LIMITS AND TOPOLOGY OF METRIC SPACES

PAUL SCHRIMPF
SEPTEMBER 26, 2013
UNIVERSITY OF BRITISH COLUMBIA
ECONOMICS 526

This lecture focuses on sequences, limits, and topology. Similar material is covered in chapters 12 and 29 of Simon and Blume, or 1.3 of Carter.

1. SEQUENCES AND LIMITS

A **sequence** is a list of elements, $\{x_1, x_2, ...\}$ or $\{x_n\}_{n=1}^{\infty}$ or sometimes just $\{x_n\}$. Although the notation for a sequence is similar to the notation for a set, they should not be confused. Sequences are different from sets in that the order of elements in a sequence matters, and the same element can appear many times in a sequence. Some examples of sequences with $x_i \in \mathbb{R}$ include

(1) $\{1, 1, 2, 3, 5, 8, ...\}$
(2) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$
(3) $\{\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, ...\}$

Some sequences, like 2, have elements that all get closer and closer to some fixed point. We say that these types of sequences converge. A sequence that does not converge diverges. Some divergent sequences like 1, increase without bound. Other divergent sequences, like 3, are bounded, but they do not converge to any single point.

To analyze sequences with elements that are not necessarily real numbers, we need to be able to say how far apart the entries in the sequence are.

Definition 1.1. A **metric space** is a set, X, and function $d : X \times X \to \mathbb{R}$ called a **metric** (or distance) such that $\forall x, y, z \in X$

(1) $d(x, y) > 0$ unless $x = y$ and then $d(x, x) = 0$
(2) (symmetry) $d(x, y) = d(y, x)$
(3) (triangle inequality) $d(x, y) \le d(x, z) + d(z, y)$.

Example 1.1. $\mathbb{R}$ is a metric space with $d(x, y) = |x - y|$.

Example 1.2. Any normed vector space is a metric space with $d(x, y) = \|x - y\|$.

The most common metric space that we will encounter will be $\mathbb{R}^n$ with the Euclidean metric, $d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Definition 1.2. A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space **converges** to x if $\forall \epsilon > 0$ $\exists N$ such that

$$d(x_n, x) < \epsilon$$

for all $n \ge N$. We call x the **limit** of $\{x_n\}_{n=1}^{\infty}$ and write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Example 1.3. The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\} = \{1/n\}_{n=1}^{\infty}$ converges. To see this, take any $\epsilon > 0$. Then $\exists$ N such that $1/N < \epsilon$. For all $n \geq N$, $d(1/n, 0) = 1/n < \epsilon$.

If a sequence does not converge, it diverges.

Example 1.4. The Fibonacci sequence, $\{1, 1, 2, 3, 5, 8, ...\}$ diverges.

Definition 1.3. *a is an* **accumulation point** *of $\{x_n\}_{n=1}^{\infty}$ if $\forall \epsilon > 0$ $\exists$ infinitely many x_i such that*

$$d(a, x_i) < \epsilon.$$

Example 1.5. The sequence $\{\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, ...\}$ has two accumulation points, 1 and -1.

The limit of any convergent sequence is an accumulation point of the sequence. In fact, it is the only accumulation point.

Lemma 1.1. *If $x_n \to x$, then x is the only accumulation point of $\{x_n\}_{n=1}^{\infty}$.*

Proof. Let $\epsilon > 0$ be given. By the definition of convergence, $\exists N$ such that

$$d(x_n, x) < \epsilon$$

for all $n \geq N$. $\{n \in \mathbb{N} : n \geq N\}$ is infinite, so x is an accumulation point.

Suppose x' is another accumulation point. Then $\forall \epsilon > 0$ $\exists N$ and N' such that if $n \geq N$ and $n \geq N'$, then $d(x_n, x) < \epsilon/2$ and $d(x_n, x') < \epsilon/2)$. By the triangle inequality, $d(x, x') \leq d(x_n, x') + d(x_n, x) < \epsilon$. Since this inequality holds for any ϵ, it must be that $d(x, x') = 0$. d is a metric, so then $x = x'$, and the limit of sequence is the sequence's unique accumulation point. □

The third example of a sequence at the start of this section, 3, shows that the converse of this lemma is false. Not every accumulation point is a limit.

Definition 1.4. Given $\{x_n\}_{n=1}^{\infty}$ and any sequence of positive integers, $\{n_k\}$ such that $n_1 < n_2 < ...$ we call $\{x_{n_k}\}$ a **subsequence** of $\{x_n\}_{n=1}^{\infty}$.

In example 3, there are two accumulation points, -1 and 1, and you can find subsequences that converge to these points.

