
  - $sexTV$ = hours of television with sexual content
  - $totalTV$ = hour of television

- Short (bivariate) regression:

$$preg = \beta_0^s + \beta_1^s sexTV + v$$

- Results:
  - Short: $\hat{\beta}_1^s \sqrt{\widehat{\text{Var}}(sexTV)} = 0.18$
  - Long: $\hat{\beta}_1 \sqrt{\widehat{\text{Var}}(sexTV)} = 0.30$

Chandra et al. (2008)

- Also report that

$$totalTV = \gamma_0 + \gamma_1 sexTV + e$$

$$\hat{\gamma}_1 \sqrt{\widehat{\text{Var}}(sexTV)} = 0.29$$

- What is β_2?
  - From omitted variable bias formula, we know

$$\begin{aligned}
\text{E}[\hat{\beta}_1^s] =& \beta_1 + \beta_2 \underbrace{\frac{\widehat{\text{Cov}}(sexTV, totalTV)}{\widehat{\text{Var}}(sexTV)}}_{=\hat{\gamma}_1} \\
=& \beta_1 + \beta_2 \hat{\gamma}_1 \\
\beta_2 =& \frac{\beta_1 - \text{E}[\hat{\beta}_1^s]}{\hat{\gamma}_1}
\end{aligned}$$

Chandra et al. (2008)

- so,

$$
\begin{aligned}
\hat{\hat{\beta}}_2 = & \frac{\hat{\beta}_1 - \hat{\beta}_1^s}{\hat{\gamma}_1} \\
= & \frac{(\hat{\beta}_1 - \hat{\beta}_1^s)\sqrt{\widehat{\mathrm{Var}}(sexTV)}}{\hat{\gamma}_1\sqrt{\widehat{\mathrm{Var}}(sexTV)}} \\
= & \frac{0.30 - 0.18}{0.29} \\
= & -0.41
\end{aligned}
$$

Section 3

Interpretation

Multiple linear regression model

$$y_i = \beta_0 + \beta_1 x_{1,i} + \beta_2 x_{2,i} + \cdots + \beta_k x_{k,i} + \epsilon_i$$

As in bivariate regression, let's assume:

MLR.1 (linear model)

MLR.2 (independence) $\{(x_{1,i}, x_{2,i}, y_i)\}_{i=1}^n$ is independent random sample

MLR.3 (rank condition) no multicollinearity

MLR.4 (exogeneity) $\mathrm{E}[\epsilon | x_{1,i}, x_{2,i}, \ldots, x_{k,i}] = 0$

Interpretation of coefficients

- β_j is partial (marginal) effect of x_j on y

$$\beta_j = \frac{\partial y_i}{\partial x_{j,i}}$$

- β_j is partial (marginal) effect of $\mathrm{E}[y|x_1, ..., x_k]$

$$\beta_j = \frac{\partial \mathrm{E}[y|x_1, ..., x_k]}{\partial x_j}$$

- Other regressors are held constant

$$\Delta y_i = \beta_0 + \beta_1 \Delta x_{1,i} + \beta_2 x_{2,i} + \cdots + \beta_k x_{k,i} + \epsilon_i$$

β_1 is the *ceteris paribus* effect of $x_{1,i}$ on y_i

Changing more than one regressor simultaneously

- Sometimes it does not make sense to change x_j without also changing x_k
- Example: age-earnings profile

$$\log wage_i = \beta_0 + \beta_1 age_i + \beta_2 age_i^2 + \epsilon_i$$

  - Cannot *age* while holding age^2 constant
  - Instead should report marginal effect

$$\frac{\partial \log wage_i}{\partial age_i}\Big|_{age_i=a} = \beta_1 + 2\beta_2 a$$

or average marginal effect

$$\overline{\frac{\partial \log wage_i}{\partial age_i}} = \beta_1 + 2\beta_2 \overline{age}$$

Changing more than one regressor simultaneously

- Example: Chandra et al. (2008) effect of exposure to sexual content on television on teen pregnancy

$$preg = \beta_0 + \beta_1 sexTV + \beta_2 totalTV + \epsilon$$

  - $sexTV$ = hours of television with sexual content
  - $totalTV$ = hour of television
  - Should we care about β_1, β_2, or some combination?

