Differential Calculus

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Section 1

Derivatives

Partial derivatives

Definition

Let $f : \mathbb{R}^n \to R$. The ith **partial derivative** of f is

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{h \to 0} \frac{f(x_{01}, ..., x_{0i}, ...x_{0n}) - f(x_0)}{h}.$$

Example

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a production function. Then we call $\frac{\partial f}{\partial x_i}$ the **marginal product** of x_i. If f is Cobb-Douglas, $f(k, l) = Ak^\alpha l^\beta$, where k is capital and l is labor, then the marginal products of capital and labor are

$$\frac{\partial f}{\partial k}(k, l) = A\alpha k^{\alpha-1} l^\beta$$
$$\frac{\partial f}{\partial l}(k, l) = A\beta k^\alpha l^{\beta-1}.$$

Example

If $u : \mathbb{R}^n \to \mathbb{R}$ is a utility function, then we call $\frac{\partial u}{\partial x_i}$ the marginal utility of x_i. If u is CRRA,

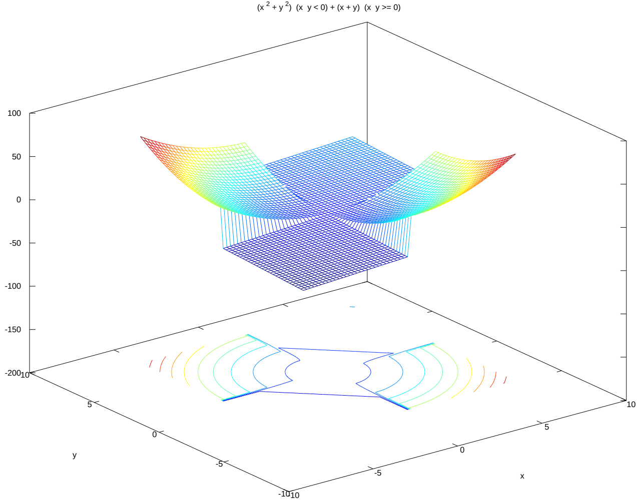
$$u(c_1, ..., c_T) = \sum_{t=1}^{T} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

then the marginal utility of consumption in period t is

$$\frac{\partial u}{\partial c_t} = \beta^t c_t^{-\gamma}.$$

Example (Demand elasticities)

- $q_1 : \mathbb{R}^3 \to \mathbb{R}$ is a demand function with three arguments: own price p_1, the price of another good, p_2, and consumer income, y
- Own price elasticity
$$\epsilon_{q_1,p_1} = \frac{\partial q_1}{\partial p_1} \frac{p_1}{q_1(p_1, p_2, y)}.$$
- Cross price elasticity
$$\epsilon_{q_1,p_2} = \frac{\partial q_1}{\partial p_2} \frac{p_2}{q_1(p_1, p_2, y)}.$$
- Income elasticity of demand
$$\epsilon_{q_1,y} = \frac{\partial q_1}{\partial y} \frac{y}{q_1(p_1, p_2, y)}.$$



$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } xy < 0 \\ x + y \text{ if } xy \geq 0 \end{cases}$$

Total derivative

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$. The **derivative** (or total derivative or differential) of f at x_0 is a linear mapping, $Df_{x_0} : \mathbb{R}^n \to \mathbb{R}^1$ such that

$$\lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - Df_{x_0} h|}{\|h\|} = 0.$$

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at x_0, then $\frac{\partial f}{\partial x_i}(x_0)$ exists for each i and

$$Df_{x_0} h = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x_0) & \cdots & \frac{\partial f}{\partial x_n}(x_0) \end{pmatrix} h.$$

Proof.

The definition of derivative says that

$$\lim_{t \to 0} \frac{|f(x_0 + e_i t) - f(x_0) - Df_{x_0}(e_i t)|}{\|e_i t\|} = 0$$

$$\lim_{t \to 0} \frac{f(x_0 + e_i t) - f(x_0) - tDf_{x_0} e_i}{|t|} = 0$$

This implies that

$$f(x_0 + e_i t) - f(x_0) = tDf_{x_0} e_i + r_i(x_0, t)$$

with $\lim_{t \to 0} \frac{|r_i(x_0,t)|}{|t|} = 0$. Dividing by t,

$$\frac{f(x_0 + e_i t) - f(x_0)}{t} = Df_{x_0} e_i + \frac{r_i(x_0, t)}{t}$$

and taking the limit

$$\lim_{t \to 0} \frac{f(x_0 + e_i t) - f(x_0)}{t} = Df_{x_0} e_i$$

□

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose its partial derivatives exist and are continuous in $N_\delta(x_0)$ for some $\delta > 0$. Then f is differentiable at x_0 with

$$Df_{x_0} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x_0) & \cdots \frac{\partial f}{\partial x_n}(x_0) \end{pmatrix}.$$

Corollary

$f : \mathbb{R}^n \to \mathbb{R}$ has a continuous derivative on an open set $U \subseteq \mathbb{R}^n$ if and only if its partial derivatives are continuous on U