Lemma 1.2. *Let a be an accumulation point of $\{x_n\}$. Then $\exists$ a subsequence that converges to a.*

Proof. We can construct a subsequence as follows. Let $\{\epsilon_k\}$ be a sequence that converges to zero with $\epsilon_k > 0 \forall k$, (for example, $\epsilon_k = 1/k$). By the definition of accumulation point, for each ϵ_k $\exists$ infinitely many x_n such that

$$d(x_n, a) < \epsilon_k \tag{1}$$

Pick any x_{n_1} such that (1) holds for ϵ_1. For $k > 1$, pick $n_k \neq n_j$ for all $j < k$ and such that (1) holds for ϵ_k. Such an n_k always exists because there are infinite x_n that satisfy (1). By construction, $\lim_{k\to\infty} x_{n_k} = a$ (you should verify this using the definition of limit). □

Convergence of sequences is often preserved by arithmetic operations, as in the following two theorems.

Theorem 1.1. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a normed vector space V. If $x_n \to x$ and $y_n \to y$, then*

$$x_n + y_n \to x + y.$$

Proof. Let $\epsilon > 0$ be given. Then $\exists\ N_x$ such that for all $n \geq N_x$,

$$d(x_n, x) < \epsilon/2,$$

and $\exists N_y$ such that for all $n \geq N_y$,

$$d(y_n, y) < \epsilon/2.$$

Let $N = \max\{N_x, N_y\}$. Then for all $n \geq N$,

$$\begin{aligned} d(x_n + y_n, x + y) = \|(x_n + y_n) - (x + y)\| \leq & \|x_n - x\| + \|y_n - y\| \\ < & \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

□

Theorem 1.2. *Let $\{x_n\}$ be a sequence in a normed vector space with scalar field $\mathbb{R}$ and let $\{c_n\}$ be a sequence in $\mathbb{R}$. If $x_n \to x$ and $c_n \to c$ then*

$$x_n c_n \to xc.$$

Proof. On problem set. □

In fact, in the next lecture we will see that if $f(\cdot, \cdot)$ is continuous, then $\lim f(x_n, y_n) = f(x, y)$. The previous two theorems are examples of this with $f(x, y) = x + y$ and $f(c, x) = cx$, respectively.

1.1. **Series.** Infinite sums or series are formally defined as the limit of the sequence of partial sums.

Definition 1.5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a normed vector space. Let $s_n = \sum_{i=1}^{n} x_i$ denote the sum of the first n elements of the sequence. We call s_n the nth partial sum. We define the sum of all the x_is as

$$\sum_{i=1}^{\infty} x_i \equiv \lim_{n \to \infty} s_n$$

This is called a(n infinite) **series**.

Example 1.6. Let $\beta \in \mathbb{R}$. $\sum_{i=0}^{\infty} \beta^i$ is called a geometric series. Geometric series appear often in economics, where β will be the subjective discount factor or perhaps $1/(1 + r)$. Notice that

$$\begin{aligned} s_n =& 1 + \beta + \beta^2 + \cdots + \beta^n \\ =& 1 + \beta(1 + \beta + \cdots + \beta^{n-1}) \\ =& 1 + \beta(1 + \beta + \cdots + \beta^{n-1} + \beta^n) - \beta^{n+1} \\ s_n(1 - \beta) =& 1 - \beta^{n+1} \\ s_n =& \frac{1 - \beta^{n+1}}{1 - \beta}, \end{aligned}$$

so,

$$\begin{aligned}\sum_{i=0}^{\infty} \beta^i &= \lim s_n \\ &= \lim \frac{1-\beta^{n+1}}{1-\beta} \\ &= \frac{1}{1-\beta} \text{ if } |\beta| < 1.\end{aligned}$$

1.2. **Cauchy sequences.** We have defined convergent sequences as ones whose entries all get close to a fixed limit point. This means that all the entries of the sequence are also getting closer together. You might imagine a sequence where the entries get close together without necessarily reaching a fixed limit.

Definition 1.6. A sequence $\{x_n\}_{n=1}^{\infty}$ is a **Cauchy** sequence if for any $\epsilon > 0$ $\exists N$ such that for all $i, j \geq N$, $d(x_i, x_j) < \epsilon$.

It turns that in $\mathbb{R}^n$ Cauchy sequences and convergent sequences are the same. This is a consequence of $\mathbb{R}$ having the least upper bound property.

Theorem 1.3. *A sequence in $\mathbb{R}^n$ converges if and only if it is a Cauchy sequence.*

There is a proof of this in Chapter 29.1 of Simon and Blume. If you want more practice with the sort of proofs in this lecture, it would be good to read that section. The convergence of Cauchy sequences in the real numbers is a consequence of the least upper bound property that we discussed in lecture 1. Cauchy sequences do not converge in all metric spaces. For example, the rational numbers are a metric space, and any sequence of rationals that converges to an irrational number in $\mathbb{R}$ is a Cauchy sequence in $\mathbb{Q}$ but has no limit in $\mathbb{Q}$. Having Cauchy sequences converge is necessary for proving many theorems, so we have a special name for metric spaces where Cauchy sequences converge.

