Matrix algebra and introduction to vector spaces

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Correction from last time

Theorem (Rouché-Capelli)

A system of linear equations with n variables has a solution if and only if the rank of its coefficient matrix, A, is equal to the rank of its augmented matrix, $\hat{A}$. If a solution exists and $\text{rank}A$ is equal to its number of columns, the solution is unique. If a solution exists and $\text{rank}A$ is less than its number of columns

$$\{s + x^* \in \mathbb{R}^n : s \in S \text{ and } Ax^* = b\}$$

where S is a linear subspace of dimension $n - \text{rank}A$ given by the set of solutions to $Ax = 0$, and x^* is a solution to $Ax = b$.

Overview

Definition

A **vector space** is a set V and a field $\mathbb{F}$ with two operations, addition $+$, which takes two elements of V and produces another element in V, and scalar multiplication $\cdot$, which takes an element in V and an element in $\mathbb{F}$ and produces an element in V, such that

1. $(V, +)$ is a commutative group, i.e. addition is close, associative, invertible, and commutative.
2. Scalar multiplication has the following properties:
   1. Closure: $\forall v \in V$ and $f \in \mathbb{F}$ we have $vf \in V$
   2. Distributivity: $\forall v_1, v_2 \in V$ and $f_1, f_2 \in \mathbb{F}$

      $$f_1(v_1 + v_2) = f_1 v_1 + f_1 v_2$$

      and

      $$(f_1 + f_2)v_1 = f_1 v_1 + f_2 v_1$$

   3. Consistent with field multiplication: $\forall v \in V$ and $f_1, f_2 \in V$ we have

Example (Euclidean space)

$\mathbb{R}^n$ over the field $\mathbb{R}$ is a vector space. Vector addition and multiplication are defined in the usual way. If $\mathbf{x}_1 = (x_{11}, ..., x_{n1})$ and $\mathbf{x}_2 = (x_{12}, ..., x_{n2})$, then

$$\mathbf{x}_1 + \mathbf{x}_2 = (x_{11} + x_{12}, ..., x_{n1} + x_{n2}).$$

Scalar multiplication is defined as

$$a\mathbf{x} = (ax_1, ..., ax_n)$$

for $a \in \mathbb{R}$ and $\mathbf{x} \in R^n$.

Example

Any linear subspace of $\mathbb{R}^n$.

Example

$(\mathbb{Q}^n, \mathbb{Q}, +, \cdot)$ is a vector space where $+$ and $\cdot$ defined as in 3.

Example

$(\mathbb{C}^n, \mathbb{C}, +, \cdot)$ where $+$ and $\cdot$ defined as in 3 except with complex addition and multiplication taking the place of real addition and multiplication.

Example

Take $V = \mathbb{R}^+$. Define "addition" as $x \oplus y = xy$ and define "scalar multiplication" as $\alpha \odot x = x^\alpha$. Then $(\mathbb{R}^+, \mathbb{R}, \oplus, \odot)$ is a vector space with identity element 1.

Vector spaces of functions

Example

Let $V =$ all functions from $[0, 1]$ to $\mathbb{R}$. For $f, g \in V$, define $f + g$ by $(f + g)(x) = f(x) + g(x)$. Define scalar multiplication as $(\alpha f)(x) = \alpha f(x)$. Then this is a vector space.

Example

The set of all continuous functions with addition and scalar multiplication defined as in 8.

Example

The set of all k times continuously differentiable functions with addition and scalar multiplication defined as in 8.

Example

The set of all polynomials with addition and scalar multiplication defined as in 8.

Example

The set of all polynomials of degree at most d with addition and scalar multiplication defined as in 8.

Example

The set of all functions from $\mathbb{R} \to \mathbb{R}$ such that $f(29481763) = 0$ with addition and scalar multiplication defined as in 8.

