Functions

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Section 1

Definition and examples

Functions

- A **function** from a set A to a set B is a rule that assigns to each $a \in A$ one and only one $b \in B$
- $f : A \to B$
- A is the domain
- B is the target space
- **Image** of A under f

$$\{y \in B : f(x) = y \text{ for some } x \in A\}$$

Examples

1. Production functions: $f : \mathbb{R}^2 \to \mathbb{R}$
   - Linear $f(x_1, x_2) = a_1 x_1 + a_2 x_2$
   - Cobb-Douglas: $f(x_1, x_2) = K x_1^{\alpha_1} x_2^{\alpha_2}$
   - Constant elasticity of substitution: $f(x_1, x_2) = K(c_1 x_1^{-a} + c_2 x_2^{-a})^{-b/a}$
2. Utility functions: $u : \mathbb{R}^T \to \mathbb{R}$
   - Constant relative risk aversion: $u(c_1, ..., c_T) = \sum_{t=1}^T \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$
   - Constant absolute risk aversion: $u(c_1, ..., c_T) = \sum_{t=1}^T \beta^t (-e^{-\alpha c_t})$
3. Demand function with constant elasticity, $D : \mathbb{R}^3 \to \mathbb{R}^2$

$$D(p_1, p_2, y) = \begin{pmatrix} M p_1^{\alpha_{11}} p_2^{\alpha_{12}} y^{\beta_1} \\ M p_1^{\alpha_{21}} p_2^{\alpha_{22}} y^{\beta_2} \end{pmatrix}$$

Visualizing function

Definition

The **level sets** of a function $f : X \to Y$ are sets of the form

$$\{x \in X : f(x) = y\}$$

for some fixed $y \in Y$.

- Indifference curves
- Isoquants

Figure : CES, $a = 2$, $b = \frac{4}{5}$

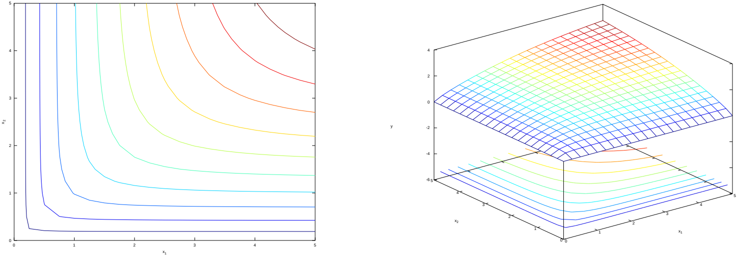


Figure : CRRA, $\gamma = 2$, $\beta = 0.95$

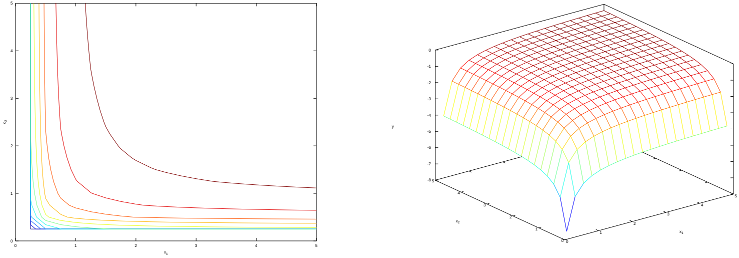
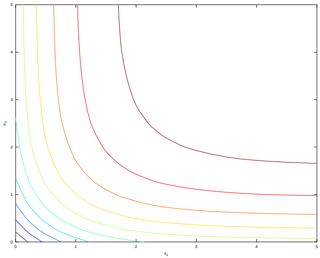
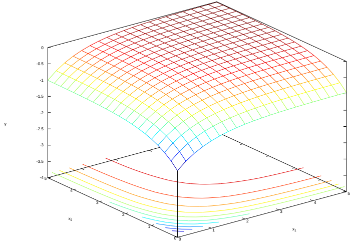


Figure : CARA, $\alpha = 1$, $\beta = 0.95$





Section 2

Special types of functions

Types of functions

Definition

A function $f : V \to W$ where V and W are vector spaces is **linear** if f preserves addition and scalar multiplication, ie

- $f(x + y) = f(x) + f(y)$
- $f(\alpha x) = \alpha f(x)$

Quadratic

- $\mathbb{R} \to \mathbb{R}$: $a_0 + a_1 x + a_2 x^2$

Definition

$q : \mathbb{R}^n \to R$ is a **quadratic** if

$$q(x_1, ..., x_n) = a_0 + \sum_{i=1, j \geq i}^{n} a_{ij} x_i x_j$$

- Written using matrix:

$$q(x_1, ..., x_n) = a_0 + x^T A x$$

where $A = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} \cdots & \frac{1}{2}a_{2n} & \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \cdots & \cdots & a_{nn} \end{pmatrix}$ (not unique)

Polynomials

Definition

A **monomial** $f : \mathbb{R}^n \to R$ is any function of the form

$$f(x_1, ..., x_n) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

where a_i are nonnegative integers.
$\sum_{i=1}^n a_i$ is the **degree** of the monomial.
A **polynomial** $f : \mathbb{R}^n \to \mathbb{R}$ is the sum of finitely many monomials, i.e.

$$f(x_1, ..., x_n) = \sum_{k=1}^{k} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}$$

The maximum degree of the monomials making up a polynomial is the degree of the polynomial.

