

# Heteroskedasticity & Dependence

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## References

- Wooldridge (2013) chapter 8
- Stock and Watson (2009) chapter 10.5
- Angrist and Pischke (2014) chapter 5

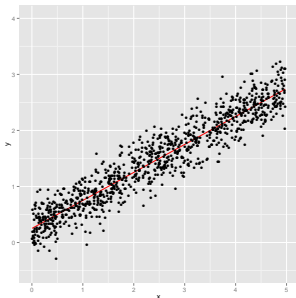
# Section 1

## Introduction

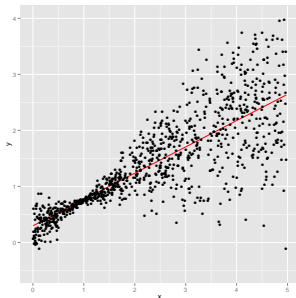
# Introduction

**MLR.5** (homoskedasticity)  $\text{Var}(\epsilon_i|X) = \sigma_\epsilon^2$  is often an implausible assumption

## Homoskedastic



## Heteroskedastic



## Code

- **Heteroskedasticity** is when  $\text{Var}(\epsilon|X)$  varies with  $X$

# Introduction

- Role of homoskedasticity:
  - Was *not* needed to show OLS unbiased and consistent
  - Was needed to calculate  $\text{Var}(\hat{\beta}|X)$
- If errors are heteroskedastic then still have  $E[\hat{\beta}_j] = \beta_j$  and  $\hat{\beta}_j \xrightarrow{p} \beta_j$  (assuming MLR.1-4), but usual standard error is invalid
- If there is heteroskedasticity, the variance of  $\hat{\beta}_1$  is different, but we can fix it
- “heteroskedasticity robust standard errors” / “heteroscedasticity consistent standard errors” / “Eicker–Huber–White standard errors”

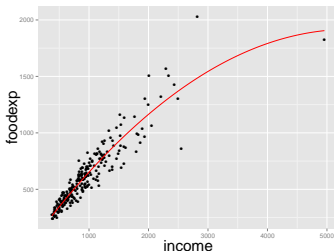
## Example: Engel curves

- An **Engel curve** describes how a consumer's purchases of a good (like food) varies with the consumer's total resources (income or total spending)
- Named after **Engel (1857)** – looked at food expenditures and income of Belgian families
  - See **Chai and Moneta (2010)** for a description of Engel's work and **Lewbel (2008)** for a brief summary of modern research on Engel curves

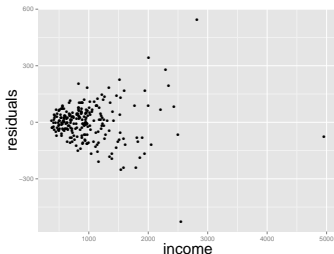
## Example: Engel (1857) curve

$$\text{foodexp} = \beta_0 + \beta_1 \text{income} + \beta_2 \text{income}^2 + \epsilon$$

Engel curve



Residuals



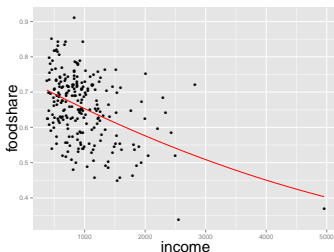
- Income and food expenditure measured in Belgian Francs



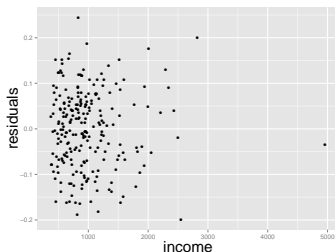
# Example: Engel (1857) curve in shares

$$\frac{\text{foodexp}}{\text{income}} = \text{foodshare} = \beta_0 + \beta_1 \text{income} + \beta_2 \text{income}^2 + \epsilon$$

Engel curve



Residuals



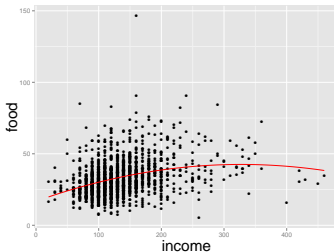
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# Engel curves with modern data

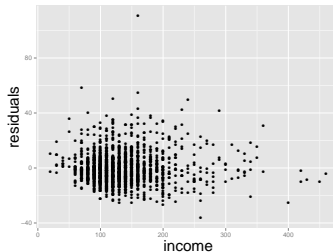
- There are still papers about estimating Engel curves
  - Banks, Blundell, and Lewbel (1997), Blundell, Duncan, and Pendakur (1998), Blundell, Browning, and Crawford (2003)
- Next slides uses data from British Family Expenditure Surveys (FES) for 1980-1982 (same data as Blundell, Duncan, and Pendakur (1998))

