Sets, Numbers, and Proofs

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Sets

- A **set** is any well-specified collection of elements
- Notation and examples:

$$a \in A \qquad A = \{4, 5, 6\}$$
$$A = \{n \in \mathbb{N} : 3 < n < 7\} \qquad A_k = \{n \in \mathbb{N} : n > k\}$$
$$B = \{A_1, A_6, A_7\} \qquad \emptyset$$

Example

[Sample space] In a random experiment, the set of all possible outcomes is called the **sample space**. E.g. for the roll of a dice, the sample space if $\{1, 2, 3, 4, 5, 6\}$. An **event** is any subset of the sample space.

Example

[Games] A game is a model of strategic decision making. A game consists of a finite set of n players, say $N = \{1, 2, ..., n\}$. Each player $i \in N$ chooses an action a_i from a set of actions A_i. The outcome of the game depends on the actions chosen by all players.

Example

[Consumption set] The **consumption set** is the set of all feasible consumption bundles. Suppose there are n commodities. A consumer chooses a consumption bundle $\mathbf{x} = (x_1, x_2, ..., x_n)$. Consumption cannot be negative, so the consumption set is a subset of $\mathbb{R}^n_+ = \{(x_1, ..., x_n) : x_1 \geq 0, x_2 \geq 0, ...x_n \geq 0\}$.

Set operations and comparisons

- **Union:** $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- **Intersect:** $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- **Minus:** $A \setminus B = \{x : x \in A \text{ and } \notin B\}$
- **Complement:** $A^c = \{x \in U \setminus A\}$
- **Subset:** $A \subseteq B$, $A \subset B$.

Sets and logic

- **Predicate**: is a function, $p(x)$, from some set, X, to $\{T, F\}$
- Truth set of $p(x)$ is $\{x \in X : p(x) = T\}$

Theorem (Equivalence of truth sets and predicates)

Let $p(x)$ and $q(x)$ be predicates and P and Q be the associated truth sets. Then if $\tilde{p}(x) = \begin{cases} T \text{ if } x \in P \\ F \text{ if } x \notin P \end{cases}$, *we have* $p(x) = \tilde{p}(x) \forall x \in X$. *Also,*

1. $p(x) \wedge q(x)$ *iff* $x \in P \cap Q$
2. $p(x) \vee q(x)$ *iff* $x \in P \cup Q$
3. $\sim p(x)$ *iff* $x \in P^c$
4. $p(x) \Rightarrow q(x)$ *iff* $P \subseteq Q$

Corollary

Let X, Y, and Z be sets contained in some universe U. The following sets from columns A and B are equivalent.

A	B
$(X \cup Y)^c$	$X^c \cap Y^c$
$(X \cap Y)^c$	$X^c \cup Y^c$
$X \cap (Y \cup Z)$	$(X \cap Y) \cup (X \cap Z)$
$X \cup (Y \cap Z)$	$(X \cup Y) \cap (X \cup Z)$

Cardinality

- How can we compare the size of infinite sets?
- Cardinality of set A denoted $|A|$.
- $|A| = |B|$ if $\exists$ one-to-one and onto mapping between A and B
- Cardinality is either
  - Finite
  - Countable $= |\mathbb{N}|$
  - Uncountable $> |\mathbb{N}|$

Lemma

$\mathbb{Z}$ is countable.

Theorem

Every infinite subset of a countable set A is countable.

Theorem

The rational numbers are ??????

Theorem

The real numbers are ??????

Remarks

- Sets bigger than $\mathbb{R}$?
  - Power set of A always has cardinality larger than A
- Sets bigger than $\mathbb{N}$ and smaller than $\mathbb{R}$?
  - Continuum hypothesis

Numbers

- Natural, integer, rational, real: where do they come from? what makes them special?
- Natural: take as given (but could construct from logic or set theory)

- Properties of addition:
  1. Closure if $a, b \in \mathbb{N}$, so is $a + b$
  2. Associative $a + (b + c) = (a + b) + c$.
  3. Identity $\exists 0$ s.t. $a + 0 = a$,
  4. Inverse $\forall a$, $\exists b$ s.t. $a + b = 0$
- Given natural numbers, if we demand addition have (3) and (4), we need integers
- If we also demand multiplication has property (4), we need rationals
- Addition and multiplication are also
  5. Commutative $a + b = b + a$
  6. Distributive $a(b + c) = ab + ac$
- A set with $+$ and $\times$ s.t. (1)-(6) is a field

More properties of rationals

- An **ordered set** is a set, A, and a relation, $<$, such that (i) $\forall a, b \in A$ either $a < b$ or $a = b$ or $a > b$; and (ii) if $a < b$ and $b < c$ then $a < c$.
- An **ordered field** is a field that is an ordered set and addition and multiplication preserve the ordering in that (i) if $b < c$ then $a + b < a + c$ (ii) if $a > 0$ and $b > 0$ then $ab > 0$

The problem with rationals

Theorem

$\sqrt{2} \notin \mathbb{Q}$.

- $s \in S$ is an **upper bound** of A if $s \geq a \forall a \in A$.
- s is a **least upper bound** of A if s is an upper bound of A and if $r < s$, then r is not an upper bound of A.
- S has the **least-upper-bound property** if whenever $A \subset S$ has an upper bound, A has a least upper bound.

Theorem

$\mathbb{Q}$ does not have the least-upper-bound property.

Real numbers

Theorem

There exists an ordered field, $\mathbb{R}$, that has the least upper bound property. $\mathbb{R}$ contains $\mathbb{Q}$. Moreoever, $\mathbb{R}$ is "unique".

Relations

Definition (Relation)

A **relation** on two sets A and B is any subset of $A \times B$, $R \subseteq A \times B$. We usually denote relations by $a \overset{R}{\sim} b$ if $(a, b) \in R$ (where $\overset{R}{\sim}$ could be some other symbol).

Properties of relations

A relation $\overset{R}{\sim}$ on A is

- **reflexive** if $a \overset{R}{\sim} a \; \forall a \in A$,
- **symmetric** if $a \overset{R}{\sim} b$ implies $b \overset{R}{\sim} a$,
- **transitive** if $a \overset{R}{\sim} b$ and $b \overset{R}{\sim} c$ implies $a \overset{R}{\sim} c$,
- **antisymmetric** if $a \overset{R}{\sim} b$ and $b \overset{R}{\sim} a$ implies $a = b$,
- **asymmetric** if $a \overset{R}{\sim} b$ implies b is not $\overset{R}{\sim} a$, and
- **complete** if either $a \overset{R}{\sim} b$ or $b \overset{R}{\sim} a$ or both $\forall a, b \in A$.

Example (Preference relation)

A consumer's preference relation, $\succeq$, is a relation on her consumption set, X. $x \succeq y$ means that the consumer likes the bundle of goods x at least as much of the bundle of goods y. We usually assume that preference relations are complete and transitive.

Equivalence

- **equivalence relation** $=$ complete,transitive, & symmetric
- **equivalence class** of x is $\sim(x) = \{a \in X : a \sim x\}$

Example (Indifference)

- Take $\succeq$, a preference relation on X
- Indifference relation $x \sim y$ if $x \succeq y$ and $y \succeq x$
- Indifference classes $=$ indifference curves