Definition 1.7. A metric space, X, is **complete** if every Cauchy sequence of points in X converges in X.

Completeness is important for so many results that complete version of vector spaces are named after the mathematicians who first studied them extensively. A **Banach space** is a complete normed vector space. A **Hilbert space** is a complete inner product space. Since any inner product space is a normed vector space with norm $\|x\| = \sqrt{\langle x, x \rangle}$, any Hilbert space is also a Banach space.

Example 1.7. $\mathbb{R}^n$ is a Hilbert space.

Example 1.8. $\ell^p = \{(x_1, x_2, ...)\ s.t.\ x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ with norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

is a Banach space.

ℓ^2 with
$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$$
is a Hilbert space.

Showing that ℓ^p is complete is slightly tricky because you have deal with a sequence of $\mathbf{x}_i \in \ell^p$, each element of which is itself an infinite sequence. You should not worry if you have difficulty following the rest of this example.

To show that ℓ^p is complete, let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Denote the elements of $\mathbf{x}_i$ by $x_{i1}, x_{i2}, ...$. First, let's show that for any n, $x_{1n}, x_{2n}, ...$ is a Cauchy sequence in $\mathbb{R}$. Let $\epsilon > 0$. Since $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N_\epsilon$ such that for all $i, j \geq N_\epsilon$,
$$\left\|\mathbf{x}_i - \mathbf{x}_j\right\| < \epsilon.$$
Since
$$\left\|\mathbf{x}_i - \mathbf{x}_j\right\|^p = \left(\sum_{m=1}^{\infty} \left|x_{im} - x_{jm}\right|^p\right)$$
All terms in the sum on the right are non-negative and the sum includes $\left|x_{in} - x_{jn}\right|$, so
$$\left|x_{in} - x_{jn}\right|^p \leq \left\|\mathbf{x}_i - \mathbf{x}_j\right\|^p$$
$$\left|x_{in} - x_{jn}\right| \leq \left\|\mathbf{x}_i - \mathbf{x}_j\right\|$$
Therefore, $\left|x_{in} - x_{jn}\right| < \epsilon$ for all $i, j \geq N_\epsilon$, i.e. $x_{1n}, x_{2n}, ...$ is a Cauchy sequence in $\mathbb{R}$. $\mathbb{R}$ is complete, so it has some limit. Denote the limit by x_n^*.

Now we will show that $\mathbf{x}^* = (x_1^*, x_2^*, ...)$ is the limit of $\{\mathbf{x}_n\}_{n=1}^{\infty}$. First, we should show that $\mathbf{x}^* \in \ell^p$. Let
$$s_m^* = \sum_{n=1}^{m} |x_n^*|^p.$$
We need to show that $\lim s_m^*$ exists. Since $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is Cauchy, $\exists j$ such that if $i \geq j$, $\left\|\mathbf{x}_i - \mathbf{x}_j\right\| < 1$. Using the triangle inequality,
$$\|\mathbf{x}_i\| \leq \left\|\mathbf{x}_i - \mathbf{x}_j\right\| + \left\|\mathbf{x}_j\right\| = 1 + \left\|\mathbf{x}_j\right\| \equiv M$$
for all $i \geq j$ and some fixed j. Thus, $\|\mathbf{x}_i\| \leq M$ for some constant M and all $i \geq j$. Then,
$$s_m^* = \lim_{i\to\infty} \sum_{n=1}^{m} |x_{in}|^p \leq M^p$$
for all m. s_m^* is a bounded weakly increasing sequence in $\mathbb{R}$, so it must converge.[1]

[1]Let $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}$ and suppose $x_1 \leq x_2 \leq x_3 \leq ...$ and $\{x_n\}_{n=1}^{\infty}$ is bounded, then we will show $\{x_n\}_{n=1}^{\infty}$ converges. Suppose not. Then the sequence has no accumulation points. In particular, x_i is not an accumulation point of the sequence for any i i.e. there is an $\epsilon > 0$ such that for all i there are finitely many j with $d(x_i, x_j) < \epsilon$. Then we can construct a subsequence by choosing j_k such that $j_k > j_{k-1}$ and $|x_{j_k} - x_{j_{k-1}}| > \epsilon$. But then
$$\begin{aligned} x_{j_k} =& x_{j_1} + (x_{j_2} - x_{j_1}) + (x_{j_3} - x_{j_2}) + ... + (x_{j_k} - x_{j_{k-1}}) \\ \geq& x_{j_1} + (k-1)\epsilon \end{aligned}$$
which is not bounded.

