Limits and topology of metric spaces

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Section 1

Sequences and limits

Sequences and limits

- **sequence** is a list of elements, $\{x_1, x_2, ...\}$ or $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$
    - Different than set
- Examples
    1. $\{1, 1, 2, 3, 5, 8, ...\}$
    2. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$
    3. $\{\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, ...\}$

Definition

A **metric space** is a set, X, and function $d : X \times X \to \mathbb{R}$ called a **metric** (or distance) such that $\forall x, y, z \in X$

1. $d(x, y) > 0$ unless $x = y$ and then $d(x, x) = 0$
2. (symmetry) $d(x, y) = d(y, x)$
3. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

Example

$\mathbb{R}$ is a metric space with $d(x, y) = |x - y|$.

Example

Any normed vector space is a metric space with $d(x, y) = \|x - y\|$.

Definition

A sequence $\{x_n\}_{n=1}^\infty$ in a metric space **converges** to x if $\forall \epsilon > 0$ $\exists N$ such that

$$d(x_n, x) < \epsilon$$

for all $n \geq N$. We call x the **limit** of $\{x_n\}_{n=1}^\infty$ and write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition

a is an **accumulation point** of $\{x_n\}_{n=1}^{\infty}$ if $\forall \epsilon > 0$ $\exists$ infinitely many x_i such that

$$d(a, x_i) < \epsilon.$$

Lemma

If $x_n \to x$, then x is an accumulation point of $\{x_n\}_{n=1}^{\infty}$.

Proof.

Let $\epsilon > 0$ be given. By the definition of convergences, $\exists N$ such that

$$d(x_n, x) < \epsilon$$

for all $n \geq N$. $\{n \in \mathbb{N} : n \geq N\}$ is infinite, so x is an accumulation point. □

Definition

Given $\{x_n\}_{n=1}^{\infty}$ and any sequence of positive integers, $\{n_k\}$ such that $n_1 < n_2 < ...$ we call $\{x_{n_k}\}$ a **subsequence** of $\{x_n\}_{n=1}^{\infty}$.

Lemma

Let a be an accumulation point of $\{x_n\}$. Then $\exists$ a subsequence that converges to a.

Sequences and arithmetic

Theorem

Let $\{x_n\}$ and $\{y_n\}$ be sequences in a normed vector space V. If $x_n \to x$ and $y_n \to y$, then

$$x_n + y_n \to x + y.$$

Proof.

Let $\epsilon > 0$ be given. Then $\exists\ N_x$ such that for all $n \geq N_x$,

$$d(x_n, x) < \epsilon/2,$$

and $\exists N_y$ such that for all $n \geq N_y$,

$$d(y_n, y) < \epsilon/2.$$

Let $N = \max\{N_x, N_y\}$. Then for all $n \geq N$,

$$d(x_n + y_n, x + y) = \|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\|$$

Sequences and arithmetic

Theorem

Let $\{x_n\}$ be a sequence in a normed vector space with scalar field $\mathbb{R}$ and let $\{c_n\}$ be a sequence in $\mathbb{R}$. If $x_n \to x$ and $c_n \to c$ then

$$x_n c_n \to xc.$$

Proof.

On problem set. □

Series

Definition

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a normed vector space. Let $s_n = \sum_{i=1}^{n} x_i$ denote the sum of the first n elements of the sequence. We call s_n the nth partial sum. We define the sum of all the x_is as

$$\sum_{i=1}^{\infty} x_i \equiv \lim_{n\to\infty} s_n$$

This is called a(n infinite) **series**.

Series

Example

Geometric series: $\sum_{i=0}^{\infty} \beta^i$ where $\beta \in \mathbb{R}$ has partial sums:

$$\begin{aligned} s_n =& 1 + \beta + \beta^2 + \cdots + \beta^n \\ =& 1 + \beta(1 + \beta + \cdots + \beta^{n-1}) \\ =& 1 + \beta(1 + \beta + \cdots + \beta^{n-1} + \beta^n) - \beta^{n+1} \\ s_n(1-\beta) =& 1 - \beta^{n+1} \\ s_n =& \frac{1 - \beta^{n+1}}{1 - \beta}, \end{aligned}$$

so,

$$\begin{aligned} \sum_{i=0}^{\infty} \beta^i =& \lim s_n \\ =& \lim \frac{1 - \beta^{n+1}}{1 - \beta} \\ =& \frac{1}{1 - \beta} \text{ if } |\beta| < 1. \end{aligned}$$

Cauchy sequences

Definition

A sequence $\{x_n\}_{n=1}^{\infty}$ is a **Cauchy** sequence if for any $\epsilon > 0$ $\exists N$ such that for all $i, j \geq N$, $d(x_i, x_j) < \epsilon$.

