Implicit and inverse function theorems

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Economics 526

October 17, 2011

Section 1

Inverse functions

Inverse functions

- $f : \mathbb{R}^n \to \mathbb{R}^m$
- When can we solve $f(x) = y$ for x?
- Use derivative and what we know about linear equations to get a local answer

Inverse functions

- If $f(a) = b$, expand f around a.

$$f(x) = f(a) = Df_a(x-a) + r_1(a, x-a) = y$$

- If r_1 is small, we almost have a system of linear equations

$$Df_a x = y - f(a) + Df_a a$$

- Know:
    - Solution exists if

$$\mathrm{rank} Df_a = \mathrm{rank}\begin{pmatrix} Df_a & y - f(a) + Df_a a \end{pmatrix}$$

    - Solution unique and if $\mathrm{rank} Df_a = n$

Theorem (Inverse function)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open set E. Let $a \in E$, $f(a) = b$, and Df_a be invertible . Then

1. *there exist open sets U and V such that $a \in U$, $b \in V$, f is one-to-one on U and $f(U) = V$, and*
2. *the inverse of f exists and is continuously differentiable on V with derivative $\left(Df_{f^{-1}(x)}\right)^{-1}$.*

Proof.

Choose λ such that $\lambda \left\|Df_a^{-1}\right\| = 1/2$. Since Df_a is continuous, there is an open neighborhood U of a such that $\|Df_x - Df_a\| < \lambda$ for all $x \in U$. For any $y \in \mathbb{R}^n$, consider $\varphi^y(x) = x + Df_a^{-1}(y - f(x))$. Note that

$$
\begin{aligned}
D\varphi^y_x = & I - Df_a^{-1} Df_x \\
= & Df_a^{-1}(Df_a - Df_x) \leq \left\|Df_a^{-1}\right\| \lambda = \frac{1}{2}
\end{aligned} \tag{1}
$$

Then, by the mean value theorem for $x_1, x_2 \in U$,

$$
\|\varphi^y(x_1) - \varphi^y(x_2)\| = \left\|D\varphi^y_{\bar{x}}(x_1 - x_2)\right\| \leq \frac{1}{2}\|x_1 - x_2\|.
$$

. φ^y is a contraction, so it has a unique fixed point. When $\varphi^y(x) = x$, it must be that $y = f(x)$. Thus for each $y \in f(U)$, there is at most one x such that $f(x) = y$. That is, f is one-to-one on U. This proves the first part of the theorem and that f^{-1} exists. □

Section 2

Contraction mappings

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}^n$. f is a **contraction mapping** on $U \subseteq \mathbb{R}^n$ if for all $x, y \in U$,

$$\|f(x) - f(y)\| \leq c \|x - y\|$$

for some $0 \geq c < 1$.

If f is a contraction mapping, then an x such that $f(x) = x$ is called a **fixed point** of the contraction mapping.

Lemma

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction mapping on $U \subseteq \mathbb{R}^n$. If $x_1 = f(x_1)$ and $x_2 = f(x_2)$ for some $x_1, x_2 \in U$, then $x_1 = x_2$.

Proof.

Since f is a contraction mapping,

$$\|f(x_1) - f(x_2)\| \leq c \|x_1 - x_2\|.$$

$f(x_i) = x_i$, so

$$\|x_1 - x_2\| \leq c \|x_1 - x_2\|.$$

Since $0 \geq c < 1$, the previous inequality can only be true if $\|x_1 - x_2\| = 0$. Thus, $x_1 = x_2$. □

Lemma

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction mapping on $U \subseteq \mathbb{R}^n$, and suppose that $f(U) \subseteq U$. Then f has a unique fixed point.

Proof.

Pick $x_0 \in U$. As in the discussion before the lemma, construct the sequence defined by $x_n = f(x_{n-1})$. Each $x_n \in U$ because $x_n = f(x_{n-1}) \in f(U)$ and $f(U) \subseteq U$ by assumption. Since f is a contraction on U, $\|x_{n+1} - x_n\| \leq c^n \|x_1 - x_0\|$, so $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, and $\{x_n\}$ is a Cauchy sequence. Let $x = \lim_{n\to\infty} x_n$. Then

$$\begin{aligned} \|x - f(x)\| &\leq \|x - x_n\| + \|f(x) - f(x_n)\| \\ &\leq \|x - x_n\| + c\,\|x - x_n\| \end{aligned}$$

$x_n \to x$, so for any $\epsilon > 0$ $\exists N$, such that if $n \geq N$, then $\|x - x_n\| < \frac{\epsilon}{1+c}$. Then,

$$\|x - f(x)\| < \epsilon$$

for any $\epsilon > 0$. Therefore, $x = f(x)$. □

Section 3

Implicit functions

Implicit functions

- Cannot always write conditions of a model as $f(x) = y$
- Often only $f(x, y) = c$.
- Using same sort of idea, can get x as a function of y.

