APPLICATIONS

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1. PORTFOLIO ANALYSIS

This section presents a basic model of portfolio analysis.[1] We will setup the model, define some special types of portfolios, and then use our results on systems of linear equations and matrices to say things about the existence of these special types of portfolios.

Suppose there are A assets and S states of nature. States of nature are things about which we are uncertain. You could think of the assets as various stocks and the states of nature could be all possible combinations of prices of the stocks tomorrow. For this example, we will assume A and S are finite so that everything can be represented by matrices, but we could allow them to be infinite and use abstract linear transformations instead. Considering an infinite number of assets is a bit of a stretch, but you could imagine bonds that mature at time t, $t+1$, $t+2$, and so on forever. Of course, in the real world we don't see bonds that mature arbitrarily far into the future. On the other hand, allowing for infinite states of nature S is pretty reasonable. In the model a state of nature is anything that affects the value of the assets. Each potential combination of values of the assets is a single state of nature.

We will think about a single period model. At the start of the period, the value of asset $i \in \{1, .., A\}$ is v_i. Then some state of nature, $s \in \{1, ..., S\}$, is drawn and the value of the asset becomes y_{si}. The realized return of asset i in state s is $R_{si} = \frac{y_{si}}{v_i}$. Let $\mathcal{R}$ be the S by A matrix consisting of the R_{si}.

The investor has wealth w_0 is choosing to buy n_i units or shares of asset i. The budget constraint is

$$\sum_{i=1}^{A} n_i v_i = w_0 \tag{1}$$

Let $x_i = \frac{n_i v_i}{w_0}$ be the share of wealth in asset i. We call $(x_1, ..., x_A)$ a portfolio. The return to a portfolio $x = (x_1, ..., x_A)$ in state s is

$$R_s = \sum_{i=1}^{A} \frac{y_{si}}{v_i} x_i = \sum_{i=1}^{A} R_{si} x_i \tag{2}$$

[1] This example is taken from sections 6.2, 7.X and 28.2 of Simon and Blume.

Definition 1.1. A portfolio is **riskless** if $x \neq 0$ and

$$\sum_{i=1}^{A} R_{1i}x_i = \cdots = \sum_{i=1}^{A} R_{Si}x_i$$

or in matrix form,

$$\underbrace{\mathcal{R}}_{S\times A}\underbrace{x}_{A\times 1} = \underbrace{c}_{S\times 1}$$

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A riskless portfolio has the same return in all states of nature.

When does a riskless portfolio exist? A riskless portfolio solves

$$\mathcal{R}x = c$$

where c is an S by 1 vector with all entries equal. We know that this equation has a solution if $c \in \mathrm{Col}(\mathcal{R})$.

If there is a riskless portfolio, then buying that portfolio ensures the same return in all states of nature. A person may not want an entirely riskless portfolio, but could be interested in insuring against risk associated with a particular state. For example, suppose s^* is a state of the world where the demand for economists has collapsed. In this state, economists' labor income will be lower. A risk averse economist would like to insure against this outcome by buying a portfolio that has a large return in state s^*.

Definition 1.2. A state s^* is **insurable** if $\exists$ portfolio x^* such that $\sum_{a=1}^{A} R_{s^*a}x_a^* > 0$ and $\sum_{a=1}^{A} R_{sa}x_a^* = 0$ for all $s \neq s^*$.

If s^* is insurable, then buying portfolio x^* lets a person transfer income from other states to s^*.

When are state insurable? If state s^* is insurable, then there must be a solution to

$$\mathcal{R}x = e_{s^*}$$

As above, state s^* is insurable if and only if $e_{s^*} \in \mathrm{Col}(\mathcal{R})$. If all states are insurable, then $\mathrm{Col}(\mathcal{R}) = \mathbb{R}^S$ and $\mathrm{rank}\mathcal{R} = S$. In particular, if all states are insurable, then there are at least as many assets as states. Also, in that case, we must also have a riskless portfolio. This makes sense, if every state is insurable, we should be able to eliminate all risk.