Theorem

A sequence in $\mathbb{R}^n$ converges if and only if it is a Cauchy sequence.

- Implied by least upper bound property of $\mathbb{R}$
- Proof in 29.1 of Simon and Blume
- Not always true, e.g. $\mathbb{Q}$

Completeness

Definition

A metric space, X, is **complete** if every Cauchy sequence of points in X converges in X.

- **Banach space** $=$ complete normed vector space.
- **Hilbert space** $=$ complete inner product space.

Section 2

Open sets

Open sets

Definition

Let X be a metric space and $x \in X$. A **neighborhood** of x is the set

$$N_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}.$$

Definition

A set, $S \subseteq X$ is **open** if $\forall x \in S$, $\exists\ \epsilon > 0$ such that

$$N_\epsilon(x) \subset S.$$

- At any point in an open, can move slightly and stay in the set.

Open sets

Example

Open sets:

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $(\infty, b) = \{x \in \mathbb{R} : x < b\}$
- The whole space
- $\emptyset$
- Open unit ball $\{x \in \mathbb{R}^n : \|x\| < 1\}$

Not open sets

Example

Not open sets:

- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- Any linear subspace of dimension $k < n$ in $\mathbb{R}^n$.
- Finite sets in $\mathbb{R}^n$
- $\mathbb{Q} \subset \mathbb{R}$

Open sets

Theorem

1. *Any union of open sets is open. (finite or infinite)*
2. *The finite intersection of open sets is open.*

Proof.

Let S_j, $j \in J$ be a collection of open sets. Pick any $j_0 \in J$. If $x \in \cup_{j \in J} S_j$, then there must be $\epsilon_{j_0} > 0$ such that $N_{\epsilon_{j_0}}(x) \subset S_{j_0}$. It is immediate that $N_{\epsilon_{j_0}}(x) \subset \cup_{j \in J} S_j$ as well. Let $S_1, .., S_k$ be a finite collection of open sets. For each i $\exists \epsilon_i > 0$ such that $N_{\epsilon_i}(x) \subset S_i$. Let $\underline{\epsilon} = \min_{i \in \{1,\dots,k\}} \epsilon_i$. Then $\underline{\epsilon} > 0$ since it is the minimum of a finite set of positive numbers. Also, $N_{\underline{\epsilon}}(x) \subset S_i$ for each i, so $N_{\underline{\epsilon}}(x) \subset \cap_{i=1}^k S_i$. □

Interior

Definition

The **interior** of a set A is the union of all open sets contained in A. It is denoted as $\text{int}(A)$.

- The interior of a set is open
- Open sets are equal to their interior

Example

Interiors of sets in $\mathbb{R}$.

1. $A = (a, b)$, $\text{int}(A) = (a, b)$.
2. $A = [a, b]$, $\text{int}(A) = (a, b)$.
3. $A = \{1, 2, 3, 4, ...\}$, $\mathrm{A} = \emptyset$

Section 3

Closed sets

Closed sets

Definition

A set $S \subseteq X$ is closed if its complement, X^c, is open.

Example

Closed sets:

1. $[a, b] \subseteq \mathbb{R}$
2. Any linear subspace of $\mathbb{R}^n$
3. $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

Not closed sets

1. Any open set
2. $\mathbb{Q} \subset \mathbb{R}$

Closed sets

Theorem

1. *The intersection of any collection of closed sets is closed.*
2. *The union of any finite collection of closed sets is closed.*

Proof.

Let C_j, $j \in J$ be a collection of closed sets. Then $\left(\cap_{j\in J} C_j\right)^c = \cup_{j\in J} C_j^c$. C_j^c are open, so by theorem 20, $\cup_{j\in J} C_j^c = \left(\cap_{j\in J} C_j\right)^c =$ is open.
The proof of part 2 is similar. □

Closed sets

Theorem

Let $\{x_n\}$ be any convergent sequence with each element contained in a set C. Then $\lim x_n = x \in C$ for all such $\{x_n\}$ if and only if C is closed.

- This is usually a more useful definition of closed sets

Proof.

First, we will show that any set that contains the limit points of all its sequences is closed. Let $x \in C^c$. Consider $N_{1/n}(x)$. If for any n, $N_{1/n}(x) \subset C^c$, then C^c could be open. If for all n, $N_{1/n}(x) \not\subset C^c$, then $\exists y_n \in N_{1/n}(x) \cap C$. The sequence $\{y_n\}$ is in C and $y_n \to x$. However, by assumption C contains the limit of any sequence within it. Therefore, there can be no such x, and C^c must be open and C is closed.