Finally, we should show that $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges to $\mathbf{x}^*$. Let $\epsilon > 0$. Since the original sequence is Cauchy, there is a N such that if $i, j > N$, then

$$\sum_{m=1}^{M} |x_{im} - x_{jm}|^p \leq \|\mathbf{x}_i - \mathbf{x}_j\|^p < \epsilon$$

for all M. Therefore,

$$\lim_{j\to\infty} \sum_{m=1}^{M} |x_{im} - x_{jm}|^p = \sum_{m=1}^{M} |x_{im} - x_m^*| < \epsilon$$

for all $i \geq N$ and all M. Thus,

$$\|\mathbf{x}_i - \mathbf{x}^*\| = \lim_{M\to\infty} \sum_{m=1}^{M} |x_{im} - x_m^*| < \epsilon$$

for all $i \geq N$, so the sequence converges.

2. Open sets

Definition 2.1. Let X be a metric space and $x \in X$. A **neighborhood** of x is the set

$$N_\epsilon(x) = \{y \in X : d(x,y) < \epsilon.$$

A neighborhood is also called an open ϵ**-ball** of x and written $B_\epsilon(x)$.

Definition 2.2. A set, $S \subseteq X$ is **open** if $\forall x \in S$, $\exists\, \epsilon > 0$ such that

$$N_\epsilon(x) \subset S.$$

For every point in an open set, you can find a small neighborhood around that point such that the neighborhood lies entirely within the set.

Example 2.1. Any open interval, $(a,b) = \{x \in \mathbb{R} : a < x < b\}$, is an open set.

Example 2.2. Any linear subspace of dimension $k < n$ in $\mathbb{R}^n$ is not open.

Theorem 2.1.

(1) *Any union of open sets is open. (finite or infinite)*
(2) *The* finite *intersection of open sets is open.*

Proof. Let S_j, $j \in J$ be a collection of open sets. Pick any $j_0 \in J$. If $x \in \cup_{j\in J} S_j$, then there must be $\epsilon_{j_0} > 0$ such that $N_{\epsilon_{j_0}}(x) \subset S_{j_0}$. It is immediate that $N_{\epsilon_{j_0}}(x) \subset \cup_{j\in J} S_j$ as well.

Let $S_1, .., S_k$ be a finite collection of open sets. For each i $\exists \epsilon_i > 0$ such that $N_{\epsilon_i}(x) \subset S_i$. Let $\underline{\epsilon} = \min_{i\in\{1,\ldots,k\}} \epsilon_i$. Then $\underline{\epsilon} > 0$ since it is the minimum of a finite set of positive numbers. Also, $N_{\underline{\epsilon}}(x) \subset S_i$ for each i, so $N_{\underline{\epsilon}}(x) \subset \cap_{i=1}^{k} S_i$. $\square$

Definition 2.3. The **interior** of a set A is the union of all open sets contained in A. It is denoted as $\text{int}(A)$.

From the previous, theorem, we know that the interior of any set is open.

Example 2.3. Here some examples of the interior of sets in $\mathbb{R}$.

(1) $A = (a,b)$, $\text{int}(A) = (a,b)$.

(2) $A = [a, b]$, $\text{int}(A) = (a, b)$.
(3) $A = \{1, 2, 3, 4, ...\}$, A $= \emptyset$

3. Closed sets

A closed set is the opposite of an open set.

Definition 3.1. A set $S \subseteq X$ is closed if its complement, X^c, is open.

Theorem 3.1.

(1) *The intersection of any collection of closed sets is closed.*
(2) *The union of any finite collection of closed sets is closed.*

Proof. Let C_j, $j \in J$ be a collection of closed sets. Then $\left(\cap_{j \in J} C_j\right)^c = \cup_{j \in J} C_j^c$. C_j^c are open, so by theorem 2.1, $\cup_{j \in J} C_j^c = \left(\cap_{j \in J} C_j\right)^c =$ is open.

The proof of part 2 is similar. □

Example 3.1 (Closed sets)**.** Some examples of closed sets include

(1) $[a, b] \subseteq \mathbb{R}$
(2) Any linear subspace of $\mathbb{R}^n$
(3) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

Closed sets can also be defined as sets that contain the limit of any convergent sequence in the set. Simon and Blume use this definition. The next theorem shows that their definition is equivalent to ours.

