Systems of linear equations

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More cardinality

- Let A and B be sets, the **Cartesion product** of A and B is
$$A \times B := \{(a,b) : a \in A, b \in B\}$$

Question

If A and B are countable is $A \times B$ countable?

More cardinality

Theorem

If A and B are countable, then so is $A \times B$.

Proof.

- A and B are countable, so we can write $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$.
- Consider

$$\begin{pmatrix} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots \\ (a_2, b_1) & (a_2, b_2) & (a2, b_3) & \cdots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

- Count along the diagonals

□

More cardinality

Question

If $A_1, A_2, A_3, ...$ are each countable is $\cup_{n=1}^{\infty} A_n$ countable?

More cardinality

Theorem

If $\{A_n\}$ are countable, then so is $\cup_{n=1}^{\infty} A_n$.

Proof.

- A_n is countable, so we can write $A_n = \{a_{1n}, a_{2n}, ...\}$
- Consider

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

- Count along the diagonals

□

Systems of linear equations

- Example:

$$
\begin{aligned}
5x_1 - 7x_2 =& 9 \\
-8x_1 + x_2 =& 0.
\end{aligned}
$$

- In general:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n =& b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n =& b_2 \\
\vdots& \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n =& b_m,
\end{aligned}
$$

Coefficient matrix

- Coefficient matrix A

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

- Augmented coefficient matrix $\hat{A}$

$$\hat{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix} = (A\mathbf{b})$$

Example: Markov model of employment

- Let $s_t = \begin{cases} 1 \text{ if employed at time } t \\ 0 \text{ if unemployed at time } t \end{cases}$
- Random process described by $P(s_t | s_{t-1}, s_{t-2}, ...)$
- **Markovian**: $P(s_t | s_{t-1}, s_{t-2}, ...) = P(s_t | s_{t-1})$
    - Probability of being employed tomorrow only depends on whether you're employed today and not the more distant past
- **Stationary distribution**: q stationary if when $P(s_t = 1) = q(1)$ and $P(s_t = 0) = q(0)$ today, then also have $P(s_{t+1} = 1) = q(1)$ and $P(s_{t+1} = 0) = q(0)$ tomorrow

$$q(s) = \sum_{s_0 \in \{0,1\}} P(s|s_0) q(s_0)$$

Example: Markov model of employment

- Stationary distribution satisfies:

$$
\begin{aligned}
q(1) =& P(1|1)q(1) + P(1|0)q(0) \\
q(0) =& P(0|1)q(1) + P(0|0)q(0) \\
1 =& q(1) + q(0)
\end{aligned}
$$

  system of linear equations for unknowns $q(1)$ and $q(2)$
- Coefficient matrix and augmented coefficient matrix $=$?

Questions to be answered:

Given a system of linear equations:

1. Does any solution exist?
2. How many solutions?
3. How can a solution be computed?

Equation/row operations

- Familiar with solving equations by:
  - Substitution
  - Elimination
- Use **equation (row) operations** that preserve set of solutions
  1. Multiply an equation by a non-zero constant,
  2. Add a multiple of one equation to another, and
  3. Interchange two equations.

Row echelon form

- **Row echelon form**: each row begins with more zeros than the row above it or the row is all zeros
- Examples:

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \end{pmatrix}$$

- Once a system is in row echelon form it is easy to solve by back substitution

Gaussian elimination

- Systematic way to transform system of equations to row echelon form
  1. Identify the first column to contain any non-zero elements, call this column c^*.
  2. Interchange rows so that a nonzero entry appears at the top of column c^*.
  3. Add a multiple of the first row to each of the rows below so that the entries in column c^* below the first row are zero.
  4. Repeat 1-2 on the submatrix consisting of the lower right part of the original matrix below the first row and to the right of column c^*. Stop if this submatrix has no columns or has no rows.

Gaussian elimination: example

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 3x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & 7x_3 & = & 9 \\ 3x_1 & + & 5x_2 & + & 13x_3 & = & 1 \end{array}$$

Transform the following system into row echelon form:

$$\begin{aligned} x + 2y - z =& 2 \\ 4y + z =& 5 \\ -2x - 4y + 2z =& 1. \end{aligned}$$

Existence of solutions

- **rank** of A is the number of nonzero rows in its row echelon form
- Is rank well-defined?

Lemma

The rank of a matrix A is always less than or equal to the number of columns of A and less than or equal to the number of rows of A.

Lemma

Let A be a coefficient matrix and $\hat{A}$ be an augmented coefficient matrix. Then $\mathrm{rank} A \leq \mathrm{rank} \hat{A}$.

Existence of solutions

Theorem (Existence of solutions)

A system of linear equations with coefficient matrix A and augmented coefficient matrix $\hat{A}$ has a solution (perhaps more than one) if and only if $\mathrm{rank} A = \mathrm{rank} \hat{A}$.

Proof.

