Constrained optimization

Paul Schrimpf

UBC
Economics 526

October 17, 2012

Section 1

First order conditions

$$\max f(x) \text{ s.t. } h(x) = c$$

- $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$
- Draw picture of $n = 2$ and $m = 1$
  - At optimum, constraint tangent to level curve of function

$$\frac{(\partial f)/(\partial x_1)(x^*)}{(\partial f)/(\partial x_2)(x^*)} = \frac{(\partial h)/(\partial x_1)(x^*)}{(\partial h)/(\partial x_2)(x^*)}$$
$$\frac{(\partial f)/(\partial x_1)(x^*)}{(\partial h)/(\partial x_1)(x^*)} = \frac{(\partial f)/(\partial x_2)(x^*)}{(\partial h)/(\partial x_2)(x^*)} \equiv \mu$$

- Rewrite as

$$\frac{\partial f}{\partial x_1}(x^*) - \mu \frac{\partial h}{\partial x_1}(x^*) = 0 \quad (1)$$
$$\frac{\partial f}{\partial x_2}(x^*) - \mu \frac{\partial h}{\partial x_2}(x^*) = 0 \quad (2)$$
$$h(x^*) - c = 0 \quad (3)$$

- Lagrangian $L(x, \mu) \equiv f(x) - \mu(h(x) - c)$.

FOC with equality constraints

Theorem

Let $f : U \to \mathbb{R}$ and $h : U \to \mathbb{R}^m$ be continuously differentiable on $U \subseteq \mathbb{R}^n$. Suppose $x^* \in \text{interior}(U)$ is a local maximizer of f on U subject to $h(x) = c$. Also assume that Dh_{x^*} has rank m. Then there exists $\mu^* \in \mathbb{R}^m$ such that (x^*, μ^*) is a critical point of the Lagrangian,

$$L(x, \mu) = f(x) - \mu^T (h(x) - c).$$

i.e.

$$\frac{\partial L}{\partial x_i}(x^*, \mu^*) = \frac{\partial f}{\partial x_i} - \mu^{*T} \frac{\partial h}{\partial x_i}(x^*) = 0$$
$$\frac{\partial L}{\partial \mu_j}(x^*, \mu^*) = h(x^*) - c = 0$$

for $i = 1, ..., n$ and $j = 1, ..., m$.

Inequality constraints

$$\max_{x \in U} f(x) \text{ s.t. } g(x) \leq b.$$

- Binding constraints, $g_j(x^*) = b_j$ are just like equality constraints
$$Df_{x^*} - \lambda_j Dg_{j,x^*} = 0.$$
- Df_{x^*} is direction f increases, so $x^* + \delta Df_{x^*}$ must violate constraint or x^* cannot be a maximizer

$$\begin{aligned} g_j(x^* + \delta Df_{x^*}^T) &> b_j \\ g_j(x^*) + \delta Dg_{x^*} Df_{x^*}^T + o(\delta^2) &> b_j \\ Dg_{j,x^*} Df_{x^*}^T &> 0 \end{aligned}$$

- Combine with first order condition to get $\lambda_j > 0$
- Thus, $\lambda_j \geq 0$ and $\lambda_j = 0$ iff $g_j(x^*) < b_j$ (complementary slackness condition)

FOC with inequality constraints

Theorem

Let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}^m$ be continuously differentiable on $U \subseteq \mathbb{R}^n$. Suppose $x^* \in \mathrm{interior}(U)$ is a local maximizer of f on U subject to $g(x) \leq b$. Suppose that the first $k \leq m$ constraints, bind

$$g_j(x^*) = b_j$$

for $j = 1...k$ and that the Jacobian for these constraints, has rank k. Then, there exists $\lambda^* \in \mathbb{R}^m$ such that for

$$L(x, \lambda) = f(x) - \lambda^T (g(x) - b).$$

we have

$$\frac{\partial L}{\partial x_i}(x^*, \lambda^*) = \frac{\partial f}{\partial x_i} - \lambda^{*T} \frac{\partial g}{\partial x_i}(x^*) = 0$$
$$\lambda_j^* \frac{\partial L}{\partial \lambda_j}(x^*, \lambda^*) = \lambda_j^* \left(g(x^*) - c\right) = 0$$
$$\lambda_j^* \geq 0$$
$$g(x^*) \leq b$$

for $i = 1, ..., n$ and $j = 1, ..., m$.

Mixed equality and inequality constraints

- Similar result

Section 2

Second order conditions

Second order conditions

- Second order expansion of $f(x)$ around x^*.

$$
\begin{aligned}
f(x^* + v) - f(x^*) =& Df_{x^*} v + v^T D^2 f_{x^*} v + r(v, x^*) \\
=& v^T D^2 f_{x^*} v + r(v, x^*)
\end{aligned}
$$

- $x^* + v$ must satisfy the constraints

$$
h(x^* + v) = h(x^*) + Dh_{x*} v + r_h(v, x^*) = c.
$$

so $Dh_{x^*} v = 0$

- x^* is a local maximizer of f subject to $h(x) = c$ if

$$
v^T D^2 f_{x^*} v \leq 0
$$

for all v such that that $Dh_{x^*} v = 0$

Second order condition for constrained maximization

Theorem

Let $f : U \to \mathbb{R}$ be twice continuously differentiable on U, and $h : U \to \mathbb{R}^l$ and $g : U \to \mathbb{R}^m$ be continuously differentiable on $U \subseteq \mathbb{R}^n$. Suppose $x^* \in \mathrm{interior}(U)$ and there exists $\mu^* \in \mathbb{R}^l$ and $\lambda^* \in \mathbb{R}^m$ such that for

$$L(x, \lambda, \mu) = f(x) - \lambda^T (g(x) - b) - \mu^T (h(x) - c).$$

the first order condition is satisfied. Let B be the matrix of the derivatives of the binding constraints evaluated at x^*. If

$$v^T D^2 f_{x*} v < 0$$

for all $v \neq 0$ such that $Bv = 0$, then x^* is a strict local constrained maximizer for f subject to $h(x) = c$ and $g(x) \leq b$.