- $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$
- Have a model that requires $f(x, y) = c$
- Know that $f(x_0, y_0) = c$
- Expand f around x_0 and y_0

$$f(x,y) = f(x_0,y_0) + \underbrace{D_x f_{(x_0,y_0)}}_{n \times n}(x - x_0) + \underbrace{D_y f_{(x_0,y_0)}}_{n \times m}(y - y_0) + r$$

- If r small enough,

$$\begin{aligned} f(x_0,y_0) + D_x f_{(x_0,y_0)}(x - x_0) + D_y f_{(x_0,y_0)}(y - y_0) \approx c \\ D_x f_{(x_0,y_0)}(x - x_0) \approx \left(c - f(x_0\right. \end{aligned}$$

a system of linear equations

Theorem (Implicit function)

Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be continuously differentiable on some open set E and suppose $f(x_0, y_0) = c$ for some $(x_0, y_0) \in E$, where $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$. If $D_x f_{(x_0,y_0)}$ is invertible, then there exists open sets $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^{n-k}$ with $(x_0, y_0) \in U$ and $y_0 \in W$ such that

1. *For each $y \in W$ there is a unique x such that $(x, y) \in U$ and $f(x, y) = c$.*
2. *Define this x as $g(y)$. Then g is continuously differentiable on W, $g(y_0) = x_0$, $f(g(y), y) = c$ for all $y \in W$, and $Dg_{y_0} = -\left(D_x f_{(x_0,y_0)}\right)^{-1} D_y f_{(x_0,y_0)}$*

Section 4

Applications

Application: Roy's identity

- $V(m,p)$ an **indirect utility function**

$$V(m,p) = \max_c U(c) \text{ s.t. } pc \leq m. \tag{2}$$

- **expenditure function**, $E(u,p)$

$$E(u,p) = \min_c pc \text{ s.t. } U(c) \geq u. \tag{3}$$

- Observe that $V(E(u,p),p) = u$ (if U continuous and $p \neq 0$)

Application: Roy's identity

- Differentiate

$$\frac{\partial V}{\partial m}(E(u,p),p)\frac{\partial E}{\partial p_i}(u,p) + \frac{\partial V}{\partial p_i}(E(u,p),p) = 0$$

$$\frac{\partial E}{\partial p_i}(u,p) = -\frac{\frac{\partial V}{\partial m}(E(u,p),}{\frac{\partial V}{\partial p_i}(E(u,p),}$$

- Shephard's lemma is

$$c_i^*(u,p) = \frac{\partial E}{\partial p_i}(u,p),$$

- Roy's identity is

$$c_i^*(m,p) = -\frac{\frac{\partial V}{\partial m}(m,p)}{\frac{\partial V}{\partial p_i}(m,p)}.$$

Comparative statics

- Finite horizon macro model.
- Production

$$y_t = A_t k_t^{\alpha}$$

- Budget

$$c_t + k_{t+1} = (1-\delta)k_t + A_t k_t^{\alpha}.$$

- Social planner's problem

$$\max_{\{c_t,k_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \text{ s.t. } c_t + k_{t+1} = (1-\delta)k_t + A_t k_t^{\alpha}.$$

- Lagrangian

$$\max_{\{c_t,k_t,\lambda_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} + \lambda_t(c_t + k_{t+1} - (1-\delta)k_t - A_t k_t^{\alpha})$$

- First order conditions

$$\begin{aligned}
[c_t] : \quad && \beta^t c_t^{-\gamma} =& \lambda_t \\
[k_t] : \quad && \lambda_{t-1} =& \lambda_t \left((1-\delta) + A_t \alpha k_t^{\alpha-1}\right) \\
[\lambda_t] : \quad && c_t + k_{t+1} =& (1-\delta)k_t + A_t k_t^{\alpha}
\end{aligned}$$

