Applications

Paul Schrimpf

UBC
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Section 1

Portfolio analysis

Portfolio analysis

- Model of portfolio analysis taken from Simon and Blume
- Good way of reviewing results about matrices and systems of linear equations

Setup

- A assets
- S states of nature
- Initial value of $i = v_i$
- After state revealed, value y_{si}
- Realized return $R_{si} = \frac{y_{si}}{v_i}$
- Matrix of returns $\underbrace{\mathcal{R}}_{S \times A}$

Setup

- Choosing budget shares x_i, $\sum_{i=1}^{A} x_i = 1$ (but individual shares could be positive or negative), call $x = (x_1, ..., x_A)$ a portfolio

Riskless portfolios

Definition

A portfolio is **riskless** if

$$\sum_{i=1}^{A} R_{1i} x_i = \cdots = \sum_{i=1}^{A} R_{Si} x_i$$

- Solution to $\mathbf{c} = (c, ..., c)$

$$\underbrace{\mathcal{R}}_{S \times A} \underbrace{x}_{A \times 1} = \underbrace{\mathbf{c}}_{S \times 1}$$

- Exists if $\mathbf{c} \in \mathrm{Col}(\mathcal{R})$

Insurable states

Definition

A state s^* is **insurable** if $\exists$ portfolio x such that $\sum_{a=1}^{A} R_{s^*a} x_a > 0$ and $\sum_{a=1}^{A} R_{sa} x_a = 0$ for all $s \neq s^*$.

- s^* insurable iff $\exists$ solution to

$$\mathcal{R}x = e_{s^*}$$

- So if $e_{s^*} \in \mathrm{Col}(\mathcal{R})$
- All states insurable iff $\mathrm{Col}(\mathcal{R}) = \mathbb{R}^S$

Duplicable portfolios

Definition

A portfolio x is **duplicable** if there is another portfolio w such that $\sum_{i=1}^{A} x_i = \sum_{i=1}^{A} w_i$ and

$$\mathcal{R}x = \mathcal{R}w.$$

- i.e. multiple solutions to

$$\begin{pmatrix} \mathcal{R} \\ 1 \cdots 1 \end{pmatrix} x = \begin{pmatrix} \mathcal{R} \\ 1 \cdots 1 \end{pmatrix} w$$
$$\tilde{\mathcal{R}}x = \tilde{\mathcal{R}}w$$

if and only if $\dim \mathcal{N}(\tilde{\mathcal{R}}) > 0$ if and only if $\operatorname{rank} \tilde{\mathcal{R}} < A$

Section 2

First and second welfare theorems

Welfare theorems

1st: Every competitive equilibrium is Pareto efficient (under some conditions)

2nd: Every Pareto efficient allocation is a competitive equilibrium (under some stronger conditions)

- We will
  - Carefully define competitive equilibrium and Pareto efficiency
  - State and prove theorems

Welfare theorems

- Proofs will use:
  - Vector spaces, linear transformations
    - Should mostly understand
  - Convexity, separating hyperplane theorem
  - Continuity — $\forall \epsilon > 0 \exists \delta > 0$ s.t. ...
    - Maybe difficult to follow, but we will see a lot more of this in calculus
- Economic issues to think about:
  - The result of the two theorems, their assumptions, and how they relate to the real world
  - General role of assumptions in proofs, so can conjecture what happens when assumptions are false

Lines, planes, and hyperplanes

Definition

A **hyperplane** in $\mathbb{R}^n$ is an $n-1$ dimensional affine subspace. Equivalently, a hyperplane is the set of solutions to a single equation with n variables.

Any hyperplane can be written in the form:

$$H_{\xi,c} = \{x : \langle \xi, x \rangle = c\}$$

where $c \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, and $\|\xi\| = 1$.

Separating hyperplanes

Definition

A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x_1, x_2 \in S$ and $\lambda \in (0,1)$, we have $x_1\lambda + x_2(1-\lambda) \in S$.