Definiteness on subspaces

Definition

Let A be an n by n symmetric matrix and B be m by n, then A is

- **Negative definite on** $\mathcal{N}(B)$ if $x^T A x < 0$ for all $x \in \mathcal{N}(B) \setminus \{0\}$
- **Positive definite on** $\mathcal{N}(B)$ if $x^T A x > 0$ for all $x \in \mathcal{N}(B) \setminus \{0\}$
- **Indefinite on** $\mathcal{N}(B)$ if $\exists\ x_1 \in \mathcal{N}(B) \setminus \{0\}$ s.t. $x_1^T A x_1 > 0$ and some other $x_2 \in \mathcal{N}(B) \setminus \{0\}$ such that $x_2^T A x_2 < 0$.

Checking definiteness using determinants

Theorem

Let A be an n by n symmetric matrix and B be m by n. Then A is negative definite on $\mathcal{N}(B)$ iff the last $n-m$ leading principal minors of

$$\begin{pmatrix} 0 & B \\ B & A \end{pmatrix}$$

alternate in sign, and the final $(n+m)$th principal minor has the same sign as $(-1)^n$.

Checking definiteness using eigenvalues

- Write $B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}$ with $\operatorname{rank} B = \operatorname{rank} B_1 = m$.

$$\begin{aligned} 0 =& Bx = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ x_1 =& -(B_1)^{-1} B_2 x_2 \end{aligned}$$

so $x \in \mathcal{N}(B)$ iff $x = \begin{pmatrix} -(B_1)^{-1} B_2 \\ I_{n-m} \end{pmatrix} x_2$ for some $x_2 \in \mathbb{R}^{n-m}$

- $x^T A x$ for $x \in \mathcal{N}(B)$

$$\begin{aligned} x^T A x =& x_2^T \begin{pmatrix} (B_1)^{-1} B_2 \\ I_{n-m} \end{pmatrix}^T \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix} \begin{pmatrix} (B_1)^{-1} B_2 \\ I_{n-m} \end{pmatrix} \\ =& x_2^T \underbrace{\left(B_2^T (B_1^T)^{-1} A_1 (B_1)^{-1} B_2 + B_2^T (B_1^T)^{-1} A_2 + A_2^T (B_1)^{-1} B_2 + A_3 \right)}_{\equiv C} x \end{aligned}$$

- A negative definite on $\mathcal{N}(B)$ iff C is negative definite on $\mathbb{R}^m$, i.e. C has negative eigenvalues

Section 3

Multiplier interpretation

Theorem (Multiplier interpretation)

Under the conditions of theorem 2, let $x^*(b, c)$ denote the solution of the constrained maximization problem,

$$\max_{x \in U} f(x) \text{ s.t. } g(x) \leq b$$
$$h(x) = c,$$

and let $\mu(b, c)$ and $\lambda(b, c)$ denote the corresponding Lagrange multipliers. The for each $j = 1..m$,

$$\frac{\partial}{\partial b_j} f(x^*(b, c)) = \lambda_j(b, c)$$

and for each $j = 1, ..., l$,

$$\frac{\partial}{\partial c_j} f(x^*(b, c)) = \mu_j(b, c).$$

Section 4

Envelope theorem

Envelope theorem

Let $f : U \times A \to \mathbb{R}$ where $U \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^k$. Consider

$$\max_{x \in U} f(x, \alpha).$$

Let $x^*(\alpha)$ be a local maximizer. Using the chain rule,

$$\begin{aligned}\frac{d}{d\alpha_j} f(x^*(\alpha), \alpha) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i^*}{\partial \alpha_j} + \frac{\partial f}{\partial \alpha_j} \\ &= \frac{\partial f}{\partial \alpha_j}(x^*(\alpha), \alpha)\end{aligned}$$

where the second line follows from the first order condition.

Envelope theorem

Let $f : U \times A \to \mathbb{R}$ and $h : U \times A \to \mathbb{R}^l$ where $U \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^k$. Consider

$$\max_{x \in U} f(x, \alpha) \text{ s.t. } h(x, \alpha) = 0.$$

Let $x^*(\alpha)$ be a local maximizer, and let $L(x^*(\alpha), \mu^*(\alpha), \alpha)$ be the Lagrangian. Using the chain rule,

$$\frac{d}{d\alpha_j} L(x^*(\alpha), \mu^*(\alpha), \alpha) = \sum_{i=1}^{n} \frac{\partial L}{\partial x_i} \frac{\partial x_i^*}{\partial \alpha_j} + \sum_{k=1}^{l} \frac{\partial L}{\partial \mu_k} \frac{\partial \mu_k^*}{\partial \alpha_j} + \frac{\partial L}{\partial \alpha_j}$$

$$= \frac{\partial L}{\partial \alpha_j}(x^*(\alpha), \mu^*(\alpha), \alpha)$$