Mean value theorem

Theorem (mean value)

Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be in $C^1(U)$ for some open U. Let $x, y \in U$ be such that the line connecting x and y, $\ell(x, y) = \{z \in \mathbb{R}^n : z = \lambda x + (1-\lambda)y, \lambda \in [0,1]\}$, is also in U. Then there is some $\bar{x} \in \ell(x, y)$ such that

$$f(x) - f(y) = Df_{\bar{x}}(x - y).$$

Results needed to prove mean value theorem I

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $K \subset \mathbb{R}^n$ be compact. Then $\exists x^* \in K$ such that $f(x^*) \geq f(x) \forall x \in K$.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$. we say that f has a local maximum at x if $\exists \delta > 0$ such that $f(y) \leq f(x)$ for all $y \in N_\delta(x)$.

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose f has a local maximum at x and is differentiable at x. Then $Df_x = 0$.

Proof of mean value theorem

Proof.

Let $g(z) = f(y) - f(z) + \frac{f(x)-f(y)}{x-y}(z-y)$. Note that $g(x) = g(y) = 0$. The set $ell(x, y)$ is closed and bounded, so it is compact. Hence, $g(z)$ must attain its maximum on $\ell(x, y)$, say at $\bar{x}$, then the previous theorem shows that $Dg_{\bar{x}} = 0$. Simple calculation shows that

$$Dg_{\bar{x}} = -Df_{\bar{x}} + \frac{f(x) - f(y)}{x - y} = 0$$

so

$$Df_{\bar{x}}(x - y) = f(x) - f(y).$$

□

Functions from $\mathbb{R}^n \to \mathbb{R}^m$

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}^m$. The **derivative** (or total derivative or differential) of f at x_0 is a linear mapping, $Df_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\| f(x_0 + h) - f(x_0) - Df_{x_0} h \|}{\| h \|} = 0.$$

- Theorems 6 and 7 sill hold
- The total derivative of f can be represented by the m by n matrix of partial derivatwives (the **Jacobian**),

$$Df_{x_0} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}.$$

Corollary (mean value for $\mathbb{R}^n \to \mathbb{R}^m$)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be in $C^1(U)$ for some open U. Let $x, y \in U$ be such that the line connecting x and y, $\ell(x, y) = \{z \in \mathbb{R}^n : z = \lambda x + (1-\lambda)y, \lambda \in [0, 1]\}$, is also in U. Then there are $\bar{x}_j \in \ell(x, y)$ such that

$$f_j(x) - f_j(y) = Df_{j\,\bar{x}_j}(x - y)$$

and

$$f(x) - f(y) = \begin{pmatrix} Df_{1\,\bar{x}_1} \\ \vdots \\ Df_{m\,\bar{x}_m} \end{pmatrix} (x - y).$$

Chain rule

- $f(g(x)) = f'(g(x))g'(x)$.

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^k \to \mathbb{R}^n$. Let g be continuously differentiable on some open set U and f be continuously differentiable on $g(U)$. Then $h : \mathbb{R}^k \to \mathbb{R}^m$, $h(x) = f(g(x))$ is continuously differentiable on U with

$$Dh_x = Df_{g(x)} Dg_x$$

Proof.

Let $x \in U$. Consider

$$\frac{\|f(g(x+d)) - f(g(x))\|}{\|d\|}.$$

Since g is differentiable by the mean value theorem, $g(x+d) = g(x) + Dg_{\bar{x}(d)} d$, so

$$\begin{aligned}\|f(g(x+d)) - f(g(x))\| = & \left\|f(g(x) + Dg_{\bar{x}(d)} d) - f(g(x))\right\| \\ \leq & \|f(g(x) + Dg_x d) - f(g(x))\| + \epsilon\end{aligned}$$

where the inequality follows from the the continuity of Dg_x and f, and holds for any $\epsilon > 0$. f is differentiable, so

$$\lim_{Dg_x d \to 0} \frac{\left\|f(g(x) + Dg_x d) - f(g(x)) - Df_{g(x)} Dg_x d\right\|}{\|Dg_x d\|} = 0$$

Using the Cauchy-Schwarz inequality, $\|Dg_x d\| \leq \|Dg_x\| \, \|d\|$, so

$$\lim_{d \to 0} \frac{\left\|f(g(x) + Dg_x d) - f(g(x)) - Df_{g(x)} Dg_x d\right\|}{\|d\|} = 0.$$

□

Higher order derivatives

- Take higher order derivatives of multivariate functions just like of univariate functions.
- If $f : \mathbb{R}^n \to \mathbb{R}^m$, then is has nm partial first derivatives. Each of these has n partial derivatives, so f has $n^2 m$ partial second derivatives, written $\frac{\partial^2 f_k}{\partial x_i \partial x_j}$.