Suppose C is closed. Then C^c is open. Let $\{x_n\}$ be in C and $x_n \to x$. Then $d(x_n, x) \to 0$, and for any $\epsilon > 0$, $\exists x_n \in N_\epsilon(x)$. Hence, there can be no ϵ neighborhood of x contained in C^c. C^c is open by assumption, so $x \notin C^c$ and it must be that $x \in C$. □

Closure

Definition

The **closure** of a set S, denoted by $\overline{S}$ (or $\mathrm{cl}(S)$), is the intersection of all closed sets containing S.

- The closure of a set is closed
- A closed set is its own closure
- Examples
  - $\overline{(a,b)} = [a,b]$
  - $\overline{\emptyset} = \emptyset$

Closure

Lemma

$\overline{S}$ is the set of limits of convergent sequences in S.

Proof.

Let $\{x_n\}$ be a convergent sequence in S with limit x. If C is any closed set containing S, then $\{x_n\}$ is in C and by theorem 26, $x \in C$. Therefore, $x \in \overline{S}$.

Let $x \in \overline{S}$. For any $\epsilon > 0$, $N_\epsilon(x) \cap S \neq \emptyset$ because otherwise $N_\epsilon(x)^c$ is a closed set containing S, but not x. Therefore, we can construct a sequence $x_n \in S \cap N_{1/n}(x)$ that converges to x and is in S. □

Boundary

Definition

The **boundary** of a set S is $\overline{S} \cap \overline{S^c}$.

- Equivalently, $\overline{S} \setminus \mathrm{int}(S)$
- Boundary can be empty, e.g. $\mathbb{Q} \subset \mathbb{R}$, $\emptyset$

Boundary

Lemma

If x is in the boundary of S then $\forall \epsilon > 0$, $N_\epsilon(x) \cap S \neq \emptyset$ and $N_\epsilon(x) \cap S^c \neq \emptyset$.

Proof.

As in the proof of lemma 28, all ϵ-neighborhoods of $x \in \overline{S}$ must intersect with S. The same applies to S^c. □

Section 4

Compact sets

Open cover

Definition

An **open cover** of a set S is a collection of open sets, $\{G_\alpha\}$ $\alpha \in \mathcal{A}$ such that $S \subset \cup_{\alpha \in \mathcal{A}} G_\alpha$.

Example

Some open covers of $\mathbb{R}$ are:

- $\{\mathbb{R}\}$
- $\{(-\infty, 1), (-1, \infty)$
- $\{..., (-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3), ...\}$
- $\{(x, y) : x < y\}$

Example

Let X be a metric space and $A \subseteq X$ and $\epsilon > 0$. $\{N_\epsilon(x)\}_{x \in A}$ is an open cover of A

Compact sets

Definition

A set K is **compact** if every open cover of K has a finite subcover.

- Finite subcover means finite $G_{\alpha_1}, ...G_{\alpha_n}$ such that $K \subseteq \cup_{i=1}^n G_{\alpha_i}$
- e.g. $G_\alpha = (\alpha - \epsilon, \alpha + \epsilon)$ for $\alpha \in [0, 1]$ is an open cover of $[0, 1]$. One finite subcover is $(-\epsilon, \epsilon), (\epsilon/2 - \epsilon, \epsilon/2 + \epsilon), ...(1 - \epsilon, \epsilon)$ i.e. $G_{\alpha_i} = (i\epsilon/2 - \epsilon, i\epsilon/2 + \epsilon)$

Example

$\mathbb{R}$ is not compact.
$\{..., (-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3), ...\}$ is an infinite cover, but if we leave out any single interval (the one beginning with n) we will fail to cover some number ($n + 1$).

Example

Let $K = \{x\}$, a set of a single point. Then K is compact. Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of K. Then $\exists\ \alpha$ such that $x \in G_\alpha$. This single set is a finite subcover.

Example

Let $K = \{x_1, ..., x_n\}$ be a finite set. Then K is compact. Let $\{G_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of K. Then for each i, $\exists\ \alpha_i$ such that $x_i \in G_{\alpha_i}$. The collection $\{G_{\alpha_1}, ...G_{\alpha_n}\}$ is a finite subcover.

Example

$(0,1) \subseteq \mathbb{R}$ is not compact. $\{(1/n, 1)\}_{n=2}^{\infty}$ is an open cover, but there can be no finite subcover. Any finite subcover would have a largest n and could not contain, e.g. $1/(n+1)$.

Compact sets

- Definition a bit abstract, but will show that a set in $\mathbb{R}^n$ is compact iff it is closed and bounded, which we will prove in the next few slides

Definition

Let X be a metric space and $S \subseteq X$. S is **bounded** if $\exists x_0 \in S$ and $r \in \mathbb{R}$ such that

$$d(x, x_0) < r$$

for all $x \in S$.