From the above we saw that if $\mathrm{rank}\mathcal{R} = S$ then there are portfolios that eliminate all risk. For this to be the case there must be at least as many (linearly independent) assets as states. Is there anything to be gained by having more assets than states? Potential realized returns are in $\mathrm{Col}(\mathcal{R})$. If $\mathrm{Col}(\mathcal{R}) = \mathbb{R}^S$ already, adding extra assets (which are rows) will not change $\mathrm{Col}(\mathcal{R})$, so there is no change in the set of possible outcomes.

Definition 1.3. A portfolio x is **duplicable** if there is another portfolio w such that $\sum_{i=1}^{A} x_i = \sum_{i=1}^{A} w_i$ and

$$\mathcal{R}x = \mathcal{R}w.$$

[2]This definition ignores the budget constraint, $\sum_{i=1}^{A} x_i = 1$. This is okay because if x is riskless, so is αx for any scalar α, and we can adjust α to meet the budget constraint.

Duplicable portfolios satisfy[3]

$$\begin{pmatrix} \mathcal{R} \\ 1 \cdots 1 \end{pmatrix} x = \begin{pmatrix} \mathcal{R} \\ 1 \cdots 1 \end{pmatrix} w$$

If $\underbrace{\widetilde{\mathcal{R}}}_{(S+1)\times A} = \begin{pmatrix} \mathcal{R} \\ 1 \cdots 1 \end{pmatrix}$, then the above equation has solutions $x \neq w$ if and only if $\dim \mathcal{N}(\widetilde{\mathcal{R}}) > 0$. This will be the case if and only if $\text{rank}\widetilde{\mathcal{R}} < A$.

2. First and second welfare theorems

You may have heard of the first and second welfare theorems before. The first welfare theorem says that under some conditions, every competitive equilibrium is Pareto efficient. The second welfare theorem says that under some stronger conditions, every Pareto efficient allocation can be achieved by some competitive equilibrium. We now have (nearly) enough mathematical tools to state these theorems very precisely and prove them in a very general setting. Do not worry if you find the proofs confusing. The proofs are somewhat long, and they use continuity and convexity, which we have not yet covered. We will be talking a lot about continuity and related concepts next week, so this should be a good preview. In terms of mathematics, the things that you should understand are parts where we use properties of linear transformations, convexity, and the separating hyperplane theorem. In terms of economics, the important things to understand are the two theorems, the assumptions of the theorems (and how they relate to the real world), and the general role of the assumptions in the proofs, so you can make an informed conjecture about what happens when the assumptions are false.

2.1. **Lines, planes, and hyperplanes.** A line in $\mathbb{R}^2$ splits $\mathbb{R}^2$ into two pieces. A plane in $\mathbb{R}^3$ splits it into two pieces. More generally, an $n-1$ dimensional affine space splits $\mathbb{R}^n$ into two pieces.

Definition 2.1. A **hyperplane** in $\mathbb{R}^n$ is an $n-1$ dimensional affine subspace. Equivalently, a hyperplane is the set of solutions to a single equation with n variables.

Any hyperplane can be written in the form:

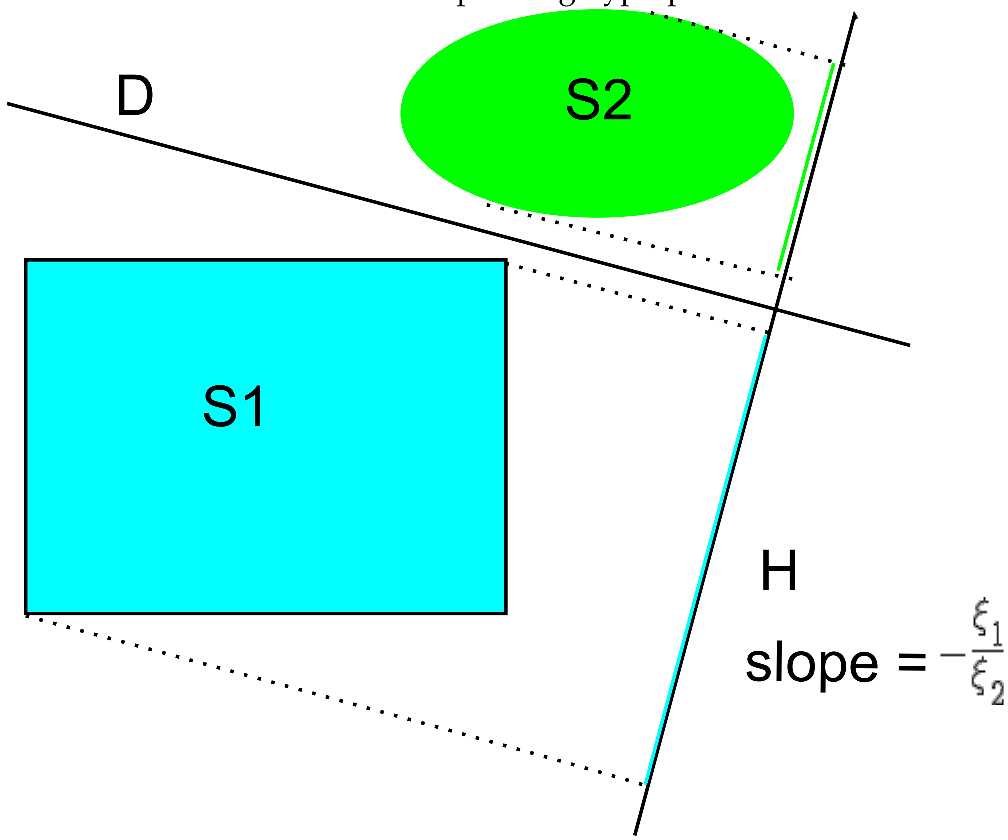
$$H_{\xi,c} = \{x : \langle \xi, x \rangle = c\}$$

where $c \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, and $\|\xi\| = 1$. Hyperplanes play an important role in optimization. There is one theorem, which we state here without proof, that is especially useful. We will use this theorem to prove the second welfare theorem. First, a definition.

Definition 2.2. A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x_1, x_2 \in S$ and $\lambda \in (0,1)$, we have $x_1\lambda + x_2(1-\lambda) \in S$.

[3]Until now, we ignored the budget constraint $\sum x_i = 1$, but now we include it. Why was it okay to ignore the budget constraint earlier? Why does it matter now?

FIGURE 1. Separating hyperplane



If a set is convex, when we draw a line between any two points in the set, the line remains entirely within the set. In $\mathbb{R}^2$, convex sets include things shaped like triangles, squares, pentagons, circles, ellipses, etc. Some non-convex shapes are stars, horseshoes, rings, etc.

Theorem 2.1 (Separating hyperplane theorem). *If S_1 and $S_2 \subseteq \mathbb{R}^n$ are convex and $S_1 \cap S_2 = \emptyset$ then there exists a hyperplane, $H_{\xi c} = \{x : \xi' x = c\}$ such that*

$$\langle s_1, \xi \rangle \leq c \leq \langle s_2, \xi \rangle$$

for all $s_1 \in S_1$ and $s_2 \in S_2$. We say that $H_{\xi,c}$ separates S_1 and S_2.

Visually, this theorem says that we can draw a hyperplane, D, between S_1 and S_2. H will then be another hyperplane orthogonal to D and the projection of S_1 on H is disjoint from the projection of S_2 on H. See figure 1 for an illustration in $\mathbb{R}^2$.

Recall that the projection of S_1 on H is the set

$$P_H S_1 = \{\langle s_1, \xi \rangle \xi : s_1 \in S_1.$$

The projections are disjoint or almost disjoint)[4] because if $\langle s_1, \xi \rangle < \langle s_2, \xi \rangle \; \forall s_1 \in S_1, s_2 \in S_2$, we can never have $\langle s_1, \xi \rangle \xi = \langle s_2, \xi \rangle \xi$.