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ *be twice continuously differentiable on some open set* U*. Then*

$$\frac{\partial^2 f_k}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f_k}{\partial x_j \partial x_i}(x)$$

for all i, j, k *and* $x \in U$*.*

Corollary

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be k times continuously differentiable on some open set U. Then

$$\frac{\partial^k f}{\partial x_1^{j_1} \times \cdots \times \partial x_n^{j_n}} = \frac{\partial^k f}{\partial x_{p(1)}^{j_{p(1)}} \times \cdots \times \partial x_{p(n)}^{j_{p(n)}}}$$

where $\sum_{i=1}^n j_i = k$ and $p : \{1, .., n\} \to \{1, ..., n\}$ is any permutation (i.e. reordering).

Taylor series

Theorem (Univarite Taylor series)

Let $f : \mathbb{R} \to \mathbb{R}$ be $k+1$ times continuously differentiable on some open set U, and let a, $a+h \in U$. Then

$$f(a+h) = f(a) + f'(a)h + \frac{f^2(a)}{2}h^2 + ... + \frac{f^k(a)}{k!}h^k + \frac{f^{k+1}(\bar{a})}{(k+1)!}h^{k+1}$$

where $\bar{a}$ is between a and h.

Theorem (Multivariate Taylor series)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be k times continuously differentiable on some open set U and $a, a+h \in U$. Then there exists a k times continuously differentiable function $r_k(a,h)$ such that

$$f(a+h) = f(a) + \sum_{\sum_{i=1}^n j_i = 1}^{k} \frac{1}{k!} \frac{\partial^{\sum j_i} f}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(a) h_1^{j_1} h_2^{j_2} \cdots h_n^{j_n} + r_k(a,h)$$

and $\lim_{h \to 0} \|r_k(a,h)\| \, \|h\|^k = 0$

Proof.

Follows from the mean value theorem. For $k = 1$, the mean value theorem says that

$$
\begin{aligned}
f(a+h) - f(a) = & Df_{\bar{a}} h \\
f(a+h) = & f(a) + Df_{\bar{a}} h \\
= & f(a) + Df_a h + \underbrace{(Df_{\bar{a}} - Df_a)h}_{r_1(a,h)}
\end{aligned}
$$

Df_a is continuous as a function of a, and as $h \to 0$, $\bar{a} \to a$, so $\lim_{h\to 0} r_1(a,h) = 0$, and the theorem is true for $k = 1$. For general k, suppose we have proven the theorem up to $k - 1$. Then repeating the same argument with the $k - 1$st derivative of f in place of f shows that theorem is true for k. □

Section 2

Functions on vector spaces

Definition

Let $f : V \to W$. The Fréchet **derivative** of f at x_0 is a continuous[1] linear mapping, $Df_{x_0} : V \to W$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Df_{x_0} h\|}{\|h\|} = 0.$$

- Just another name for total derivative

[1] If V and W are finite dimensional, then all linear functions are continuous. In infinite dimensions, there can be discontinuous linear functions.

Example

Let $V = \mathcal{L}^\infty(0,1)$ and $W = \mathbb{R}$. Suppose f is given by

$$f(x) = \int_0^1 g(x(\tau), (\tau)) d\tau$$

for some continuously differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$. Then Df_x is a linear transformation from V to $\mathbb{R}$. How can we calculate Df_x?

Definition

Let $f : V \to W$, $v \in V$ and $x \in U \subseteq V$ for some open U. The **directional derivative** (or Gâteaux derivative when V is infinite dimensional) in direction v at x is

$$df(x; v) = \lim_{\alpha \to 0} \frac{f(x + \alpha v) - f(x)}{\alpha}.$$

where $\alpha \in \mathbb{R}$ is a scalar.

Relationship between directional and total derivative

Lemma

If $f : V \to W$ is Fréchet differentiable at x, then the Gâteaux derivative, $df(x; v)$, exists for all $v \in V$, and

$$df(x; v) = Df_x v.$$

Lemma

If $f : V \to W$ has Gâteaux derivatives that are linear in v and "continuous" in x in the sense that $\forall \epsilon > 0\ \exists \delta > 0$ such that if $\|x_1 - x\| < \delta$, then

$$\sup_{v \in V} \frac{\|df(x_1; v) - df(x; v)\|}{\|v\|} < \epsilon$$

then f is Fréchet differentiable with $Df_{x_0} v = df(x; v)$.

Calculating Fréchet derivative

Example

Let $V = \mathcal{L}^\infty(0,1)$ and $W = \mathbb{R}$. Suppose f is given by

$$f(x) = \int_0^1 g(x(\tau),(\tau))d\tau$$

- Directional (Gâteaux) derivatives:

$$\begin{aligned} df(x;v) = & \lim_{\alpha\to 0} \frac{\int_0^1 g(x(\tau)+\alpha v(\tau),\tau)d\tau}{\alpha} \\ = & \int_0^1 \frac{\partial g}{\partial x}(x(\tau),\tau)v(\tau)d\tau \end{aligned}$$

- Check that continuous and linear in v
- Or guess and verify that

$$Df_x(v) = \int_0^1 \frac{\partial g}{\partial x}(x(\tau),\tau)v(\tau)d\tau$$

satisfies

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Df_x(h)\|}{\|h\|} = 0$$