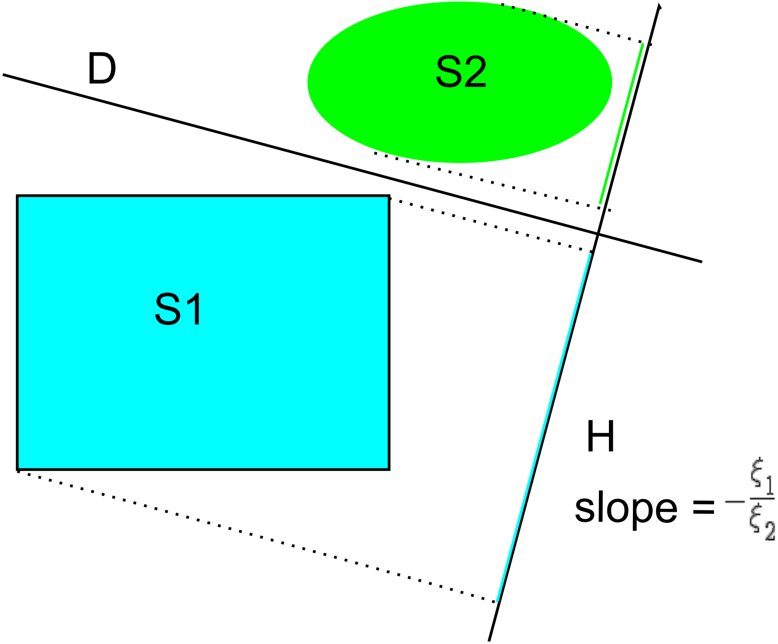
Theorem (Separating hyperplane theorem)

If S_1 and $S_2 \subseteq \mathbb{R}^n$ are convex. If $S_1 \cap S_2 = \emptyset$ then there exists a hyperplane, $H_{\xi c} = \{x : \xi' x = c\}$ such that

$$\langle s_1, \xi \rangle \leq c \leq \langle s_2, \xi \rangle$$

for all $s_1 \in S_1$ and $s_2 \in S_2$. We say that $H_{\xi,c}$ separates S_1 and S_2.

- Draw picture


D
S2
S1
H
slope = $-\frac{\xi_1}{\xi_2}$

Setup

- Commodities $s \in S$, a normed vector space (e.g. S linear subspace of $\mathbb{R}^n$)
- Consumers $i = 1, ..., I$ with consumption possibility set $X_i \subseteq S$ (think of as physical constraint, not budget constraint)
- Preference relation $\succeq_i$ s.t.
  1. (total) $\forall x, z \in X_i$, either $x \succeq_i z$ or $z \succeq_i x$ or both,
  2. (transitive) $\forall x, w, z \in X_i$, if $x \succeq_i w$ and $w \succeq_i z$ then $x \succeq_i z$,
  3. (reflexive) $\forall x \in X_i$, $x \succeq_i x$.

  Denote strict preference $\succ_i$, and indifference $\simeq_i$
  - Could come from utility function i.e. $x \succeq_i z$ iff $u_i(x) \geq u_i(z)$, but does not have to

Setup

- Firms $j = 1, ..., J$ produce y_j from within production possibility sets $Y_j \subseteq S$
    - Example: 3 commodities, food f, clothing c, and labor l. Production function $f = F^f(l)$, and $c = F^c(l)$,

$$Y_j = \{(f, c, l) \in S : l \leq 0 \wedge f \leq F^f(\alpha|l|) \wedge c \leq F^c((1-\alpha)|l|) \text{ for s}$$

- $I + J$ tuple $((x_i), (y_j))$ is an **allocation**
- Allocation is feasible if $x_i \in X_i \forall i$, $y_j \in Y_j \forall j$, and the market clears, $\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} y_j$.

