Sets and their properties

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Much (perhaps all) of mathematics is about studying sets of objects with particular properties.

Section 1 introduces sets and some related concepts. Section 1.4 briefly discusses cardinality and introduces countable and uncountable sets. Section 2 is about relations, especially orders, which are used to state Arrow's impossibility theorem. The appendix section A is about familiar sets of numbers, including the integers, rationals, and real numbers. The properties of these sets of numbers that make them distinct are discussed.

References. Section 1 on sets is partly based on chapter 1 of Carter (2001). Any similar high-level mathematical economics textbook covers similar material. Examples include De la Fuente (2000), Ok (2007), and Corbae, Stinchcombe, and Zeman (2009). Textbooks on real analysis, such as Rudin (1976) and Tao (2006), also typically start with a section about sets.

Section 1.4 about cardinality is largely based on chapter 2 Rudin (1976). Chapter B of Ok (2007) covers similar material. Weeks 2 and 3 of the notes of Tao (2003) (on which Tao (2006) is based) also cover cardinality.

Section 2 about relations is based on chapter 1.2 of Carter (2001). Arrow's impossibility theorem first appeared in Arrow (1950). Feldman (1974) is a more approachable, simplified proof of the theorem.

The appendix section A is based on Rudin (1976), but any textbook on real analysis will cover similar material. Tao (2006) (or the note version Tao (2003)) is especially detailed and careful in its construction of the real numbers.

1. Sets

A **set** is any well-specified collection of elements.[1] Sets are conventionally denoted by capital letters, and elements of a set are usually denoted by lower case letters. The notation, $a \in A$, means that a is a member of the set A. A set can be defined by listing its elements inside braces. For example,

$$A = \{4, 5, 6\}$$

[1] "Well-specified" is somewhat ambiguous, and this ambiguity can lead to trouble such as Russell's paradox or Cantor's paradox. We'll ignore these paradoxes, but rest assured that they can be avoided by more carefully defining "well-specified."

means that A is a set of three elements with members 4, 5, and 6. The members of a set need not be explicitly listed. Instead, they can be defined by some logical relation. For example, the same set A could be written

$$A = \{n \in \mathbb{N} : 3 < n < 7\} \tag{1}$$

where $\mathbb{N} = \{1, 2, 3, ...\}$ is the natural numbers. The expression in (1) could be read as, "the set of natural numbers, n, such that 3 is less than n is less than 7." Sometimes $|$ will be used to mean "such that" instead of $:$. The elements of sets need not be simple things like numbers. For example, if $A_k = \{n \in \mathbb{N} : n > k\}$ is the set of natural numbers greater than k, then you could have a set of sets, $B = \{A_1, A_{10}, A_6\}$. Sets are unordered, so the previous definition of B is the same as $B = \{A_1, A_6, A_{10}\}$. Also, sets do not contain duplicates, so for example, $\{1, 1, 2\} \equiv \{1, 2\}$. Sets can be empty. The empty set, also called the null set, is denoted by $\emptyset$ or, less commonly, $\{\}$.

1.1. **Economic examples.** Sets appear all over economics.[2]

Example 1.1. [Sample space] In a random experiment, the set of all possible outcomes is called the **sample space**. E.g. for the roll of a dice, the sample space if $\{1, 2, 3, 4, 5, 6\}$. An **event** is any subset of the sample space.

Example 1.2. [Games] A game is a model of strategic decision making. A game consists of a finite set of n players, say $N = \{1, 2, ..., n\}$. Each player $i \in N$ chooses an action a_i from a set of actions A_i. The outcome of the game depends on the actions chosen by all players.

Example 1.3. [Consumption set] The **consumption set** is the set of all feasible consumption bundles. Suppose there are n commodities. A consumer chooses a consumption bundle $\mathbf{x} = (x_1, x_2, ..., x_n)$. Consumption cannot be negative, so the consumption set is a subset of $\mathbb{R}^n_+ = \{(x_1, ..., x_n) : x_1 \geq 0, x_2 \geq 0, ...x_n \geq 0\}$.

1.2. **Set operations.** Given two sets A and B, a new set can be formed with the following operations:

(1) **Union:** $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
(2) **Intersect:** $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
(3) **Minus:** $A \setminus B = \{x : x \in A \text{ and } \notin B\}$
(4) **Product:** $A \times B = \{(x, y) : x \in A, y \in B\}$
(5) **Power set:** $\mathcal{P}(A)$ = set of all subsets of A

Often, we will discuss sets that are all subsets of some universal set, U. In this case, the **complement** of A in U is $A^c = U \setminus A$. If we have an indexed collection of sets, $\{A_k\}_{k \in \mathcal{K}}$, we may take the union or intersection of all these sets and denote it as $\cup_{k \in \mathcal{K}} A_k$ or $\cap_{k \in \mathcal{K}} A_k$.

[2]These examples come from chapter 1 of Carter.

1.3. **Set relations.** If every element of A is also in B, then we say that B contains A and write $B \supseteq A$, or A is a subset of B and write $A \subseteq B$. If, additionally, there exists $b \in B$ such that $b \notin A$, then we say that A is a proper subset of B, which is denoted by $A \subset B$ or $B \supset A$.

Example 1.4 (1.2 Games continued)**.** In a game subsets of players are called **coalitions**. The set of all coalitions is the power set of the set of players, $\mathcal{P}(N)$.