Example

Let $1 \leq p < \infty$ and let $\mathcal{L}^p(0,1)$ be the set of functions from $(0,1)$ to $\mathbb{R}$ such that $\int_0^1 |f(x)|^p dx$ is finite. Then $\mathcal{L}^p(0,1)$ with the field $\mathbb{R}$ and addition and scalar multiplication defined as

$$
\begin{aligned}
(f+g)(x) =& f(x) + g(x) \\
(\alpha f)(x) =& \alpha f(x)
\end{aligned}
$$

is a vector space.

Definition

Let V be a vector space and $v_1, ..., v_k \in V$. A linear combination of $v_1, ..., v_k$ is any vector

$$c_1 v_1 + ... + c_k v_k$$

where $c_1, ..., c_k \in \mathbb{F}$.

Question

How can we be sure that $c_1 v_1 + ... + c_k v_k \in V$?

Definition

Let V be a vector space and $W \subseteq V$. The **span** of W is the set of all finite linear combinations of elements of W.

Lemma

*The **span** of any $W \subseteq V$ is a linear subspace.*

Example

Let V be the vector space of all functions from $[0, 1]$ to $\mathbb{R}$ as in example 8. The span of $\{1, x, ..., x^n\}$ is the set of all polynomials of degree less than or equal n.

Definition

A set of vectors $v_1, ..., v_k \in V$, is **linearly independent** if the only solution to

$$\sum_{j=1}^{k} c_j v_j = 0$$

is $c_1 = c_2 = ... = c_k = 0$.

Definition

The **dimension** of a vector space, V, is the cardinality of the largest set of linearly independent elements in V.

Definition

A **basis** of a vector space V is any set of linearly independent vectors $b_1, ..., b_k$ such that the span of $b_1, ..., b_k$ is V.

Example

A basis for $\mathbb{R}^n$ is $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, ..., 0, 1)$. This basis is called the standard basis of $\mathbb{R}^n$.

Example

What is the dimension of each of the examples of vector spaces above? Can you find a basis for them?

Basis gives coordinates

Lemma

Let $\{b_1, ..., b_k\}$ be a basis for a vector space V. Then $\forall v \in V$ there exists a unique $v_1, ..., v_k \in \mathbb{F}$ and such that $v = \sum_{i=1}^k v_i b_i$

Proof.

- B spans V, so such $(v_1, ..., v_k)$ exist.
- Suppose there exists another such $(v_1', ..., v_k')$. Then

$$
\begin{aligned}
v = \sum v_i b_i &= \sum v_i' b_i \\
\sum v_i b_i - \sum v_i' b_i &= 0 \\
\sum (v_i - v_i)' b_i &= 0.
\end{aligned}
$$

□

Dimension $= |$Basis$|$

Lemma

If B is a basis for a vector space V and $I \subseteq V$ is a set of linearly independent elements then $|I| \leq |B|$.

Corollary

Any two bases for a vector space have the same cardinality.

Definition

Let V and W be vector spaces over the field $\mathbb{F}$. V and W are **isomorphic** if there exists a one-to-one and onto function, $I : V \to W$ such that

$$I(v^1 + v^2) = I(v^1) + I(v^2)$$

for all $v^1, v^2 \in V$, and

$$I(\alpha v) = \alpha I(v)$$

for all $v \in V$, $\alpha \in \mathbb{F}$. Such an I is called an **isomorphism**.

$\mathbb{R}^n$ is the "unique" n-dimensional vector space

Theorem

Let V be an n-dimensional vector space over the field $\mathbb{F}$. Then V is isomorphic to $\mathbb{F}^n$.

Definition

A **linear transformation** (aka linear function) is a function, A, from a vector space $(V, \mathbb{F}, +, \cdot)$ to a vector space $(W, \mathbb{F}, +, \cdot)$ such that $\forall v_1, v_2 \in V$,

$$A(v_1 + v_2) = Av_1 + Av_2$$

and

$$A(fv_1) = fAv_1$$

for all scalars f.