Compact sets

Lemma

Let X be a metric space and $K \subseteq X$. If K is compact, then K is closed.

Proof.

Let $x \in K^c$. The collection $\{N_{d(x.y)/3}(y)\}$, $y \in K$ is an open cover of K. K is compact, so there is a finite subcover, $N_{d(x,y_1)/3}(y_1), ..., N_{d(x,y_n)/3}(y_n)$. For each i, $N_{d(x,y_i)/3}(y_i) \cap N_{d(x,y_i)/3}(x) = \emptyset$, so

$$\cap_{i=1}^{n} N_{d(x,y_i)/3}(x)$$

is an open neighborhood of x that is contained in K^c. K^c is open, so K is closed. □

Lemma

Let $K \subseteq X$ be compact. Then K is bounded.

Proof.

Pick $x_0 \in K$. $\{N_r(x_0)\}_{r \in \mathbb{R}}$ is an open cover of K, so there must be a finite subcover. The finite subcover has some maximum r^*. Then $K \subseteq N_{r^*}(x_0)$, so K is bounded. □

Theorem (Heine-Borel)

A set $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

1. Compact $\Rightarrow$ closed and bounded shown by last two lemmas (regardless of whether $S \subseteq \mathbb{R}^n$ or some other space)

□

Proof.

2 Closed and bounded $\Rightarrow$ compact

- Bounded so subset of cube, $[-a, a]^n$, for some a

2.1 Show $[-a, a]^n$ compact

- Suppose not, infinite cover of $[-a, 0]$ or $[0, a]$, repeat k times to get infinite cover of closed interval I_k of length $a2^{-k}$
- $\cap_{k=1}^{\infty} I_k \neq \emptyset$ because I_k closed and nested
- But eventually $I_k \subset N_\epsilon(x)$ for any $\epsilon > 0$ and we have a finite subcover

2.2 Closed subset of compact set is compact

□

- Always compact $\Rightarrow$ close and bounded
- In $\mathbb{R}^n$ closed and bounded $\Rightarrow$ compact
- In infinite dimensional spaces, a set can be closed and bounded but not compact

Example

$\ell^\infty = \{(x_1, x_2, ...) : \sup_i |x_i| < \infty$ with norm $\|x\| = \sup_i |x_i|$

- $e_i =$ all 0s except for a 1 in the ith position
- $E = \{e_i\}_{i=1}^\infty$ is closed and bounded
- E is not compact

Sequential compactness

Definition

Let X be a metric space and $K \subseteq X$. K is **sequentially compact** if every sequence in K has an accumulation point in K.

Example

- $[0, 1] \subseteq \mathbb{R}$ is sequentially compact
- $\mathbb{N} \subseteq \mathbb{R}$ is not sequentially compact
- $(0, 1) \subseteq \mathbb{R}$ is not sequentially compact

Compact $\Rightarrow$ sequentially compact

Lemma

Let X be a metric space and $K \subseteq X$ be compact. Then K is sequentially compact.

Proof.

- Given $\{x_n\}_{n=1}^{\infty}$, construct Cauchy sub-sequence:
  - Pick any $\epsilon > 0$, $N_\epsilon(x)$, $x \in K$ is an open cover of K, so there is a finite subcover, so $\exists\, x^*$ s.t. infinite $x_n \in N_\epsilon(x^*)$
  - Let n_1 be smallest n s.t. $x_n \in N_\epsilon(x^*)$
  - Repeat with $\tilde{K} = \overline{N_\epsilon(x^*)} \cap K$ instead of K and $\epsilon/2$ instead of ϵ
- Conclude K sequentially compact

$\square$

Sequentially compact $\Rightarrow$ compact

Theorem

Let X be a metric space and $K \subseteq X$. K is compact if and only if K is sequentially compact.

Proof.

- Already showed compact $\Rightarrow$ sequentially compact
- See notes for proof that sequentially compact $\Rightarrow$ compact

$\square$

Theorem (Bolzano-Weierstrass)

A set $S \subseteq \mathbb{R}^n$ is closed and bounded if and only if it is sequentially compact.

- In $\mathbb{R}^n$, compact, sequentially compact, and closed and bounded are all equivalent

Corollary

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Compactness

- $S \subseteq \mathbb{R}^n$ compact if
  1. Every open cover has a finite subcover
  2. Closed and bounded
  3. Every sequence in S has a convergent subsequence with its limit in S
- In metric spaces that are not $\mathbb{R}^n$, 1 and 3 are still equivalent, but 2 is not