Theorem 3.2. *Let $\{x_n\}$ be any convergent sequence with each element contained in a set C. Then* $\lim x_n = x \in C$ *for all such $\{x_n\}$ if and only if C is closed.*

Proof. First, we will show that any set that contains the limit points of all its sequences is closed. Let $x \in C^c$. Consider $N_{1/n}(x)$. If for any n, $N_{1/n}(x) \subset C^c$, then C^c is open, and C is closed as desired. If for all n, $N_{1/n}(x) \not\subset C^c$, then $\exists y_n \in N_{1/n}(x) \cap C$. The sequence $\{y_n\}$ is in C and $y_n \to x$. However, by assumption C contains the limit of any sequence within it. Therefore, there can be no such x, and C^c must be open and C is closed.

Suppose C is closed. Then C^c is open. Let $\{x_n\}$ be in C and $x_n \to x$. Then $d(x_n, x) \to 0$, and for any $\epsilon > 0$, $\exists x_n \in N_\epsilon(x)$. Hence, there can be no ϵ neighborhood of x contained in C^c. C^c is open by assumption, so $x \notin C^c$ and it must be that $x \in C$. □

Definition 3.2. The **closure** of a set S, denoted by $\overline{S}$ (or $\text{cl}(S)$), is the intersection of all closed sets containing S.

Example 3.2. If S is closed, $\overline{S} = S$.

Example 3.3. $\overline{(0, 1]} = [0, 1]$

Lemma 3.1. *$\overline{S}$ is the set of limits of convergent sequences in S.*

Proof. Let $\{x_n\}$ be a convergent sequence in S with limit x. If C is any closed set containing S, then $\{x_n\}$ is in C and by theorem 3.2, $x \in C$. Therefore, $x \in \overline{S}$.

Let $x \in \overline{S}$. For any $\epsilon > 0$, $N_\epsilon(x) \cap S \neq \emptyset$ because otherwise $N_\epsilon(x)^c$ is a closed set containing S, but not x. Therefore, we can construct a sequence $x_n \in S \cap N_{1/n}(x)$ that converges to x and is in S. $\square$

Example 3.4. $\overline{\{1/n\}_{n\in\mathbb{N}}} = \{0, 1, 1/2, 1/3, ...\}$

Definition 3.3. The **boundary** of a set S is $\overline{S} \cap \overline{S^c}$.

Example 3.5. The boundary of $[0, 1]$ is $\{0, 1\}$.

Example 3.6. The boundary of the unit ball, $\{x \in \mathbb{R}^2 : \|x\| < 1\}$ is the unit circle, $\{x \in \mathbb{R}^2 : \|x\| = 1\}$.

Lemma 3.2. *If x is in the boundary of S then $\forall \epsilon > 0$, $N_\epsilon(x) \cap S \neq \emptyset$ and $N_\epsilon(x) \cap S^c \neq \emptyset$.*

Proof. As in the proof of lemma 3.1, all ϵ-neighborhoods of $x \in \overline{S}$ must intersect with S. The same applies to S^c. $\square$

4. Compact sets

Definition 4.1. An **open cover** of a set S is a collection of open sets, $\{G_\alpha\}$ $\alpha \in \mathcal{A}$ such that $S \subset \cup_{\alpha\in\mathcal{A}} G_\alpha$.

Example 4.1. Some open covers of $\mathbb{R}$ are:

- $\{\mathbb{R}\}$
- $\{(-\infty, 1), (-1, \infty)$
- $\{..., (-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3), ...\}$
- $\{(x, y) : x < y\}$

The first two are finite open covers since they consist of finitely many open sets. The third is a countably infinite open cover. The fourth is an uncountably infinite open cover.

Example 4.2. Let X be a metric space and $A \subseteq X$. The set of open balls of radius ϵ centered at all points in A is an open cover of A. If A is finite / countable / uncountable, then this open cover will also be finite / countable / uncountable.

Open covers of the form in the previous example are often used to prove some property applies to all of A by verifying the property in each small $N_\epsilon(x)$. Unfortunately, this often involves taking a maximum or sum of something for each set in the open cover. When the open cover is infinite, it can be hard to ensure that the infinite sum or maximum stays finite. When we have a finite open cover, we know that things will remain finite.

Definition 4.2. A set K is **compact** if every open cover of K has a finite subcover.

By a finite subcover, we mean that there is finite set $G_{\alpha_1}, ...G_{\alpha_k}$ such that $S \subset \cup_{j=1}^k G_{\alpha_j}$. Compact sets are a generalization of finite sets. Many facts that are obviously true of finite sets are also true for compact sets, but not true for infinite sets that are not compact. Suppose we want to show a set has some property. If the set is compact, we can cover it with a finite number of small ϵ balls and then we just need to show that each small ball has the property we want. We will see many concrete examples of this technique in the next few weeks.

Example 4.3. $\mathbb{R}$ is not compact. $\{..., (-3,-1), (-2,0), (-1,1), (0,2), (1,3), ...\}$ is an infinite cover, but if we leave out any single interval (the one beginning with n) we will fail to cover some number $(n+1)$.

