Monotone comparative statics

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Section 1

Introduction

References

- Amir (2005) "Supermodularity and Complementarity in Economics: An Elementary Survey," *Southern Economic Journal* http://www.jstor.org/stable/20062066
- Carter sections 2.2.3, 6.2.3
- Milgrom and Shannon (1994) "Monotone Comparative Statics" *Econometrica* http://www.jstor.org/stable/2951479
- Athey, Milgrom, and Roberts *Robust Comparative Statics*

Introduction

- Problem:

$$x^*(\alpha) \in \arg\max_x f(x,\alpha) \text{ s.t. } g(x,\alpha) \leq 0$$

  How does $x^*(\alpha)$ vary with α

- Approaches:
  1. Explicitly solve for $x^*(\alpha)$
     - Often not possible
  2. Implicit function theorem
     - Apply to first order condition, if constraint does not bind,

$$0 = \frac{\partial f}{\partial x}(x^*, \alpha)$$
$$\frac{dx^*}{d\alpha} = -\left(\frac{\partial^2 f}{\partial x \partial \alpha}\right)^{-1} \frac{\partial^2 f}{\partial \alpha^2}$$

     - Requires f to be smooth, $\frac{\partial^2 f}{\partial x \partial \alpha} \neq 0$, constraint not binding, $x^*(\alpha)$ unique maximizer

Introduction

- Original problem:
$$\max_{x\in\mathcal{X}} f(x,\alpha) \text{ s.t. } g(x,\alpha)\leq 0$$
    - Same as
    $$\max_{x\in\mathcal{X}} h(f(x,\alpha)) \text{ s.t. } g(x,\alpha)\leq 0$$
    for any strictly increasing $h:\mathbb{R}\to\mathbb{R}$ including non differentiable and non continuous h
    - $$\max_{y\in\mathcal{Y}} f(k(y),\alpha)) \text{ s.t. } g(k(y),\alpha)\leq 0$$
    for any surjective $k:\mathcal{Y}\to\mathcal{X}$ including non differentiable and non continuous k
- $\Rightarrow$ Conclusions of comparative statics should not depend on smoothness, uniqueness of x^*, etc

Section 2

Monotone maximum theorem

Simple setup

$$x^*(\alpha) = \arg\max_x f(x,\alpha) \text{ s.t. } x \in [g(\alpha), h(\alpha)]$$

- $x, \alpha \in \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$, $g, h : \mathbb{R} \to \mathbb{R}$
- $x^*(\alpha)$ is set valued
- Define $\bar{x}(\alpha) = \sup_{x \in x^*(\alpha)} x$, $\underline{x}(\alpha) = \inf_{x \in x^*(\alpha)} x$
- Goal: find conditions for $\bar{x}(\alpha)$ to be increasing in α

$$x^*(\alpha) = \arg\max_x f(x,\alpha) \text{ s.t. } x \in [g(\alpha), h(\alpha)]$$

- $f(x,\alpha) = x$ or $-x$ implies $x^*(\alpha) = \{h(\alpha)\}$ or $\{g(\alpha)\}$, so need $h(\alpha)$ and $g(\alpha)$ to both be increasing in α
- Let $\alpha' > \alpha$ and suppose $\bar{x}(\alpha') < \bar{x}(\alpha)$ we want to find flexible assumptions that make this impossible
- $h(\alpha) \leq h(\alpha') \leq \bar{x}(\alpha') < \bar{x}(\alpha) \leq g(\alpha) \leq g(\alpha')$
- Maximization implies

$$f(\bar{x}(\alpha),\alpha) - f(\bar{x}(\alpha'),\alpha) \geq 0$$

and

$$f(\bar{x}(\alpha),\alpha') - f(\bar{x}(\alpha'),\alpha') \leq 0$$

- Define: f has **increasing differences** in x, α if for all $x' > x$, $\alpha' > \alpha$

$$f(x', \alpha') - f(x, \alpha') \geq f(x', \alpha) - f(x, \alpha)$$

- Increasing differences implies:

$$0 \geq f(\bar{x}(\alpha), \alpha') - f(\bar{x}(\alpha'), \alpha') \geq f(\bar{x}(\alpha), \alpha) - f(\bar{x}(\alpha'), \alpha) \geq 0$$

i.e. $f(\bar{x}(\alpha), \alpha') = f(\bar{x}(\alpha'), \alpha')$

- But then $\bar{x}(\alpha) \in x^*(\alpha')$ and $\bar{x}(\alpha) > \bar{x}(\alpha')$, contradicting $\bar{x}(\alpha') = \sup\{x^*(\alpha')\}$

Monotone maximum theorem on $\mathbb{R}$

Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g, h : \mathbb{R} \to \mathbb{R}$. *Define*

$$x^*(\alpha) = \mathop{\arg\max}_{x \in [g(\alpha), h(\alpha)]} f(x, \alpha).$$

If

1. f *has increasing differences*
2. g *and* h *are increasing*

then $\bar{x}(\alpha) = \sup\{x^*(\alpha)\}$ *and* $\underline{x}(\alpha) = \inf\{x^*(\alpha)\}$ *are increasing in* α

Increasing differences

- f has **increasing differences** in x, α if for all $x' > x$, $\alpha' > \alpha$

$$f(x', \alpha') - f(x, \alpha') \geq f(x', \alpha) - f(x, \alpha)$$

- Equivalently

$$f(x', \alpha') - f(x', \alpha) \geq f(x, \alpha') - f(x, \alpha)$$

- If f is twice continuously differentiable, then f has increasing differences iff $\frac{\partial^2 f}{\partial x \partial \alpha} \geq 0 \ \forall x, \alpha$

Example: normal good

$$\max_{x_1 \geq 0, x_2 \geq 0} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq m$$

- u increasing in x_2 implies constraint binds

$$\max_{x_1} u\left(x_1, \frac{m - p_1 x_1}{p_2}\right) \text{ s.t. } x_1 \in [0, m/p_1]$$

- Question: when is x_1 a normal good, i.e. $x_1^*(m)$ is increasing?
- Monotone maximum theorem says if $u\left(x_1, \frac{m-p_1x_1}{p_2}\right)$ has increasing difference in (x_1, m)
    - If u smooth, has increasing differences iff

$$\frac{1}{p_2}\frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_2^2}\frac{p_1}{p_2^2} \geq 0$$

- Conclusion holds even if
    - Demand integer constrained
    - Constraint binds sometimes but not always