- Linear transformation from $V \to V$ is called a **linear operator**
- Linear transformation from $V \to \mathbb{R}$ is called a **linear functional**

Examples

Example

Any isomorphism

Example

The identity operator: $I : V \to V$ defined by $I(v) = v$

Example

The zero transformation: $0_T : V \to W$ defined by $0_T(v) = 0_W$

Example

$f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f((x_1, x_2)) = x_1$

Theorem

For any linear transformation, A, from $\mathbb{R}^n$ to $\mathbb{R}^m$ there is an associated m by n matrix,

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} is defined by $Ae_j = \sum_{i=1}^m a_{ij} e_i$. Conversely, for any m by n matrix, there is an associated linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $Ae_j = \sum_{i=1}^n a_{ij}$.

Proof.

- Let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$
- $b_1, b_2, .., b_n$ basis for $\mathbb{R}^n$
- $\forall v \in V \; \exists \alpha_j \in \mathbb{R}$ s.t. $v = \sum_{j=1}^n \alpha_j b_j$
- $Av = \sum_{j=1}^n \alpha_j A b_j$ so only need Ab_j to determine A
- $d_1, ..., d_m$ basis for $\mathbb{R}^m$, so

$$Ab_j = \sum_{i=1}^m a_{ij} d_i .$$

$\square$

Other examples of linear transformations

Example (Integral operator)

Let $k(x, y)$ be a function from $(0, 1)$ to $(0, 1)$ such that $\int_0^1 \int_0^1 k(x, y)^2 dxdy$ is finite. Define $K : \mathcal{L}^2(0, 1) \to \mathcal{L}^2(0, 1)$ by

$$(Kf)(x) = \int_0^1 k(x, y)f(y)dy$$

Then K is a linear transformation.

Other examples of linear transformations

Example (Conditional expectation)

X and Y are real valued random variables with joint pdf $f_{xy}(x,y)$ and marginal pdfs $f_x(x) = \int_{\mathbb{R}} f(x,y)dy$ and $f_y(y) = \int_{\mathbb{R}} f(x,y)dx$. Define the vector spaces

$$V = \mathcal{L}^2(\mathbb{R}, f_y) = \{g : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{\mathbb{R}} f_y(y)g(y)^2 dy < \infty\}$$

and

$$W = \mathcal{L}^2(\mathbb{R}, f_x) = \{g : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{\mathbb{R}} f_x(x)g(x)^2 dx < \infty\}$$

The conditional expectation function is $\mathcal{E} : V \to W$ defined by

$$(\mathcal{E}g)(x) = E[g(Y)|X = x] = \int_{\mathbb{R}} \frac{f_{xy}(x,y)}{f_x(x)f_y(y)} g(y)f_y(y)dy.$$

Other examples of linear transformations

Example (Differential operator)

Let $C^\infty(0,1)$ be the set of all infinitely differentiable functions from $(0,1)$ to $\mathbb{R}$. It can easily be shown that $C^\infty(0,1)$ is a vector space. Let $D : C^\infty(0,1) \to C^\infty(0,1)$ be defined by

$$(Df)(x) = \frac{df}{dx}(x)$$

Then D is a linear transformation.

Addition

- $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$
- Linear transformation implies $(A+B)x = Ax + Bx$

$$\begin{aligned}(A+B)e_i =& Ae_i + Be_i \\ =& \sum_{j=1}^n a_{ij}e_j + \sum_{j=1}^n b_{ij}e_j \\ =& \sum_{j=1}^n (a_{ij}+b_{ij})e_j,\end{aligned}$$

- so $A+B = \begin{pmatrix} a+b_{11} & \cdots & a+b_{1n} \\ \vdots & \ddots & \vdots \\ a+b_{m1} & \cdots & a+b_{mn} \end{pmatrix}$.