Example 4.4. Let $K = \{x\}$, a set of a single point. Then K is compact. Let $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of K. Then $\exists\ \alpha$ such that $x \in G_\alpha$. This single set is a finite subcover.

Example 4.5. Let $K = \{x_1, ..., x_n\}$ be a finite set. Then K is compact. Let $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of K. Then for each i, $\exists\ \alpha_i$ such that $x_i \in G_{\alpha_i}$. The collection $\{G_{\alpha_1}, ...G_{\alpha_n}\}$ is a finite subcover.

Example 4.6. $(0,1) \subseteq \mathbb{R}$ is not compact. $\{(1/n, 1)\}_{n=2}^{\infty}$ is an open cover, but there can be no finite subcover. Any finite subcover would have a largest n and could not contain, e.g. $1/(n+1)$.

Example 4.7. Let $x \in V$, a normed vector space. Let $K = \{x\frac{1}{2}, x\frac{2}{3}, x\frac{3}{4}, ...\}$. Then K is not compact. Consider the open cover $N_{\|x\|\frac{1}{3(n+2)^2}}(x\frac{n}{n+1})$ for $n = 1, 2, ...$ Assuming $x \neq 0$, each of these neighborhoods contains exactly one point of K, so there is no finite subcover.

Before using compactness, let's investigate how being compact relates to other properties of sets, such as closed/open.

Lemma 4.1. *Let X be a metric space and $K \subseteq X$. If K is compact, then K is closed.*

Proof. Let $x \in K^c$. The collection $\{N_{d(x.y)/3}(y)\}$, $y \in K$ is an open cover of K. K is compact, so there is a finite subcover, $N_{d(x,y_1)/3}(y_1), ..., N_{d(x,y_n)/3}(y_n)$. For each i, $N_{d(x,y_i)/3}(y_i) \cap N_{d(x,y_i)/3}(x) = \emptyset$, so

$$\cap_{i=1}^{n} N_{d(x,y_i)/3}(x)$$

is an open neighborhood of x that is contained in K^c. K^c is open, so K is closed. □

Lemma 4.2. *Let X be a metric space, $C \subseteq K \subseteq X$. If K is compact and C is closed. Then C is also compact.*

Proof. Let $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover for C. Then $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ plus C^c is an open cover for K. Since K is compact there is a finite subcover. Since $C \subseteq K$, the finite subcover also covers C. Therefore, C is compact. □

The definition of compactness is somewhat abstract. We just saw that compact sets are always closed. Another property of compact sets is that they are bounded.

Definition 4.3. Let X be a metric space and $S \subseteq X$. S is **bounded** if $\exists x_0 \in S$ and $r \in \mathbb{R}$ such that

$$d(x, x_0) < r$$

for all $x \in S$.

A bounded set is one that fits inside an open ball of finite radius. For subsets of $\mathbb{R}$ this definition is equivalent to there being a lower and upper bound for the set. For subsets of a normed vector space, if S is bounded then there exists some M such that $\|x\| < M$ for all $x \in S$.

Lemma 4.3. *Let $K \subseteq X$ be compact. Then K is bounded.*

Proof. Pick $x_0 \in K$. $\{N_r(x_0)\}_{r \in \mathbb{R}}$ is an open cover of K, so there must be a finite subcover. The finite subcover has some maximum r^*. Then $K \subseteq N_{r^*}(x_0)$, so K is bounded. □

This lemma along with lemma 4.1 show that if a set is compact then it is also closed and bounded. In $\mathbb{R}^n$, the converse is also true.

Theorem 4.1 (Heine-Borel)**.** *A set $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. We already showed that if S is compact, then it is closed and bounded.

Now suppose S is closed and bounded. Since S is bounded, it is a subset of some n-dimensional cube, say $[-a, a]^n$ (i.e. the set of all vectors $x = (x_1, .., x_n)$ with $-a \leq x_i \leq a$). We will show $[-a, a]^n$ is compact, and then use the fact that a closed subset of a compact set is compact.