# Engel curves with modern data: food

## Engel curve



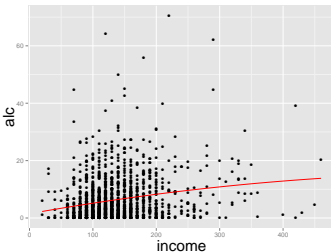
## Residuals



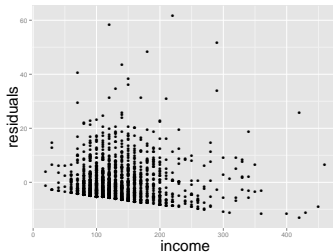
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# Engel curves with modern data: alcohol

## Engel curve



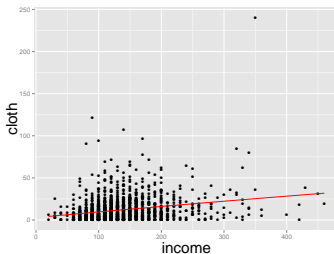
## Residuals



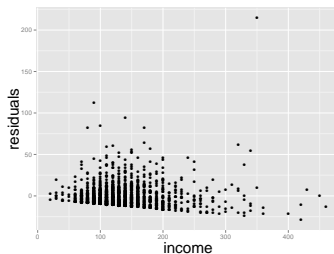
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# Engel curves with modern data: clothing

## Engel curve



## Residuals



Code

## Section 2

# Consequences of heteroskedasticity

## Consequences of heteroskedasticity

- OLS still unbiased and consistent assuming
  - MLR.1** (linear model)
  - MLR.2** (independence)  $\{(x_{1,i}, x_{2,i}, y_i)\}_{i=1}^n$  is an independent random sample
  - MLR.3** (rank condition) no multicollinearity: no  $x_{j,i}$  is constant and there is no exact linear relationship among the  $x_{j,i}$
  - MLR.4** (exogeneity)  $E[\epsilon_i | x_{1,i}, \dots, x_{k,i}] = 0$

- Homoskedastic-only standard errors are incorrect,

$$\text{Var}(\hat{\beta}_j | X) \neq \frac{\sum_{i=1}^n \sigma_{\epsilon}^2}{\sum_{i=1}^n \tilde{x}_{j,i}^2} \text{ and } \text{plim} \frac{\sum_{i=1}^n \hat{\epsilon}_i^2}{\sum_{i=1}^n \tilde{x}_{j,i}^2} \neq \text{Var}(\hat{\beta}_j)$$

- $t$  (and  $F$ ) statistics formed using homoskedasticity-only standard errors do not have  $t$  (or  $F$ ) distributions (not even asymptotically)
- Hypothesis tests and confidence intervals formed using homoskedasticity-only standard errors are invalid
- OLS is not BLUE

## Section 3

# $\text{Var}(\hat{\beta})$ with heteroskedasticity



## Var( $\hat{\beta}$ ) with heteroskedasticity

- **Central limit theorem:** for i.i.d. data, if  $E[y_i] = \mu$  and  $\text{Var}(y_i) = \sigma^2$  for all  $i$  then

$$\sqrt{n}(\bar{y}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$$

- Recall how to show  $\hat{\beta}$  is asymptotically normal in bivariate regression

$$\begin{aligned}\sqrt{n}(\hat{\beta}_1 - \beta_1) &= \sqrt{n} \left( \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}) y_i}{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} - \beta_1 \right) \\ &= \sqrt{n} \left( \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}) (\beta_0 + \beta_1 x_i + \epsilon_i)}{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2} - \beta_1 \right) \\ &= \frac{\sqrt{n} \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}) \epsilon_i}{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}\end{aligned}$$

## $\text{Var}(\hat{\beta})$ with heteroskedasticity

- Already showed that  $\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \xrightarrow{p} \text{Var}(x)$
- Need to apply CLT to  $\sqrt{n} \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}) \epsilon_i$ . First note that

$$\sqrt{n} \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}) \epsilon_i = \sqrt{n} \frac{1}{n} \sum_{i=1}^n (x_i - E[x]) \epsilon_i + \underbrace{(E[x] - \bar{x})}_{\xrightarrow{p} 0} \underbrace{\sqrt{n} \frac{1}{n} \sum_{i=1}^n \epsilon_i}_{\xrightarrow{d} \mathcal{N}(0, \text{Var}(\epsilon))}$$