Addition properties

1. Associative: $A + (B + C) = (A + B) + C$,
2. Commutative: $A + B = B + A$,
3. Identity: $A + \mathbf{0} = A$, where $\mathbf{0}$ is an m by n matrix of zeros, and
4. Invertible $A + (-A) = \mathbf{0}$ where

$$-A = \begin{pmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{m1} & \cdots & -a_{mn} \end{pmatrix}.$$

Scalar multiplication

- Linear transformation requires $A\alpha x = \alpha Ax$
- so,

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & \ddots & \vdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{pmatrix}$$

The space of matrices is a vector space

- $L(\mathbb{R}^n, \mathbb{R}^m) \equiv$ all m by n matrices $\equiv$ all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ with addition and multiplication as above is a vector space
  - Question: $L(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to what other vector space that we have seen?
- $L(V, W) \equiv$ all linear transformations from $V \to W$ is a vector space

Matrix multiplication

- Multiplication $\equiv$ composition of linear transformations
- $A : \mathbb{R}^n \to \mathbb{R}^m$, $B : \mathbb{R}^p \to \mathbb{R}^n$.
- Consider $A(Be_k)$

$$
\begin{aligned}
A(Be_k) =& A(\sum_{j=1}^n b_{jk} e_i) \\
=& \sum_{j=1}^n b_{jk} A e_i \\
=& \sum_{j=1}^n b_{jk} \left(\sum_{l=1}^m a_{lj} e_l \right) \\
=& \sum_{l=1}^m \left(\sum_{j=1}^n a_{lj} b_{jk} \right) e_l \\
=& \begin{pmatrix} \sum_{j=1}^n a_{1j} b_{j1} & \cdots & \sum_{j=1}^n a_{1j} b_{jp} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^n a_{mj} b_{j1} & \cdots & \sum_{j=1}^n a_{mj} b_{jp} \end{pmatrix} e_l \\
=& (AB) e_l.
\end{aligned}
$$

Multiplication properties

1. Associative: $A(BC) = (AB)C$
2. Distributive: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
3. Identity: $AI_n = A$ where A is m by n and I_n is the identity linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $I_n x = x \forall x \in \mathbb{R}^n$
4. Not commutative

Definition

A real **inner product space** is a vector space over the field $\mathbb{R}$ with an additional operation called the inner product that is function from $V \times V$ to $\mathbb{R}$. We denote the inner product of $v_1, v_2 \in V$ by $\langle v_1, v_2 \rangle$. It has the following properties:

1. Symmetry: $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$
2. Linear: $\langle av_1 + bv_2, v_3 \rangle = a \langle v_1, v_3 \rangle + b \langle v_2, v_3 \rangle$ for $a, b \in \mathbb{R}$
3. Positive definite: $\langle v, v \rangle \geq 0$ and equals 0 iff $v = 0$.

Example

$\mathbb{R}^n$ with the **dot product**, $xcdoty = \sum_{i=1}^n x_i y_i$, is an inner product space.

Example

$\mathcal{L}^2(0,1)$ with $\langle f, g \rangle \equiv \int_0^1 f(x) g(x) dx$ is an inner product space.

Transpose

Definition

Given a linear transformation, A, from a real inner product space V to a real inner product space W, the **transpose** of A, denoted A^T (or often A') is a linear transformation from W to V such that $\forall v \in V, w \in W$

$$\langle Av, w \rangle = \left\langle v, A^T w \right\rangle .$$

Transpose for matrices

- $$\begin{aligned} \langle Av, w \rangle &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} v_j \right) w_i \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} w_i v_j \end{aligned}$$
- $$\begin{aligned} \left\langle v, A^T w \right\rangle &= \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{m} a^T_{ji} w_i \right) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a^T_{ji} w_i v_j \end{aligned}$$
- If $\langle Av, w \rangle = \langle v, A^T w \rangle$, for any v and w we must have $a^T_{ji} = a_{ij}$
- The transpose of a matrix simply swaps rows for

Transpose properties

1. $(A+B)^T = A^T + B^T$
2. $(A^T)^T = A$
3. $(\alpha A)^T = \alpha A^T$
4. $(AB)^T = B^T A^T$.
5. $\text{rank} A = \text{rank} A^T$

Transpose and dual space

Definition

Let V be a vector space. The **dual space** of V, denote V^* is the set of all (continuous) linear functionals, $v^* : V \to \mathbb{R}$.