Let's just show $[-a, a]^n$ is compact for $n = 1$. The argument for larger n is similar, but the notation is more cumbersome. If $[-a, a]$ is not compact, then there is an infinite open cover with no finite subcover, say $\{G_\alpha\}_{\alpha \in \mathcal{A}}$. If we cut the interval into two halves, $[-a, 0]$ and $[0, a]$, at least one of them must have no finite subcover. We can repeat this argument many times to get nested closed intervals of length $a/(2^k)$ for any k. Call the kth interval I_k. We claim that $\cap_{k=1}^{\infty} I_k \neq \emptyset$. To show this take the sequence of lower endpoints of the intervals, call it $\{x_n\}_{n=1}^{\infty}$. This is a Cauchy sequence, so it converges to some limit, x_0. Also, for any k, $\{x_n\}_{n=k}^{\infty}$ is a sequence in I_k. I_k is closed so $x_0 \in I_k$. Thus $x_0 \in \cap_{k=1}^{\infty} I_k$. On the other hand, $I_k \subset \cup_{\alpha \in \mathcal{A}} G_\alpha$ for all k. Therefore, x_0 must be in some open G_α as part of this cover. Then $\exists \epsilon > 0$ such that $N_\epsilon(x_0) \subset G_\alpha$. However, for $k > 1/\epsilon$, $I_k \subset N_\epsilon(x_0) \subset G_\alpha$, and then I_k has a finite subcover. Therefore, $[-a, a]$ must be compact. The argument for $n > 1$ is very similar. For $n = 2$, we would divide the square $[-a, a]^2$ into four smaller squares. For $n = 3$, we would divide the cube into eight smaller cubes. In general we would divide the hypercube $[-a, a]^n$ into 2^n hypercubes with half the side length. □

You may wonder whether closed and bounded sets are always compact. We know that all finite dimensional real vector spaces are isomorphic to $\mathbb{R}^n$. In any such space, sets are compact iff they are closed and bounded. However, in infinite dimensional spaces, there are closed and bounded sets that are not compact. The argument in the previous proof does not apply to infinite-dimensional spaces because an infinite dimensional hypercube can only be divided into infinitely many hypercubes with half the side length.

Example 4.8. $\ell^\infty = \{(x_1, x_2, ...) : \sup_i |x_i| < \infty$ with norm $\|x\| = \sup_i |x_i|$ is a normed vector space. Let e_i be the element of all 0s except for the ith position, which is 1. Then $E = \{e_i\}_{i=1}^{\infty}$ is closed and bounded. However, E is not compact because $\{N_{1/2}(e_i)\}_{i=1}^{\infty}$ is an open cover with no finite subcover.

We saw that closed sets contain the limit points of all their convergent sequences. There is also a relationship between compactness and sequences.

Definition 4.4. Let X be a metric space and $K \subseteq X$. K is **sequentially compact** if every sequence in K has an accumulation point in K.

Sometimes this definition is written as: K is sequentially compact if every sequence in K has a subsequence that converges in K. Compactness implies sequential compactness.

Lemma 4.4. *Let X be a metric space and $K \subseteq X$ be compact. Then K is sequentially compact.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K. Pick any $\epsilon > 0$, $N_\epsilon(x)$, $x \in K$ is an open cover of K, so there is a finite subcover. Therefore, one of the ϵ neighborhoods must contain an infinite number of the elements from the sequence. Call this neighborhood $N_\epsilon(x_1^*)$. Pick the smallest n such that $x_n \in N_\epsilon(x_1^*)$ and call it n_1. $\overline{N_\epsilon(x_1^*)} \cap K$ is a closed subset of the compact set K, so is itself compact. Repeat the above argument with $\epsilon/2$ in place of ϵ and $\overline{N_\epsilon(x_1^*)} \cap K$ in place of K to find an n_2, n_3, etc. Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ is a Cauchy sequence, so it converges. Its limit is an accumulation point. Its limit must be in K because K is compact, and so, closed. Therefore, K is sequentially compact. □

In $\mathbb{R}^n$, a set is sequentially compact iff it is compact iff it is closed and bounded.

Theorem 4.2 (Bolzano-Weierstrass)**.** *A set $S \subseteq \mathbb{R}^n$ is closed and bounded if and only if it is sequentially compact.*

Proof. Let S be closed and bounded. By the Heine-Borel theorem (4.1), S is compact. By lemma 4.4, S is sequentially compact.

Let S be sequentially compact. Let $\{x_n\}$ be a convergent sequence in S. Its limit is an accumulation point, so it must be in S. Therefore, S is closed. To show S is bounded, pick $x_0 \in S$. Suppose $\exists x_1 \in S$ such that $d(x_1, x_0) \geq 1$, and $x_2 \in S$ such that $d(x_2, x_0) \geq 2$ etc. This sequence is not Cauchy because of the reverse triangle inequality,

$$d(x_i, x_j) \geq \left| d(x_i, x_0) - d(x_j, x_0) \right| = |i - j|$$

Therefore, this would be a sequence in S with no accumulation points. Therefore, it must not always be possible to find such x_n. In other words, S must be bounded. □

Comment 4.1. This theorem is sometimes stated as "each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence." As an exercise, you may want to verify that this statement is equivalent to the one above.