[4]Really this argument only shows disjointness if S_1 and S_2 are topologically closed, something that we will discuss later.

2.2. **Setup.** We have some set of commodities, S, which we will assume is a normed vector space. For example, if a world with n goods, S could be $\mathbb{R}^n$ and for each $s = (s_1, ..., s_n) \in S$, s_j represents the quantity of the jth good. These goods include everything that is bought or sold, including things like food or clothing that we usually think of as goods, and things like labor and land. There are I consumers, indexed by i. Each consumer chooses goods from a feasible set $X_i \subseteq S$. These X_i are feasible consumption sets, not budget sets. It is supposed to represent the physical constraints of the world. For example if there are three goods: food, clothing, and labor measured in days of labor per day, then X_i might be $[0, \infty) \times [0, \infty) \times [0, 1]$. Each consumer has preferences over X_i represented by a preference relation, $\succeq_i$ that has the following properties:

(1) (total) $\forall x, z \in X_i$, either $x \succeq_i z$ or $z \succeq_i x$ or both,
(2) (transitive) $\forall x, w, z \in X_i$, if $x \succeq_i w$ and $w \succeq_i z$ then $x \succeq_i z$,
(3) (reflexive) $\forall x \in X_i$, $x \succeq_i x$.

In words, $x \succeq_i z$ means person i likes the bundle of goods x as much as or more than the bundle of goods z. If you wish, you can think of the preference relation coming from a utility function, $u_i(x) : X_i \to \mathbb{R}$ and $x \succeq_i z$ means $u_i(x) \geq u_i(z)$. If $x \succeq_i z$ but $z \not\succeq_i x$, then we say that x is strictly preferred to z and write $x \succ_i z$. If $x \succeq_i z$ and $z \succeq_i x$ we say that person i is indifferent between x and z and write $x \simeq_i z$.

There are also J firms indexed by j. Each firm j chooses production y_j from production possibility set $Y_j \subseteq S$. The firm will produce positive quantities of its outputs and negative quantities of its inputs. Continuing with the example of three goods, if the firm produces $F^f(l)$ units of food from l units of labor and $F^c(l)$ units of clothing, then production possibility set could be written:

$$Y_j = \{(f, c, l) \in S : l \leq 0 \wedge f \leq F^f(\alpha|l|) \wedge c \leq F^c((1-\alpha)|l|) \text{ for some } \alpha \in [0, 1]\}.$$

Firms produce goods and consumers consume goods. For the market to **clear** we must have sum of production equal to the sum of consumption, i.e.

$$\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j$$

We call the $I + J$-tuple of all x_i and y_j, $((x_1, ..., x_I), (y_1, ..., y_J))$ (which we will sometimes shorted by just writing $((x_i), (y_j)))$ an **allocation**. An allocation is **feasible** if $x_i \in X_i \forall i$, $y_j \in Y_j \forall j$, and $\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j$.

Definition 2.3. An allocation, $((x_i^0), (y_j^0))$, is **Pareto efficient** (or Pareto optimal) if it is a feasible and there is no other feasible allocation, $((x_i), (y_j))$, such that $x_i \succeq_i x_i^0$ for all i and $x_i \succ_i x_i^0$ for some i.

This definition is just a mathematical way of stating the usual verbal definition of Pareto efficient. An allocation is Pareto efficient if there is no other allocation that makes at least one person better off and no one worse off.

We are going to be comparing competitive equilibria to Pareto efficient allocations. To do that we must first define a competitive equilibrium. A price system is a continuous linear transformation[5], $p : S \to \mathbb{R}$, i.e. $p \in S^*$. In the case where $S = \mathbb{R}^n$, a price system is just a $1 \times n$ matrix. The entries in this price matrix are the prices of each of the n goods. px for $x \in S$ represents the total expenditure needed to purchase the bundle of goods x.