- Let  $w_i = (x_i - E[x]) \epsilon_i$ , can apply CLT to  $w_i$  because
  - $E[w_i] = E[x_i \epsilon_i] = 0$
  - Observations are independent
  - Assume  $\text{Var}(w_i) = E[(x_i - E[x])^2 \epsilon_i^2]$  exists

then  $\frac{1}{\sqrt{n}} \sum_{i=1}^n (x_i - E[x]) \epsilon_i \xrightarrow{d} \mathcal{N}(0, E[(x_i - E[x])^2 \epsilon_i^2])$

- Can conclude that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (x_i - \bar{x}) \epsilon_i \xrightarrow{d} \mathcal{N}(0, E[(x_i - E[x])^2 \epsilon_i^2])$$

## $\text{Var}(\hat{\beta})$ with heteroskedasticity

- By Slutsky's theorem,

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = \frac{\sqrt{n} \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}) \epsilon_i}{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$$
$$\xrightarrow{d} N\left(0, \frac{E[(x_i - E[x])^2 \epsilon_i^2]}{\text{Var}(x)^2}\right)$$

or equivalently,

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{E[(x_i - E[x])^2 \epsilon_i^2]}{n \text{Var}(x)^2}}} \xrightarrow{d} N(0, 1)$$

## $\text{Var}(\hat{\beta})$ with heteroskedasticity

- By Slutsky's theorem can replace  $\sqrt{\frac{E[(x_i - E[x])^2 \epsilon_i^2]}{\text{Var}(x)^2}}$  by consistent estimators, and

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \hat{\epsilon}_i^2}{n \left( \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right)^2}}} \xrightarrow{d} N(0, 1)$$

- Similar reasoning applies to multivariate regression

$$\frac{\hat{\beta}_j - \beta_j}{\sqrt{\frac{\frac{1}{n} \sum_{i=1}^n \tilde{x}_{j,i}^2 \hat{\epsilon}_i^2}{n \left( \frac{1}{n} \sum_{i=1}^n \tilde{x}_{j,i}^2 \right)^2}}} \xrightarrow{d} N(0, 1)$$

## $\text{Var}(\hat{\beta})$ with heteroskedasticity

- $\sqrt{\frac{\frac{1}{n} \sum_{i=1}^n \tilde{x}_{j,i}^2 \hat{\epsilon}_i^2}{n \left( \frac{1}{n} \sum_{i=1}^n \tilde{x}_{j,i}^2 \right)^2}}$  is called the heteroskedasticity robust standard error or the Eicker-Huber-White standard error
  - Statistics: **Eicker (1967)**, **Huber (1967)**
  - Econometrics: **White (1980)**

# R: Heteroskedasticity robust standard errors

```
1  ## calculating Heteroskedasticity robust standard errors
2  engelCurve <- lm(foodexp ~ income, data=engel)
3  ## calculate by hand
4  hetSE <- sqrt(mean( (engel$income - mean(engel$income))^2 *
5                    residuals(engelCurve)^2) /
6                    (nrow(engel)*mean( (engel$income - mean(engel$income))^2 )^2 ))
7
8  ## use "sandwich" package
9  library(sandwich)
10 sqrt(vcovHC(engelCurve, type="HC0")[2,2])
11
12 ## test H_0: each coefficient = 0 separately
13 library(lmtest)
14 coeftest(engelCurve, vcov=vcovHC(engelCurve, type="HC0"))
15 ## compare with homoskedastic standard errors
16 coeftest(engelCurve)
17
18 ## we would do F-tests with
19 ## waldtest(engelCurve, vcov=vcovHC(engelCurve, type="HC0")) or
20 ## lrtest(engelCurve, vcov=vcovHC(engelCurve, type="HC0"))
21
22 ## Or even easier, just use lfe package
23 library(lfe)
24 engelCurve2 <- felm(foodexp ~ income, data=engel)
25 summary(engelCurve2, robust=TRUE)
```

Code

## Section 4

# Examples

## Growth, GDP, and education

	GDP 1960	Education 1960	Both
(Intercept)	1.796*** (0.378) [0.456]	0.958** (0.418) [0.454]	0.895** (0.389) [0.428]
rgdp60	0.047 (0.095) [0.083]		-0.485*** (0.146) [0.149]
years school		0.247*** (0.089) [0.084]	0.640*** (0.144) [0.154]
Num. obs.	65	65	65