Example

The dual space of $\mathbb{R}^n$ is the set of $1 \times n$ matrices. In fact, for any finite dimensional vector space, the dual space is the set of row vectors from that space.

Example

Let $1 \leq p \leq \infty$, define

$$\ell^p = \{(x_1, x_2, ...) : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

and

$$\ell^\infty = \{(x_1, x_2, ...) : \max_{i \in \mathbb{N}} |x_i| < \infty\}$$

What is the dual space of ℓ_∞?

Example

Dual space of $V = \mathcal{L}^2(\mathbb{R}, f_x) = \{g : \mathbb{R} \to \mathbb{R} \text{ such that } \int_{\mathbb{R}} f_x(x) g(x)^2 dx < \infty\}$?

- Let $h \in \mathcal{L}^2(\mathbb{R}, f_x)$, define

$$h^*(g) = \int_{\mathbb{R}} f_x(x) g(x) h(x) dx.$$

  then if $h^*(g)$ is finite for all g, $h^* \in V^*$
- Can show h^* is finite for g, h & $V^* = \{h^* : h \in V\}$
- The mapping $h \to h^*$ is an isomorphism between V and V^*

Dual space definition of transpose

Definition

If $A : V \to W$ is a linear transformation, then the **transpose** (or dual) of A is $A^T : W^* \to V^*$ defined by $(A^T w^*)v = w^*(Av)$.

- This definition is the same as the previous one when V and W are inner product spaces
  - Show that if V, W are inner product spaces then V^* is isomorphic to V, W^* is isomorphic to W
  - Show definitions are same

Types of matrices

Definition

A **column** matrix is any m by 1 matrix.

Definition

A **row** matrix is any 1 by n matrix.

Definition

A **square** matrix has the same number of rows and columns.

Definition

A **diagonal** matrix is a square matrix with non-zero entries only along its diagonal, i.e. $a_{ij} = 0$ for all $i \neq j$.

Definition

An **upper triangular** matrix is a square matrix that has non-zero entries only on or above its diagonal, i.e. $a_{ij} = 0$ for all $j > i$. A **lower triangular** matrix is the transpose of an upper triangular matrix.

Definition

A matrix is **symmetric** if $A = A^T$.

Definition

A matrix is **idempotent** if $AA = A$.

Definition

A **permutation** matrix is a square matrix of 1's and 0's with exactly one 1 in each row or column.

Definition

A **nonsingular** matrix is a square matrix whose rank equals its number of columns.

Definition

An **orthogonal** matrix is a square matrix such that $A^T A = I$.

Invertibility

Definition

Let A be a linear transformation from V to W. Let B be a linear transfromation from W to V. B is a **right inverse** of A if $AB = I_V$. Let C be a linear tranfromation from V to W. C is a **left inverse** of A if $CA = I_W$.

Lemma

If A is a linear transformation from V to V and B is a right inverse, and C a left inverse, then $B = C$.

Lemma

Let A be a linear tranformation from V to V, and suppose A is invertible. Then A is nonsingular and the unique solution to $Ax = b$ is $x = A^{-1}b$.

Lemma

If A is nonsingular, then A^{-1} exists.

Corollary

A square matrix A is invertible if and only if $\mathrm{rank} A$ is equal to its number of columns.

Properties of matrix inverse

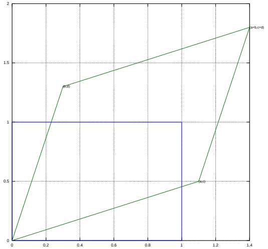
1. $(AB)^{-1} = B^{-1}A^{-1}$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(A^{-1})^{-1} = A$

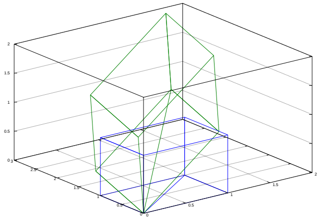
Determinants

- Determinant: geometry and invertibility
- Invert 2 by 2 matrix by Gauss-Jordan elimination:

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{pmatrix}$$
$$\simeq \begin{pmatrix} a & b & 1 & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$
$$\simeq \begin{pmatrix} a & 0 & \frac{ad}{ad-bc} & \frac{-ba}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$
$$\simeq \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc}. \end{pmatrix}$$

- Needed $ad - bc \neq 0$.