Definition

A function $f : V \to W$ which V and W are real vector spaces is **homogenous of degree k** if

$$f(tx) = t^k f(x)$$

for all $x \in V$, $t \in \mathbb{R}$.

Example

Linear functions are homogenous of degree 1.

Example

A production function that is homogenous of degree 1 has constant returns to scale because doubling each of the inputs doubles the output. A production function that is homogenous of degree less than 1 has decreasing returns to scale. A production of that is homogenous of degree greater than 1 has increasing returns to scale.

Example

An affine transformation, $f(x) = Ax + b$, is not homogenous if $b \neq 0$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$. f is **strictly increasing** if for all $x_1 > x_2$, $f(x_1) > f(x_2)$.
f is **strictly decreasing** if for all $x_1 > x_2$, $f(x_1) < f(x_2)$.
f is **strictly monontonic** if it is either strictly increasing or decreasing.
If the strict inequalities ($<$ and $>$) are replaced with weak inequalities ($\leq$ and $\geq$), then we would say f is **weakly** increasing / decreasing / monotonic.

Definition

Let $f : V \to \mathbb{R}$ where V is a vector space. f is **homothetic** if $\exists$ a homogenous $g : V \to \mathbb{R}$ and a monotonic $h : \mathbb{R} \to \mathbb{R}$ asuch that $h \circ g : V \to R$ defined by $(h \circ g)(x) = h(g(x))$ is equal to f.

Section 3

Continuous functions

Continuous functions

Definition

A function $f : X \to Y$ where X and Y are metric spaces is **continuous** if whenever $\{x_n\}_{n=1}^{\infty}$ converges to x in X, then $f(x_n) \to f(x)$ in Y.

- No jumps or holes.

$\epsilon - \delta$ definition of continuity

Lemma

$f : X \to Y$ is continuous at x if and only if for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$.

Proof.

On problem set. □

Topological definition of continuity

Definition

Let $f : X \to Y$. The **preimage** of $V \subseteq Y$ is the set in X, $f^{-1}(V)$ defined by

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

Lemma

$f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open for all open $V \subseteq Y$.

Corollary

$f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is closed for all closed $V \subseteq Y$.

Continuity and arithmetic

Theorem

Let $f : X \to Y$ and $g : X \to Y$ be continuous and X and Y be vector spaces. Then $(f+g)(x) = f(x) + g(x)$ is continuous.

Proof.

If f and g are continuous, then by definition $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ whenever $x_n \to x$. From the previous lecture the limit of a (finite) sum is the sum of limits, so $f(x_n) + g(x_n) \to f(x) + g(x)$, and $f + g$ is continuous. □

- Same for subtraction, multiplication, etc

Continuity and composition

Definition

$(f \circ g)(x) = f(g(x))$ is the **composition** of f and g.

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be continuous where X, Y, and Z are metric spaces. Then $f \circ g$ is continuous.

Proof.

Let $x_n \to x$. g is continuous, so $g(x_n) \to g(x)$. f is also continuous, so $f(g(x_n)) \to f(g(x))$. □

One-to-one

Definition

$f : X \to Y$ is **one-to-one** or **injective** if for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2)$$

if and only if $x_1 = x_2$.

- f is injective if for each $y \in Y$, the set $\{x : f(x) = y\}$ is either a singleton or empty
- If f is one-to-one, then $f(x) = b$ has at most one solution

Onto

Definition

$f : X \to Y$ is **onto** or **surjective** if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.

- If f is onto, then $f(x) = b$ has at least one solution

Definition

When f is one-to-one and onto, f is **bijective**.

Definition

If $f : X \to Y$ is bijective, then the **inverse** of f, written f^{-1} satisfies

$$f(f^{-1}(y)) = y$$

and

$$f^{-1}(f(x)) = x.$$

Section 4

Correspondences

Correspondence

Definition

A **correspondence** from a set X to a set Y, is a rule that assigns to each a $x \in X$ a subset of Y. We denote a correspondence by $\phi : X \rightrightarrows Y$.

Example (Budget correspondence)

- n goods with prices $p \in \mathbb{R}^n$.
- Income of m, a consumer can afford $\chi(p, m) = \{x \in X \subseteq \mathbb{R}^n : p'x \leq m\}$
- Consumer's problem

$$\max_{x \in \chi(p,m)} u(x)$$

- Indirect utility function

$$v(p, m) = \max_{x \in \chi(p,m)} u(x).$$

- The demand correspondence (usually function) is

$$x^*(p, m) = \arg\max_{x \in \chi(p,m)} u(x).$$

Continuity

INSERT PICTURE OF upper and lower hemicontinuous

Continuity

Definition

A correspondence, $\phi : X \rightrightarrows Y$ is **upper hemicontinuous** at x if for all sequences $x_n \to x$ and $y_n \in \phi(x_n)$ with $y_n \to y$, then $y \in \phi(x)$.

Definition

A correspondence, $\phi : X \rightrightarrows Y$ is **lower hemicontinuous** at x if for all sequences $x_n \to x$ and $y \in \phi(x)$, there exists a subsequence, x_{nk} and $y_k \in \phi(x_{nk})$ with $y_k \to y$.

Definition

A correspondence is **continuous** at x if it is both upper and lower hemicontinuous at x