TABLE 1 Bivariate Analyses Predicting Pregnancy After Baseline Among Youths Who Ever Had Sex ($N = 718$)

Baseline Predictors	β	P
Exposure to sex on television[a]	0.18	.280
Covariates		
Total television exposure[a]	−0.23	.162
Age	0.20	.108
Lower grades	0.05	.749
Parent education[a]	−0.16	.223
Educational aspirations[a]	−0.13	.405
Hispanic[b]	0.04	.923
Black[b]	0.95[c]	.011
Female	0.99[d]	.002
Lives in 2-parent household	−1.50[d]	<.001
Deviant or problem behavior[a]	0.42[c]	.012
Intention to have children before age 22 y	1.03[c]	.013

[a] Coefficients reflect the association with pregnancy for each 1-SD increase or decrease in the predictor.

[b] Comparison group is all other races.

[c] $P < .05$.

[d] $P < .01$.

TABLE 2 **Multivariate Logistic Regression Analyses Predicting Pregnancy After Baseline Among Youths Who Ever Had Sex ($N = 718$)**

Baseline Predictors	β	P
Exposure to sex on television[a]	0.44[b]	.034
Covariates		
Total television exposure[a]	−0.42[b]	.022
Age	0.28[b]	.022
Lower grades	0.21	.288
Parent education[a]	0.00	.999
Educational aspirations[a]	−0.14	.446
Hispanic[c]	0.86	.084
Black[c]	1.20[b]	.011
Female	1.20[d]	.001
Lives in 2-parent household	−1.50[d]	<.001
Deviant or problem behavior[a]	0.43[b]	.014
Intention to have children before age 22	0.61	.279

[a] Coefficients reflect the association with pregnancy for each 1-SD increase or decrease in the predictor.

[b] $P < .05$.

[c] Comparison group is white and races other than Hispanic or black.

[d] $P < .01$.

Section 4

OLS estimation

OLS estimation

$$(\hat{\beta}_0, \ldots, \hat{\beta}_k) = \arg\min_{b_0,\ldots,b_k} \sum_{i=1}^{n} (y_i - b_0 - b_1 x_{1,i} - \cdots - b_k x_{k,i})^2$$

- First order conditions:

$$\begin{aligned}
\sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{1,i} - \cdots - \hat{\beta}_k x_{k,i}\right) &= 0 \\
\sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{1,i} - \cdots - \hat{\beta}_k x_{k,i}\right) x_{1,i} &= 0 \\
\vdots &= \vdots \\
\sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{1,i} - \cdots - \hat{\beta}_k x_{k,i}\right) x_{k,i} &= 0
\end{aligned}$$

OLS estimation

or with $\hat{\epsilon}_i = \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{1,i} - \cdots - \hat{\beta}_k x_{k,i}\right)$,

$$\sum_{i=1}^{n} \hat{\epsilon}_i = 0$$

$$\sum_{i=1}^{n} \hat{\epsilon}_i x_{j,i} = 0 \text{ for } j = 1, 2, \ldots, k$$

- We choose $\hat{\beta}_0, \ldots, \hat{\beta}_k$ so that residuals and regressors are uncorrelated (orthogonal)
- First order conditions are a system of linear equations in $\hat{\beta}_0, \ldots, \hat{\beta}_k$

Section 5

Partitioned regression

Partitioned regression

- A useful representation of $\hat{\beta}_j$ (e.g. $j = 1$)
  - Regress $x_{1,i}$ on other regressors

$$x_{1,i} = \hat{\gamma}_0 + \hat{\gamma}_2 x_{2,i} + \cdots + \hat{\gamma}_k x_{k,i} + \tilde{x}_{1,i}$$

    where $\tilde{x}_{1,i}$ is the OLS residual
  - Then

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n \tilde{x}_{1,i} y_i}{\sum_{i=1}^n \tilde{x}_{1,i}^2}$$

  - Note that $\frac{1}{n} \sum_{i=1}^n \tilde{x}_{1,i} = 0$, so $\hat{\beta}_1 =$ slope coefficient from regressing y on $\tilde{x}_1$

Proof outline

- Substitute $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1,i} + \cdots + \hat{\beta}_k x_{k,i} + \hat{\epsilon}_i$ into $\frac{\sum_{i=1}^n \tilde{x}_{1,i} y_i}{\sum_{i=1}^n \tilde{x}_{1,i}^2}$
- Use the following facts to simplify:
  1. $\sum_{i=1}^n \tilde{x}_{1,i} = 0$
  2. $\sum_{i=1}^n \tilde{x}_{1,i} x_{j,i} = 0$ for $j = 2, ..., k$
  3. $\sum_{i=1}^n \tilde{x}_{1,i} x_{1,i} = \sum_{i=1}^n \tilde{x}_{1,i}^2$
  4. $\sum_{i=1}^n \tilde{x}_{1,i} \hat{\epsilon}_i = 0$
- See handout version of slides for details