\*\*\*  $p < 0.01$ , \*\*  $p < 0.05$ , \*  $p < 0.1$

Homoskedastic standard errors in parentheses

Heteroskedastic standard errors in brackets



# Test scores and student teacher ratios

	1	2	3
$\beta_0$	698.93*** (9.47) [10.46]	638.73*** (7.45) [7.37]	640.32*** (5.77) [6.06]
<i>student teacher</i>	-2.28*** (0.48) [0.52]	-0.65* (0.35) [0.36]	-0.07 (0.28) [0.29]
Income		1.84*** (0.09) [0.12]	1.49*** (0.07) [0.10]
English learners			-0.49*** (0.03) [0.03]
Num. obs.	420	420	420

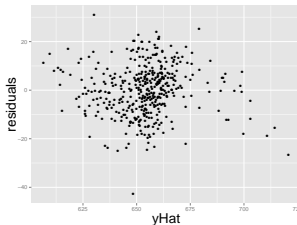
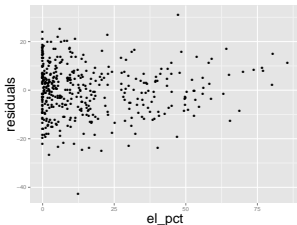
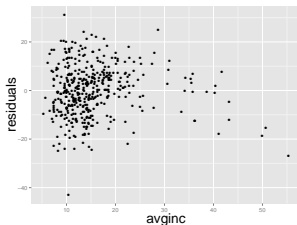
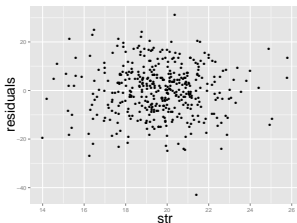
## Section 5

# Detecting heteroskedasticity

## Detecting heteroskedasticity

- Generally best to assume heteroskedasticity and use heteroskedasticity consistent standard errors
- If you have homoskedasticity, but use heteroskedastic standard errors, the heteroskedasticity consistent standard error converges to the homoskedasticity-only standard error, so in large samples it will make no difference
- But if you assume homoskedastic when there is heteroskedasticity, your standard errors will be inconsistent, and your hypothesis tests and confidence intervals will be invalid
- Nonetheless occasionally might want to check for heteroskedasticity
  - Visually: plot  $\hat{\epsilon}_i$  against  $x_{j,i}$  and/or  $\hat{y}_i$
  - Formally: can test  $H_0$  : homoskedasticity, see **Wooldridge (2013)** chapter 8 for details

# Example: Test scores and student teacher ratios



## Section 6

# Heteroskedasticity and efficiency

# Heteroskedasticity and efficiency

- The Gauss-Markov theorem requires homoskedasticity
- When there is heteroskedasticity, OLS is *not* the best linear unbiased estimator
- What is the best linear unbiased estimator with heteroskedasticity?
- Model:  $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$
- Assume MLR.1-4
- Suppose we know  $\text{Var}(\epsilon_i|x_i) = \sigma^2 h(x_i)$  for some known function  $h$
- Consider dividing each observation by  $\sqrt{h(x_i)}$

$$\frac{y_i}{\sqrt{h(x_i)}} = \beta_0 \frac{1}{\sqrt{h(x_i)}} + \beta_1 \frac{x_i}{\sqrt{h(x_i)}} + \frac{\epsilon_i}{\sqrt{h(x_i)}}$$
$$y_i^* = \beta_0 x_{0,i}^* + \beta_1 x_{1,i}^* + \epsilon_i^* \quad (1)$$

# Heteroskedasticity and efficiency

- Then
  - Assumptions MLR.1 (linear model), MLR.2 (independent observations), MLR.3 (rank condition), and MLR.4  $E[\epsilon_i^* | x_i^*] = 0$  hold for (1)
  - (1) is homoskedastic

$$\begin{aligned}\text{Var}(\epsilon_i^* | x_i^*) &= \text{Var} \left( \frac{\epsilon_i}{\sqrt{h(x_i)}} \middle| x_i \right) \\ &= \frac{1}{h(x_i)} h(x_i) \sigma^2 = \sigma^2\end{aligned}$$

- So OLS of (1) is BLUE
- This is called **weighted least squares** (WLS)
- If do not know  $h(x_i)$  but can estimate it this is called **feasible generalized least squares** (FGLS)

# Heteroskedasticity and efficiency

- In most empirical work do not know  $h(x_i)$  and often cannot estimate it well, so usually just use OLS



## Section 7

# Standard errors for dependent data

# Standard errors for dependent data

- Assume:

**MLR.1** (linear model)

**MLR.2 (not too dependent)**  $\{(x_{j,i}, y_i)\}_{i=1}^n$  is not too dependent

**MLR.3** (rank condition) no multicollinearity: no  $x_{j,i}$  is constant and there is no exact linear relationship among the  $x_{j,i}$

**MLR.4** (strict exogeneity)  $E[\epsilon_i | \{x_{1,j}, \dots, x_{k,j}\}_{j=1}^n] = 0$

- As with heteroskedasticity, when observations are not independent (and not too dependent):
  - OLS remains unbiased and consistent
  - Usual homoskedastic-only or heteroskedasticity robust standard errors are incorrect
  - If you do not correct standard errors, hypothesis tests and confidence intervals are invalid
  - OLS is not BLUE

# Standard errors for dependent data

- To correct standard errors for dependent data, we need to know something about the form of dependence
- Common forms of dependence:
  - **Clustering**: observations can be organized into clusters (groups); pairs of observations in the same cluster are dependent, pairs of observations in different clusters are independent
  - **Autocorrelation** (or serial correlation): observations are taken over time;  $y_t$  is correlated with  $y_{t-k}$
- Can correct standard errors for clustering and autocorrelation
- Clustered standard errors and autocorrelation consistent standard errors are often much larger than standard errors that assume independence

# Clustering

- **Clustering:** observations can be organized into clusters (groups); pairs of observations in the same cluster are dependent, pairs of observations in different clusters are independent

## Clustering - examples

- **Blimpo (2014)** (from 2015 midterm)
  - Observations of students test scores, about 1500 students from 100 schools
  - Students' test scores in the same school unlikely to be independent, but reasonable to think students in different schools are independent
- **Rodrik (2013)** (from the 2013 midterm)

$$\Delta v_{ij} = \beta_u \log v_{ij} + \delta_0 + \delta_1 \text{Industry}_{1,ij} + \dots + \\ + \dots + \delta_{J-1} \text{Industry}_{J-1,ij} + \epsilon_{ij}$$

- Data on 20 industries from 118 countries
- $\log v_{ij}$  = log labor productivity in manufacturing industry  $i$  in country  $j$  ten years in the past
- $\Delta v_{ij}$  be the annual growth rate of labor productivity in industry  $i$  in country  $j$  over the past ten years

# Clustering - examples

- Unrealistic to assume to assume that industries within the same country are independent, so  $E[\epsilon_{ij}\epsilon_{ik}] \neq 0$
- More reasonable to assume that observations in different countries are independent

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + \epsilon_{ij}$$

- $J$  clusters,  $n(j)$  observations in each cluster
- Assume observations in different clusters are independent (observations in the same cluster can be arbitrarily dependent)

- Then

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{\sum_{j=1}^J \left( \sum_{i=1}^{n(j)} (x_{ij} - \bar{x}) \hat{\epsilon}_{ij} \right)^2}{\frac{1}{n} \left( \sum_{i,j} (x_{ij} - \bar{x})^2 \right)^2}}} \xrightarrow{d} N(0, 1)$$

as  $J \rightarrow \infty$

- Can use these standard errors and perform  $t$  and  $F$  tests as usual

# Clustering

- These standard errors allow heteroskedasticity and clustering
- This is an asymptotic result for a large number of clusters, many empirical applications have relatively few clusters, there is no consensus on the best thing to do when you have few clusters, but there are number of papers about it, see [Angrist and Pischke \(2009\)](#) chapter 8
- For clustered standard errors in R use the `felm` from the `lfe` package



## Autocorrelation

- Observations over time

$$y_t = \beta_0 + \beta_1 x_t + \epsilon_t$$

- Unlikely that  $\epsilon_t$  independent of  $\epsilon_{t-k}$
- Need to correct standard error of  $\hat{\beta}$
- Correct formula called **Newey-West** or **autocorrelation consistent** or **HAC** (heteroskedasticity and autocorrelation consistent) standard errors

$$\widehat{\text{se}}(\hat{\beta}_1) = \frac{\sum_{\ell=-(T^{1/3})}^{T^{1/3}} \left[ \left(1 - \frac{|\ell|}{T^{1/3}}\right) \frac{1}{T} \sum_{t=1}^T (x_t - \bar{x}) \hat{\epsilon}_t (x_{t+\ell} - \bar{x}) \hat{\epsilon}_{t+\ell} \right]}{\left( \sum_{t=1}^T (x_t - \bar{x})^2 \right)^2}$$

- See **Wooldridge (2013)** section 12.5 for details, or **Mikusheva and Schrimpf (2007)** lectures 2 & 3, & recitation 2 for more discussion

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