Definition 2.4. An allocation, $((x_i^0), (y_j^0))$, along with a price system, p, is a **competitive equilibrium** if

(C1) The allocation is feasible
(C2) For each i and $x \in X_i$ if $px \leq px_i^0$ then $x_i^0 \succeq_i x$,
(C3) For each j if $y \in Y_j$ then $py \leq py_j^0$

Condition C2 says that each consumer must be choosing the most preferred bundle of goods that he or she can afford. If the preference relation comes from a utility function, C2 says that consumers maximize their utility given prices. Similarly, condition C3 says that producers maximize profits.

2.3. **First welfare theorem.** The first welfare theorem requires one additional condition on preferences.

Definition 2.5. Preference relation $\succ_i$ has the **local non-satiation condition** if for each $x \in X_i$ and $\epsilon > 0$ $\exists x' \in X_i$ such that $\|x - x'\| \leq \epsilon$ and $x' \succ_i x$.

This condition says that given any bundle of goods you can find another bundle very close by that is preferred. If the preference relation comes from utility function, the utility function having a non-zero derivative everywhere implies local non-satiation. The intuition for why the first welfare theorem requires local non-satiation is that local non-satiation rules out the following scenario. Suppose person i does not care about clothing at all. Then you take clothes away from person i, making person i no worse off, and give them to someone else, making that person better off. However, there is nothing in the definition of a competitive equilibrium that prevents person i from having clothes.

Theorem 2.2 (First welfare theorem)**.** *If $((x_i^0), (y_j^0))$ and p is a competitive equilibrium and all consumers' preferences have the local non-satiation condition, then $((x_i^0), (y_i^0))$ is Pareto efficient.*

Proof. We will prove it by contradiction. Suppose that a competitive equilibrium is not Pareto efficient. Then there exists another feasible allocation[6], $((x_i), (y_j))$, such that there is at least one $x_{i^*} \succ_{i^*} x_{i^*}^0$. The contrapositive of condition C2 in the definition of competitive equilibrium implies that then $px_{i^*} > px_{i^*}^0$. For all other $i \neq i^*$ it must be that $x_i \succeq_i x_i^0$. When $x_i \succ_i x_i^0$, by the same argument as above, $px_i > px_i^0$. When $x_i \simeq_i x_i^0$, then we will show that local non-satiation implies $px_i \geq px_i^0$. If not and $px_i < px_i^0$, then by continuity[7]

[5] All linear transformations on finite dimensional vector spaces are continuous, so matrices are always continuous linear transformations. There are discontinuous linear operators on infinite dimensional vector spaces (differentiation on C^∞ is one example), but they are beyond the scope of this course. Also, they are not needed for this proof, unless you want allow economies with an infinite number of goods.

[6] This sort of allocation is called a Pareto improvement.

[7] We have not yet defined continuity, so do not worry if you find this part of the proof confusing. A function $f : V \to W$ where V and W are normed vector spaces is continuous if $\forall \epsilon > 0$ $\exists \delta > 0$ such that

of p there exists some $\delta > 0$ such that for all x' with $\|x_i - x'\| < \delta$, we have

$$|px_i - px'| < |px_i - px_i^0|$$

and in particular,

$$px' < px_i^0.$$

Additionally since preferences are locally non-satiated, there exists some $\tilde{x}$ with $\|x_i - \tilde{x}\| < \delta$ and $\tilde{x} \succ_i x_i \simeq_i x_i^0$. However, then we also have $\tilde{x} \succ_i x_i^0$ and $p\tilde{x} < px_i^0$, which contradicts x_i^0 and p being part of a competitive equilibrium. Thus, we can conclude that $px_i \geq px_i^0$.