Definition

An allocation, $((x_i^0), (y_j^0))$, is **Pareto efficient** if it is a feasible and there is no other feasible allocation, $((x_i), (y_j))$, such that $x_i \succeq_i x_i^0$ for all i and $x_i \succ_i x_i^0$ for some i.

Definition

A price system is a continuous linear transformation $p : S \to \mathbb{R}$. (If S is n-dimensional then p is any $1 \times n$ matrix)

Definition

An allocation, $((x_i^0), (y_j^0))$, along with a price system, p, is a **competitive equilibrium** if

- C1 The allocation is feasible
- C2 (Utility maximization) For each i and $x \in X_i$ if $px \leq px_i^0$ then $x_i^0 \succeq_i x$,
- C3 (Profit maximization) For each j if $y \in Y_j$ then $py \leq py_j^0$

first welfare theorem

Definition

preference relation $\succ_i$ has the **local non-satiation condition** if for each $x \in x_i$ and $\epsilon > 0$ $\exists x' \in x_i$ such that $\|x - x'\| \leq \epsilon$ and $x' \succ_i x$.

Theorem (first welfare theorem)

if $((x_i^0), (y_j^0))$ and p is a competitive equilibrium and all consumers' preferences have the local non-satiation condition, then $((x_i^0), (y_i^0))$ is pareto efficient.

proof of first welfare theorem

- will show contradiction
- suppose competitive equilibrium not pareto efficient, then $\exists$ a pareto improvement $((x_i),(y_j))$ another feasible allocation s.t.
  - $\exists$ at least one i^* with $x_{i^*} \succ_i x_i^0$
    - the contrapositive of C2 implies $px_{i^*} > px_i^0$
  - all other $x_i \succeq_i x_i^0$
    - local non-satiation and continuity of p imply $px_i \geq px_i^0$ (next slide)

proof of first welfare theorem

- local non-satiation and continuity of p imply $px_i \geq px_i^0$
  - if $px_i < px_i^0$, then local non-satiation implies $\exists\, x_i'$ very close to x_i such that $x_i' \succ_i x_i \succeq_i x_i^0$.
  - p continuous means that if x_i' is close enough to x_i, then px_i' will be close enough to px_i that $px_i' < px_i^0$ as well.
  - but $x_i' \succ_i x_i$ and $px_i' < px_i^0$ contradicts the definition of competitive equilibrium.
- so in this pareto improvement, we have

$$\sum_{i=1}^{i} px_i > \sum_{i=1}^{i} px_i^0 \tag{1}$$

next, we will show this contradicts profit maximization

proof of first welfare theorem

- start with market clearing, apply p, rearrange:

$$\sum_{i=1}^{i} x_i = \sum_{j=1}^{j} y_j \tag{2}$$

$$p(\sum_{i=1}^{i} x_i) = \sum_{i=1}^{i} p x_i = \sum_{j=1}^{j} p y_j = p(\sum_{j=1}^{j} y_j) \tag{3}$$

$$\tag{4}$$

- same thing for x_i^0 and y_j^0 with inequality from last slide gives

$$\sum_{j=1}^{j} p y_j > \sum_{j=1}^{j} p y_j^0 \tag{5}$$

but then there is at least on j with $p y_j > p y_j^0$, contradicting C3

Second welfare theorem

Definition

A preference relation, $\succeq_i$, is **convex** if whenever $x \succeq_i z$ and $y \succeq_i z$, then $\lambda x + (1-\lambda) y \succeq_i z$ for all $\lambda \in [0,1]$.

Definition

A preference relation, $\succeq_i$, is **continuous** if for any $x \succ_i z$ there exists a $\delta > 0$ such that for all x' with $\|x - x'\| < \delta$ we have $x' \succ_i z$.

Second welfare theorem

Theorem (Second welfare theorem)

Assume the preferences of each consumer are convex, locally non-satiated, and continuous, and that X_i is convex and non-empty. Also assume that Y_j is convex and non-empty for each firm j.