See notes. □

Consider the system:

$$\begin{aligned} 4y + z &= 5 \\ x + 2y - z &= 2 \\ -8y - 2z &= -10. \end{aligned}$$

- What if $b_3 \neq -10$?

Multiple solutions means infinite solutions

Lemma

Suppose $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are two distinct solutions to the system of equations* $A\mathbf{x} = \mathbf{b}$*. Then the system of equations has (uncountably) infinitely many solutions.*

Solution existence for any b

Theorem (Solution existence)

A system of linear equations with coefficient matrix A will have a solution for any choice of $b_1, ..., b_m$ if and only if $\text{rank}A$ is equal to the number of rows of A.

Corollary

*For any system of equations with more equations than variables, (i.e. an overdetermined system) there exists a choice of **b** such that no solutions exist.*

Uniqueness

Theorem (Solution uniqueness)

Any system of equations with coefficient matrix A has at most one solution for any $b_1, ..., b_m$ if and only if $\mathrm{rank} A$ equals the number of columns of A.

Corollary

If $\mathrm{rank} A$ is less than the number of columns of A then either no solutions exists or multiple solutions exists.

Corollary

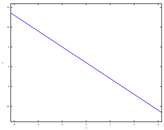
A is nonsingular (always has a unique solution) if and only if A has an equal number of columns and rows (A is square) and has rank equal to its number of columns (or rows).

Summary

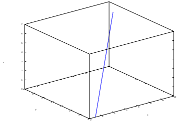
	$\text{rank}A = \text{rank}\hat{A}$		$\text{rank}A < \text{rank}\hat{A}$
	$\text{rank}A = \text{columns}(A)$	$\text{rank}A < n$	
0			X
1	X		
∞		X	

Sets of solutions

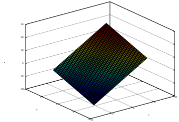
$$-2x + y = 1$$



$$\begin{aligned} x - y - z = & \quad 0 \\ -2x + y + 3z = & \quad 1 \end{aligned}$$



$$-2x - y + z = 2$$



Definition

The set $S \subseteq \mathbb{R}^n$ is called a **linear subspace** if it is closed under (i) scalar multiplication and (ii) addition in other words, if

(i) for every $(x_1, ..., x_n) \in S$ and $a \in \mathbb{R}$, we have $(ax_1, ax_2, ..., ax_n) \in S$, and

(ii) for every $(x_1, ..., x_n) \in S$ and $(y_1, ..., y_n) \in S$, we have $(x_1 + y_1, ..., x_n + y_n) \in S$

Definition

A set of vectors in $\mathbb{R}^n$, $\{\mathbf{x}_j = (x_1^j, ..., x_n^j)\}_{j=1}^J$, is **linearly independent** if the only solution to

$$\sum_{j=1}^{J} c_j \mathbf{x}_j = 0$$

is $c_1 = c_2 = ... = c_J = 0$.

Definition

The dimension of a linear subspace $S \subseteq \mathbb{R}^n$ is the cardinality of the largest set of linearly independent elements in S.

Theorem (Rouché-Capelli)

A system of linear equations with n variables has a solution if and only if the rank of its coefficient matrix, A, is equal to the rank of its augmented matrix, $\hat{A}$. If a solution exists and $\mathrm{rank} A$ is equal to its number of columns, the solution is unique. If a solution exists and $\mathrm{rank} A$ is less than its number of columns, there are infinite solutions. In this case the set of solutions is of the form[1]

$$\{s + x^* \in \mathbb{R}^n : s \in S \text{ and } Ax^* = b\}$$

where S is the linear subspace of dimension $n - \mathrm{rank} A$ defined by $S = \{s \in \mathbb{R}^n : As = 0\}$ and x^* is any single solution to $Ax = b$.

[1] A set of this form is called an affine subspace. It is a linear subspace that has been shifted so that it no longer necessarily contains the origin.

Example: Markov model of employment

- Employment $s_t \in \{u, e\}$.
- Markovian: $\mathrm{P}(s_t | s_{t-1}, s_{t-2}, ...) = \mathrm{P}(s_t | s_{t-1})$
- Stationary distribution: (π_e, π_u) such that if $s_t = e$ with probability π_e and u with probability π_u, then s_{t+1} has same distribution, i.e.

$$\begin{aligned} \mathrm{P}(e|e)\pi_e + \mathrm{P}(e|u)\pi_u =& \pi_e \\ \mathrm{P}(u|e)\pi_e + \mathrm{P}(u|u)\pi_u =& \pi_u \end{aligned}$$

Example: Markov model of employment

- Stationary distribution is a solution to:

$$\begin{aligned}(p_{ee} - 1)\pi_e + p_{eu}\pi_u =& 0 \\ p_{ue}\pi_e + (p_{uu} - 1)\pi_e =& 0 \\ \pi_e + \pi_u =& 1.\end{aligned}$$

- Questions:
  1. Does any solution exist?
  2. How many solutions exist?
  3. How can a solution be computed?