Simon and Blume also prove this theorem in chapter 29.2. They do not prove the Heine-Borel theorem first though, so their proof is of the Bolzano-Weierstrass theorem is longer. Perhaps unsurprisingly, the details of their proof are somewhat similar to our proof of the Heine-Borel theorem.

In $\mathbb{R}^n$, compactness, sequential compactness, and closed and bounded are all the same. In general metric spaces, this need not be true. We saw above that in infinite dimensional normed vector spaces, there are closed and bounded sets which are not compact. However it is always true that sequential compactness and compactness are the same for metric spaces. We already showed that compactness implies sequential compactness. The proof that sequential compactness implies compactness is a somewhat long and difficult, and you may want to skip it unless you are especially interested.

Theorem 4.3. *Let X be a metric space and $K \subseteq X$. K is compact if and only if K is sequentially compact.*

Proof. Lemma 4.4 shows that if K is compact, then K is sequentially compact.

Suppose every sequence in K has a convergent subsequence with a limit point in K. Let G_α, $\alpha \in \mathcal{A}$ be an open cover of K. $\mathcal{A}$ could be uncountable, so we will begin by showing that there must be a countable subcover. Let $n = 1$. Pick $x_1 \in K$. If possible choose $x_2 \in K$ such that $d(x_1, x_2) \geq 1/n$. Repeat this process, choosing x_j in K such that $d(x_j, x_i) \geq 1/n$ for each $i < j$. Eventually this will no longer be possible because otherwise we could construct a sequence with no convergent subsequence. When it is no longer possible, set $n = n + 1$. This gives a countable collection of open neighborhoods $N_{1/n}(x_i)$ that cover K for each n and get arbitrarily small as n increases. Call these neighborhoods η_j for $j = 1, 2, ...$ Let J be set of all η_j such that $\eta_j \subseteq G_\alpha$ for some α. J is a subset of a countable set, so J is countable. Note that $\cup_{j\in J}\eta_j \supset K$ because if $x \in K$, then $x \in G_\alpha$ for some α, and then $\exists\epsilon$ such that $N_\epsilon(x) \in G_\alpha$ and $\exists j$ s.t. $\eta_j \subset N_\epsilon(x)$. Finally, for each $j \in J$ choose G_{α_j} such that $\eta_j \subseteq G_{\alpha_j}$. Such α_j exist by construction. Also $\cup_{j\in J}G_{\alpha_j} \supset \cup_{j\in J}\eta_j \supset K$. So G_{α_j} is a countable subcover.

If G_{α_j} has no finite subcover, then for each n,

$$F_n = (\cup_{i=1}^n G_{\alpha_i})^c \cap K$$

is not empty (if it were empty, then $\cup_{i=1}^n G_{\alpha_i}$ would be a finite subcover). Choose $x_n \in F_n$. Then $\{x_n\}$ is a sequence in K, and it must have a convergent subsequence with a limit, x_0, in K. However, each $F_{i+1} \subset F_i$ and F_i are all closed. Therefore, the sequence $\{x_j\}_{j=i}^\infty$ is also in F_i and so is its limit. Then $x_0 \in \cap_{i=1}^\infty F_i$. However,

$$\cap_{i=1}^\infty F_i = (\cup_{i=1}^\infty G_{\alpha_i})^c \cap K,$$

but G_{α_i} is a countable cover of K, which implies

$$\cap_{i=1}^\infty F_i = (\cup_{i=1}^\infty G_{\alpha_i})^c \cap K = \emptyset$$

and we have a contradiction. Therefore, G_{α_j} must have a finite subcover, and K is compact.

□

Comment 4.2. There are non-metric spaces where sequential compactness and compactness are not equivalent. One can define open sets on a space without a metric by simply specifying which sets are open and making it such that theorem 2.1 holds. Such a space is called a topological space. You can then define closed sets, compact sets, and sequential compactness in terms of open sets. On the problem set, you will see that you can define continuity of functions in terms of open and closed sets. Topology is the branch of mathematics that studies topological spaces. One interesting observation is that on $\mathbb{R}^n$, if a set is open with respect to some p-norm, then it is also open with respect to any other p-norm. Thus, we say that $\mathbb{R}^n$ with the p-norms are topologically equivalent or homeomorphic. Properties like continuity and compactness are the same regardless of what p-norm we use.

As far as I know, topological spaces that are not metric spaces do not come up very often in economics, so we will not be studying them.

To review, in $\mathbb{R}^n$ a set is compact if any of the following three things hold:

(1) For every open cover there exists a finite subcover,

(2) Every sequence in the set has a convergent subsequence, or
(3) The set is closed and bounded.

In infinite dimensional spaces, closed and bounded sets need not be compact, but compact sets are always closed and bounded. In any metric space, a set is compact iff it is sequentially compact.