Suppose $((x_i^e), (y_j^e))$ is a Pareto efficient allocation such that for any price system, p, there is always a cheaper bundle of goods, i.e. $\exists x_i \in X_i$ s.t. $px_i < px_i^e$ for each i. Then there exists a price system, p^e such that $((x_i^e), (y_j^e))$ and p^e is a competitive equilibrium.

Compared to first welfare theorem three more assumptions:

1. Convexity of preferences, consumption possibility sets, and production possibility sets
2. Continuity of preferences
3. Cheaper good condition

Proof of second welfare theorem

- Price system will come from separating hyperplane theorem, first need two disjoint, convex sets
- Let $V_i = \{x \in X_i : x \succ_i x_i^e\}$, set sum of V_i be

$$V = \{\chi \in S : \chi = \sum_{i=1}^{I} x_i \text{ for some } x_i \in V_i\}$$

  V_i convex, so V convex

- Let

$$Y = \{\psi \in S : \psi = \sum_{i=j}^{J} y_j \text{ for some } y_j \in Y_j\}$$

  also convex.

Proof of second welfare theorem

- Show Y and V disjoint
  - Suppose $\chi \in Y \cap V$. Then $\exists x_i \in V_i$ and $y_j \in Y_j$ such that $\chi = \sum_{i=1}^{I} x_i = \sum_{j=1}^{J}$, is feasible and $x_i \succ_i x_i^e$
  - Contradicts $((x_i^e), (y_i^e))$ being Pareto efficient
- Separating hyperplane theorem $\Rightarrow \exists p$ s.t.

$$p\chi \geq p\psi \tag{6}$$

- Next, show $((x_i^e), (y_j^e))$ with p is a competitive equilibrium

Proof of second welfare theorem

- Show $p\chi \geq p\chi^e = p\psi^e \geq p\psi$ for all $\chi \in V$, $\psi \in Y$
  - $\psi^e \in Y$, so $p\chi \geq p\psi^e \; \forall \chi \in V$.
  - $\chi^e = \sum x_i^e = \sum y_j^e = \psi^e$, so

$$p\chi \geq p\chi^e = p\psi^e \tag{7}$$

  - Local non-satiation implies we can find $x_i' \succ_i x_i^e$ very close to x_i^e. Continuity of p means that for x_i' close enough we have $|p\chi^e - p\chi'| < \epsilon$ for any $\epsilon > 0$. Therefore $\forall \psi \in Y$

$$p\chi^e + \epsilon > p\chi' \geq p\psi \tag{8}$$

so $p\chi^e \geq p\psi$

Proof of second welfare theorem

- Show $px_i \geq px_i^e$ for each i and all $x_i \in V_i$
  - If not there $px_i + \epsilon < px_i^e$ with $x_i \in V_i$.
  - Local non-satiation and continuity of p means we can choose other $x_k \in V_k$ such that $\left|\sum_{k \neq i}(px_k - px_k^e)\right| < \epsilon/2$
  - But then $p\chi < p\chi^e$ contradicting the previous slide
- Same reasoning but using convexity of Y_j instead of local non-satiation means $py_j \leq y_j^e$ for each j and all $y_j \in Y_j$ i.e. firms are profit maximizing, C3 holds

Proof of second welfare theorem

- Have shown: if $x_i \succ_i x_i^e$ then $px_i \geq px_i^e$
- Need: if $x_i \succ_i x_i^e$ then $px_i > px_i^e$
- Use cheaper good condition
  - If $px_i = px_i^e$ and $\exists x_i' \in X_i$ s.t. $px_i' < px_i^e$, then for all $\lambda \in (0,1)$,

$$p(\lambda x_i + (1-\lambda)x_i') < px_i^e \tag{9}$$

  - Continuous preferences implies for λ close to 0,

$$\lambda x_i + (1-\lambda)x_i' \succ_i x_i^e \tag{10}$$

    so $\lambda x_i + (1-\lambda)x_i' \in V_i$, but that contradicts what we have shown.