"Partialling out"

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n \tilde{x}_{1,i} y_i}{\sum_{i=1}^n \tilde{x}_{1,i}^2}$$

1. Regress x_1 against the other controls (regressors) and keep the residuals $\tilde{x}_1$, which is the "part" of x_1 that is uncorrelated with the other regressors
2. Regress y against $\tilde{x}_1$

$\hat{\beta}_1$ measures the effect of x_1 after the effects of $x_2, \ldots, x_k$ have been partialled out

Example: California test scores and student teacher ratios

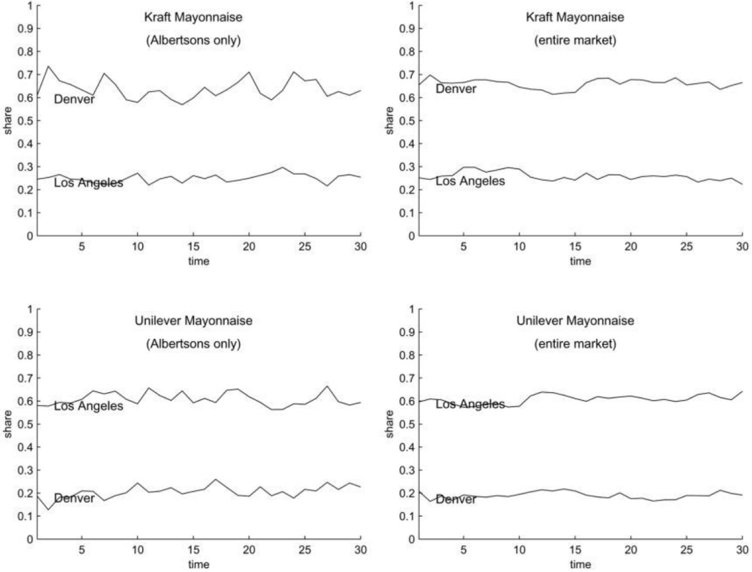
```
## long regression
summary(lm(testscr ~ str + avginc + calw_pct +
## partialling out
tilde.str <- residuals(lm(str ~ avginc + calw_p
                                    data=ca))
mean(tilde.str) ## should be 0
cov(tilde.str, ca$avginc) ## should be 0
sum(tilde.str*ca$str)
sum(tilde.str^2) ## should equal previous line

## should equal long regression coefficient
cov(tilde.str, ca$testscr)/var(tilde.str)
sum(tilde.str*ca$testscr)/sum(tilde.str^2)

## =long regression coefficient, but wrong stan
summary(lm(ca$testscr ~ tilde.str))
```

Section 6

Example: Bronnenberg, Dhar, and Dubé (2009)

Example: Bronnenberg, Dhar, and Dubé (2009)

- Looks at market shares of brands of consumer packaged goods (CPG) across markets and time
    - CPG = beer, coffee, ketchup, etc.
- Market shares from AC Nielsen scanner data
    - This type of data has been used very frequently in IO during the last decade
    - AC Nielsen distributes bar code scanners to a sample of consumers, consumers record every purchase by scanning bar codes
    - 4-week intervals, June 1992-May 1995
- Results
    - Market shares variable across geographic markets, but persistent over time within each market
    - Geographic market shares strongly correlated with first mover advantage
        - e.g. Miller (founded in Milwaukee) most popular beer in Milwaukee, Budweiser (founded in St. Louis) most popular beer in St. Louis


Kraft Mayonnaise
(Albertsons only)
Denver
Los Angeles
Kraft Mayonnaise
(entire market)
Denver
Los Angeles
Unilever Mayonnaise
(Albertsons only)
Los Angeles
Denver
Unilever Mayonnaise
(entire market)
Los Angeles
Denver
share
time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
5
10
15
20
25
30

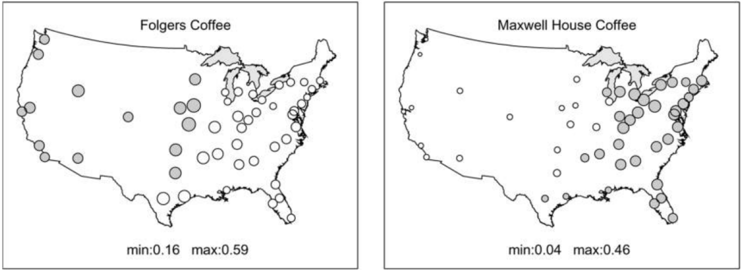


FIG. 2.—The joint geographic distribution of share levels and early entry across U.S. markets in ground coffee. The areas of the circles are proportional to share levels. Shaded circles indicate that a brand locally moved first.