- Suppose A_t changes unexpectedly at time $T-1$
- Want to find the change in c_{T-1}, c_T, and k_T
- Relevant first order conditions

$$
\begin{aligned}
0 =& F(c_T, c_{T-1}, k_T, A_T, A_{T-1}, c_{T-2}, k_{T-1}) \\
=& \begin{pmatrix} c_{T-1} + k_T - (1-\delta)k_{T-1} - A_{T-1}k_{T-1}^{\alpha} \\ c_T - (1-\delta)k_T - A_T k_T^{\alpha} \\ c_{T-1}^{-\gamma} - c_T^{-\gamma}\beta\left((1-\delta) + A_T \alpha k_T^{\alpha-1}\right) \end{pmatrix}
\end{aligned}
$$

- The implicit function theorem says that

$$\begin{pmatrix} \frac{\partial c_{T-1}}{\partial A_{T-1}} \\ \frac{\partial c_T}{\partial A_{T-1}} \\ \frac{\partial k_T}{\partial A_{T-1}} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial c_{T-1}} & \frac{\partial F_1}{\partial c_T} & \frac{\partial F_1}{\partial k_T} \\ \frac{\partial F_2}{\partial c_{T-1}} & \frac{\partial F_2}{\partial c_T} & \frac{\partial F_2}{\partial k_T} \\ \frac{\partial F_3}{\partial c_{T-1}} & \frac{\partial F_3}{\partial c_T} & \frac{\partial F_3}{\partial k_T} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial A_{T-1}} \\ \frac{\partial F_2}{\partial A_{T-1}} \\ \frac{\partial F_3}{\partial A_{T-1}} \end{pmatrix}$$

$$= - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\gamma c_{T-1}^{-\gamma-1} & \gamma c_T^{-\gamma-1} \beta \left((1-\delta) + A_T \alpha k_T^{\alpha-1} \right) \end{pmatrix}$$

- Gaussian elimination:

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -(1-\delta) - \\
-\gamma c_{T-1}^{-\gamma-1} & \gamma c_T^{-\gamma-1}\beta\left((1-\delta)+A_T\alpha k_T^{\alpha-1}\right) & -c_T^{-\gamma}\beta A_T\alpha
\end{pmatrix}
$$

$$
\simeq \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -(1-\delta) - A \\
0 & \gamma c_T^{-\gamma-1}\beta\left((1-\delta)+A_T\alpha k_T^{\alpha-1}\right) & -c_T^{-\gamma}\beta A_T\alpha(\alpha-1)k
\end{pmatrix}
$$

$$
\simeq \begin{pmatrix}
1 & 0 & 1 & k_{T-1}^{\alpha} \\
0 & 1 & -(1-\delta) - A_T\alpha k_T^{\alpha-1} & 0 \\
0 & 0 & E & \gamma c_{t-1}^{-\gamma-1} k_{T-1}^{\alpha}
\end{pmatrix}
$$

where

$$
E = \left(\gamma c_T^{-\gamma-1}\beta\left((1-\delta)+A_T\alpha k_T^{\alpha-1}\right)\right)\left((1-\delta)+A_T\alpha k_T^{\alpha-1}\right)
$$

$$
- c_T^{-\gamma}\beta A_T\alpha(\alpha-1)k_T^{\alpha-2} + \gamma c_{T-1}^{-\gamma-1}.
$$

- Assume $\alpha \leq 1$, so that $E > 0$. Then,

$$\frac{\partial k_T}{\partial A_{T-1}} = \frac{\gamma c_{T-1}^{-\gamma-1} k_{T-1}^{\alpha}}{E} > 0$$

- $$\frac{\partial c_T}{\partial A_{T-1}} = \frac{\partial k_T}{\partial A_{T-1}} \left((1-\delta) + A_T \alpha_T k_T^{\alpha-1} \right),$$

- $$\begin{aligned} \frac{\partial c_{T-1}}{\partial A_{T-1}} =& k_{T-1}^{\alpha} - \frac{\partial k_T}{\partial A_{T-1}} \\ =& \frac{k_{T-1}^{\alpha} E - \gamma c_{T-1}^{-\gamma-1} k_{T-1}^{\alpha}}{E} \\ =& \frac{k_{T-1}^{\alpha} \left(\gamma c_T^{-\gamma-1} \beta \left((1-\delta) + A_T \alpha k_T^{\alpha-1} \right) \right) \left((1-\delta) + \right.}{E} \\ 0 \leq & \frac{\partial c_{T-1}}{\partial A_{T-1}} < k_{T-1}^{\alpha} \end{aligned}$$