(a+b,c+d)
(b,d)
(a,c)
0
0.5
1
1.5
2
0
0.2
0.4
0.6
0.8
1
1.2
1.4



Definition

Let A be an n by n matrix consisting of column vectors $a_1, ..., a_n$. The determinant of A is the unique function such that

1. $\det I_n = 1$.
2. As a function of the columnes, $\det$ is an alternating form: $\det(A) = 0$ iff $a_1, ..., a_n$ are linearly dependent.
3. As a function of the columnes, $\det$ is multi-linear:

$$\det(a_1, ..., ba_j + cv, ..., a_n) = b\det(A) + c\det(a_1, ..., v, ...a_n)$$

- 1 natural, needed for volume interpretation
- 2 ensures $\det A = 0$ iff A singular

Lemma

Let A be an n by n matrix. The A is singular if and only if the columns of A are linearly dependent.

Corollary

A is nonsingular if and only if $\det A \neq 0$.

- 3 is related to volume interpretation
- Consider diagonal matrices, volume interpretation require multi-linearity

Definition

The **determinant** of a square matrix A is defined recursively as

1. For 1 by 1 matrices, $\det A = a_{11}$
2. For n by n matrices,

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{-1,-j}$$

where $A_{-i,-j}$ is the $n-1$ by $n-1$ matrix obtained by deleting the ith row and jth column of A.

- **minor:** $\det A_{-i,-j}$
- **cofactor:** $(-1)^{i+j} \det A_{-i,-j}$

Determinant properties

Theorem

The two definitions of the determinant, (62) and (65), are equivalent.

1. $\det A^T = \det A$
2. $\det(AB) = (\det A)(\det B)$
3. $\det A^{-1} = (\det A)^{-1}$
4. Usually, $\det(A+B) \neq \det A + \det B$
5. If A is diagonal, $\det A = \prod_{i=1}^{n} a_{ii}$
6. If A is upper or lower triangular $\det A = \prod_{i=1}^{n} a_{ii}$.

Theorem

Let A be nonsingular. Then,

1. $A^{-1} = \frac{1}{\det A} \begin{pmatrix} \det A_{-1,-1} & \cdots & (-1)^{1+n} \det A_{-n,-1} \\ \vdots & \ddots & \vdots \\ (-1)^{1+n} \det A_{-1,-n} & \cdots & (-1)^{n+n} \det A_{-n,-n} \end{pmatrix}$
2. *(**Cramer's rule**) The unique solution to $Ax = b$ is*

$$x_i = \frac{\det B_i}{\det A}$$

where B_i is the matrix A with the ith column replaced by b.

Computational efficiency

- Calculate determinant as defined above in $d(n)$ steps

$$d(n) = nd(n-1) + 2n$$
$$= 2n! \sum_{k=1}^{n} \frac{1}{(n-k)!}$$

- Big O notation: $d(n) = O(f(n))$ if $\exists n_0$ such that

$$d(n) \leq Mf(n)$$

  for some constant M and all $n \geq n_0$
- $d(n) = O(n!)$
- Cramer's formula $= O((n+1)!)$

- Gaussian elimination in $g(n)$ steps

$$g(n) = 2\sum_{k=1}^{n} k(k-1)$$
$$= \frac{2}{3}(n^3 - n) = \qquad O(n^3)$$

- Back substitute: $\sum_{k=1}^{n} k = \frac{1}{2}n(n-1)$ step
- Total: $O(n^3)$