At this point we have shown that if $((x_i^0), (y_j^0))$ is a competitive equilibrium that is not Pareto efficient, then there is some other allocation $((x_i), (y_j))$ that is feasible and has $x_i \succeq_i x_i^0$, which implies that $px_i \geq px_i^0$. Each consumer spends (weakly) more in this alternative, Pareto improving allocation. Now we will show that each consumer spending at least as much contradicts profit maximization. The total expenditure of consumers in the alternate allocation must be greater than in the competitive equilibrium because there is one consumer who is spending strictly more. That is,

$$\sum_{i=1}^{I} px_i > \sum_{i=1}^{I} px_i^0 \tag{3}$$

The price system is a linear transformation, so

$$\sum_{i=1}^{I} px_i = p\left(\sum_{i=1}^{I} x_I\right)$$

Both allocations are feasible, and, in particular, market clearing so

$$\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j$$

Applying p to both sides,

$$\begin{aligned} p\left(\sum_{i=1}^{I} x_i\right) &= p\left(\sum_{j=1}^{J} y_j\right) \\ &= \sum_{j=1}^{J} py_j. \end{aligned}$$

Identical reasoning would show that

$$\sum_{i=1}^{I} px_i^0 = \sum_{j=1}^{J} py_j^0.$$

whenever $\|x - y\| < \delta$ we also have $\|f(x) - f(y)\| < \epsilon$. In this proof, V is S , W is $\mathbb{R}$, f is p, and ϵ is $|px_i - px_i^0|$.

Substituting into (3) we get

$$\sum_{j=1}^{J} py_j > \sum_{j=1}^{J} py_j^0. \tag{4}$$

But this contradicts profit maximization (C3) since $y_j \in Y_j$ and we cannot have (4) if $py_j \leq py_j^0$. Therefore, we conclude that there can be no Pareto improvement from a competitive equilibrium, i.e. any competitive equilibrium is Pareto efficient. □

2.4. **Second welfare theorem.** The second welfare theorem is the converse of the first welfare theorem. The second welfare theorem says that any Pareto efficient allocation can be achieved by some competitive equilibrium. The second welfare theorem does not hold quite as generally as the first welfare theorem.

Definition 2.6. A preference relation, $\succeq_i$, is **convex** if whenever $x \succeq_i z$ and $y \succeq_i z$, then $\lambda x + (1-\lambda)y \succeq_i z$ for all $\lambda \in [0,1]$.

Alternatively, a preference relation is convex if the set $\{x \in X_i : x \succeq_i z\}$ is convex for each z. Whenever you have seen convex indifference curves, the associated preference relation is convex. If the preference relation is generated by a concave (more generally quasi-concave) utility function, then the preference relation is convex.

Definition 2.7. A preference relation, $\succeq_i$, is **continuous** if for any $x \succ_i z$ there exists a $\delta > 0$ such that for all x' with $\|x - x'\| < \delta$ we have $x' \succ_i z$.

A continuous preference relation can be generated by a continuous utility function.

Theorem 2.3 (Second welfare theorem)**.** *Assume the preferences of each consumer are convex, locally non-satiated, and continuous, and that X_i is convex and non-empty. Also assume that Y_j is convex and non-empty for each firm j.*

Suppose $((x_i^e), (y_j^e))$ is a Pareto efficient allocation such that for any price system, p, there is always a cheaper bundle of goods, i.e. $\exists x_i \in X_i$ s.t. $px_i < px_i^e$ for each i. Then there exists a price system, p^e such that $((x_i^e), (y_j^e))$ and p^e is a competitive equilibrium.

Proof. We are going to construct the price system by applying the separating hyperplane theorem. Let $V_i = \{x \in X_i : x \succ_i x_i^e\}$ be the set of x strictly preferred by person i. Let

$$V = \{\chi \in S : \chi = \sum_{i=1}^{I} x_i \text{ for some } x_i \in V_i\}$$

be the set of sums of elements from each V_i. The convexity of X_i and the preference relation implies that V_i is convex for each i. That, in turn, implies that V is convex.[8] Similarly, if

$$Y = \{\psi \in S : \psi = \sum_{i=j}^{J} y_j \text{ for some } y_j \in Y_j\}$$

is the sum of each firms' production possibility set, then Y is convex.