$$share_{im} = \beta_0 + \beta_1 \underbrace{brandA_{im}}_{=1 \text{ if good } i \text{ is brand A}} + \beta_2 \underbrace{earlyEntry_{im}}_{=1 \text{ if good } i \text{ entered } m \text{ first}} + \epsilon_{im}$$

Variable	Entry Effect (1)	Brand Effects (2)	Entry and Brand Effects (3)
Beer (N = 94):			
Intercept	.141 (.010)	.149 (.011)	.139 (.011)
Budweiser		.118 (.016)	.020 (.026)
Miller			
Early entry	.134 (.014)		.117 (.026)
R^2	.483	.372	.487
Coffee (N = 150):			
Intercept	.139 (.011)	.059 (.014)	.052 (.011)
Folgers		.251 (.020)	.206 (.015)
Maxwell House		.197 (.020)	.088 (.018)
Hills Bros.			
Early entry	.208 (.019)		.175 (.015)
R^2	.440	.533	.755
Ketchup (N = 50):			
Intercept			.388 (.019)
Heinz			
Early entry			.072 (.025)
R^2			.149

- How are results in the three columns for beer related?

$$\hat{\beta}_1^s = \frac{\sum_{i=1}^n (x_{1,i} - \bar{x}_1) y_i}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2}$$

- Substitute $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1,i} + \hat{\beta}_2 x_{2,i} + \hat{\epsilon}_i$

$$\hat{\beta}_1^s = \frac{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)(\hat{\beta}_0 + \hat{\beta}_1 x_{1,i} + \hat{\beta}_2 x_{2,i} + \hat{\epsilon}_i)}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2}$$

$$= \frac{\hat{\beta}_0 \overbrace{\left(\sum_{i=1}^n x_{1,i} - \bar{x}_1 \right)}^{=0} + \hat{\beta}_1 \left(\sum_{i=1}^n (x_{1,i} - \bar{x}_1) x_{1,i} \right)}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2} +$$

$$+ \frac{\hat{\beta}_2 \left(\sum_{i=1}^n (x_{1,i} - \bar{x}_1) x_{2,i} \right)}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2} +$$

$$+ \frac{\overbrace{\left(\sum_{i=1}^n x_{1,i} \hat{\epsilon}_i \right)}^{=0 \text{FOC}} - \overbrace{\left(\bar{x}_1 \sum_{i=1}^n \hat{\epsilon}_{1,i} \right)}^{=0 \text{FOC}}}{\sum_{i=1}^n (x_{1,i} - \bar{x}_1)^2}$$

- $$\hat{\beta}_1^s = \hat{\beta}_1 + \hat{\beta}_2 \frac{\widehat{\text{Cov}}(x_1, x_2)}{\widehat{\text{Var}}(x_1)}$$
  $$= \hat{\beta}_1 + \hat{\beta}_2 \hat{\gamma}_1^2$$
  where $\hat{\gamma}_1^2$ is coefficient on x_1 in a regression of x_2 on x_1
- "Short ($\hat{\beta}_1^s$) equals long ($\hat{\beta}_1$) plus the effect of omitted ($\hat{\beta}_2$) times the regression of the omitted on the included ($\hat{\gamma}_1^2$)" (Angrist and Pischke, 2009)
- Similarly,
  $$\hat{\beta}_2^s = \hat{\beta}_2 + \hat{\beta}_1 \underbrace{\hat{\gamma}_2^1}_{=\frac{\widehat{\text{Cov}}(x_1,x_2)}{\widehat{\text{Var}}(x_1)}}$$

- In this example, $\hat{\beta}^s_{bud} = 0.118$, $\hat{\beta}^s_{entry} = 0.134$, $\hat{\beta}_{bud} = 0.020$, and $\hat{\beta}_{entry} = 0.117$, so

$$\frac{\widehat{\mathrm{Cov}}(bud, entry)}{\widehat{\mathrm{Var}}(entry)} = \frac{\hat{\beta}^s_{entry} - \hat{\beta}_{entry}}{\hat{\beta}_{bud}} = \frac{0.134 - 0.117}{0.02} = 0.85$$

$$\frac{\widehat{\mathrm{Cov}}(bud, entry)}{\widehat{\mathrm{Var}}(bud)} = \frac{\hat{\beta}^s_{bud} - \hat{\beta}_{bud}}{\hat{\beta}_{entry}} = \frac{0.118 - 0.02}{0.17} = 0.84$$

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