[8]It might be a good exercise to prove these claims.

We have two convex sets. Now, we just need to show that they are disjoint, and then we can apply the separating hyperplane theorem. Suppose $\chi \in Y \cap V$. Then $\exists x_i \in V_i$ and $y_j \in Y_j$ such that $\chi = \sum_{i=1}^{I} x_i = \sum_{j=1}^{J}$. This is feasible allocation, and $x_i \succ_i x_i^e$ by construction. This contradicts $((x_i^e), (y_i^e))$ being Pareto efficient. Therefore, $Y \cap V = \emptyset$. o

Now, by the separating hyperplane theorem, $\exists p$ such that[9]

$$p\chi \geq p\psi \tag{5}$$

for all $\chi \in V$ and $\psi \in Y$. Now we need to verify that $((x_i^e), (y_j^e))$ with p is a competitive equilibrium. It is feasible because $((x_i^e), (y_j^e))$ is Pareto efficient, and feasible by definition.

We now show that (5) holds with equality at $\chi^e = \sum_{i=1}^{I} x_i^e$ and $\psi^e = \sum_{j=1}^{J} y_j^e$. On the one hand, $\chi^e = \psi^e \in Y$, so we must have

$$p\chi \geq p\chi^e$$

for all $\chi \in V$. On the other hand, for any$\delta > 0$, by local non-satiation, we can find x_i such that $x_i \succ_i x_i^e$ and $\left\| x_i - x_i^e \right\| < \delta / I$. It follows that $\left\| \sum_{i=1}^{I} x_i - \sum_{i=1}^{T} x_i^e \right\| < \delta$. p is continuous, so

$$\left| p\left(\sum_{i=1}^{e} x_i\right) - p\left(\sum_{i=1}^{I} x_i\right) \right| < \epsilon,$$

and we can choose $\epsilon > 0$ to be as small as we want. Then, for any $\epsilon > 0$,

$$p\chi^e + \epsilon \geq p\psi$$

for all $y \in Y$. Since this is true for any ϵ, it must be that $p\chi^e \geq p\psi$. Therefore, we have now shown that

$$p\chi \geq p\chi^e = p\psi^e \geq p\psi \tag{6}$$

for all $\chi \in V$ and $\psi \in Y$.

It must be then also be that $px_i \geq px_i^e$ for each i and all $x_i \in V_i$. If not, then there is an $\epsilon > 0$ such that $px_i + \epsilon < px_i^e$, and then using local non-satiation we can choose x_k for $k \neq i$ such that $x_k \in V_k$ and

$$\left| \sum_{k \neq i} px_k - \sum_{k \neq i} px_k^e \right| < \epsilon/2$$

and then

$$\sum_{k=1}^{I} px_k + \epsilon/2 < \sum_{k=1}^{I} px_k^e.$$

Similarly, we must have $py_j^e \geq py_j$ for all $y_j \in Y_j$, which proves that profit maximization, (C3), holds.

We have nearly shown that utility maximization, (C2), also holds. We have shown that for each i if $x_i \succ_i x_i^e$ then $px_i \geq px_i^e$. To strengthen it to the form in the definition, we need to show that $px_i > px_i^e$. We will use the continuity of preferences and the cheaper good condition. Suppose $px_i = px_i^e$ and $\exists x_i' \in X_i$ such that $px_i' < px_i^e$. Then for any $\lambda \in (0,1)$, $p(\lambda x_i' + (1-\lambda)x_i') < px_i^e$. Also, by the continuity of preferences, for λ close

[9] In the notation of theorem 2.1, p is ξ.

enough to 0, $\lambda x_i + (1-\lambda)x_i' \succ_i x_i^e$. However, then $\lambda x_i + (1-\lambda)x_i' \in V_i$ contradicting $p(\lambda x_i + (1-\lambda)x_i') < px_i^e$. Therefore, if the cheaper good exists, we must have $px_i < px_i^e$. □