The **action space** of a game is the set of all possible outcomes or combinations of actions, $A = A_1 \times A_2 \times ... \times A_n$. An element of A, $a = (a_1, a_2, ..., a_n)$ is called an **action profile**.

1.4. **Cardinality .** [3] Sometimes, we want to compare the size of two sets. This is easy when sets are finite; we simply count how many elements each has. It is not so easy to compare the size of infinite sets. Consider, for example, the natural numbers, $\mathbb{N}$, the integers $\mathbb{Z}$, rationals, $\mathbb{Q}$, and real numbers, $\mathbb{R}$. Let $|A|$ denote the "size" of A (we will define it precisely later). We know that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R},$$

so it seems sensible to say that

$$|\mathbb{N}| < |\mathbb{Z}| < |\mathbb{Q}| < |\mathbb{R}|.$$

On the other hand, the even integers are a subset of $\mathbb{Z}$, but since we can write the set of even integers as $\{2x : x \in \mathbb{Z}\}$, it doesn't seem like there are any more integers than even integers. It was questions like these that led Georg Cantor to pioneer set theory in the 1870's.

A function (aka mapping), $f : A \to B$ is called **one-to-one** (aka injective) if for every $b \in B$ the set $\{a : f(a) = b\}$ is either a singleton or emptyo. f is called **onto** (aka surjective) if $\forall b \in B \; \exists a \in A : f(a) = b$. If there exists a one-to-one mapping of A onto B (aka bijection or one-to-one correspondence), then we say that A and B have the same **cardinal number** (or cardinality) and write $|A| = |B|$. Let $J_n = \{1, ..., n\}$. A is **finite** if $|A| = |J_n|$. A is **countable** if $|A| = |\mathbb{N}|$. A is **uncountable** if A is neither finite nor countable. You should verify that the relation $|A| = |B|$ is reflexive ($|A| = |A|$), symmetric ($|A| = |B|$ implies $|B| = |A|$), and transitive (if $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$).

Lemma 1.1. *$\mathbb{Z}$ is countable.*

Proof. We can construct a bijection between $\mathbb{Z}$ and $\mathbb{N}$ as follows:

$$\begin{array}{lllllllll} \mathbb{Z}: & 0, & -1, & 1, & 2, & -2, & 3, & -3, \dots \\ \mathbb{N}: & 1, & 2, & 3, & 4, & 5, & 6, & 7, \dots \end{array}$$

Or as a formula, $f : \mathbb{N} \to \mathbb{Z}$ with

$$f(n) = \begin{cases} (n-1)/2 \text{ if } n \text{ odd} \\ -n/2 \text{ if } n \text{ even.} \end{cases}$$

□

[3] This section based on Chapter 2 Rudin (1976).

Theorem 1.1. *Every infinite subset of a countable set A is countable.*

Proof. A is countable, so there exists a bijection from A to $\mathbb{N}$. We can use this mapping to arrange the elements of A in a sequence, $\{a_n\}_{n=1}^{\infty}$[4]. Let B be an infinite subset of A. Let n_1 be the smallest number such that $a_{n_1} \in B$. Given n_{k-1}, let n_k be the smallest number greater than n_{k-1} such that $a_{n_k} \in B$. Such an n_k always exists since B is infinite. Also, $B = \{a_{n_k}\}_{k=1}^{\infty}$ since otherwise there would be a $b \in B$, but $b \notin A$. Thus, $f(k) = a_{n_k}$ is a one-to-one correspondence between B and $\mathbb{N}$. □

Theorem 1.2. *The rational numbers are countable.*

Proof. Consider the following arrangement of positive rational numbers:

$$\begin{array}{ccccc} 1/1 & 2/1 & 3/1 & 4/1 & \cdots \\ 1/2 & 2/2 & 3/2 & 4/2 & \cdots \\ 1/3 & 2/3 & 3/3 & 4/3 & \cdots \\ \vdots & & & \ddots & \end{array}$$

Starting in the top left and going back and forth diagonally, we get the following sequence:

$$1/1, 1/2, 2/1, 1/3, 2/2, 3/1, \ldots$$

Adding zero and the negative rationals, we can write e.g.

$$\begin{aligned} &0, 1/1, -1/1, 1/2, -1/2, 2/1, -2/1, 1/3, -1/3, 2/2, -2/2, 3/1, \ldots \\ =&q_1, q_2, q_3, q_4, \ldots \end{aligned}$$

Continuing on in this way, we could list all rational numbers. Some of these fractions represent the same number and can be removed. Thus, we obtain a correspondence between the rationals and an infinite subset of $\mathbb{N}$. However, by theorem 1.1, this subset is countable, so the rationals are also countable. □

Theorem 1.3. *The real numbers are uncountable.*

Proof. (Cantor's diagonal argument) We have not rigorously defined the real numbers, so we will take for granted the following: every infinite decimal expansion, (e.g. 0.135436080...) represents a unique real number in $[0, 1)$, except for expansions that end in all zeros or nines, which are equivalent[5].

We will use proof by contradiction to prove the theorem. Proof by contradiction is a common technique that works by showing that if the theorem were false, then we could prove something that contradicts what we know is true.

[4]By this notation, we mean an infinite ordered list of elements of A, i.e. $a_1, a_2, a_3, \ldots$.

[5]E.g. 0.199... = 0.200...

Suppose the theorem is false. Then we can construct a surjective mapping from $\mathbb{N}$ to $(0,1)$. That is we can list all real numbers in $(0,1)$ as

$$\begin{array}{llllll} r_1 & = 0. & d_{11} & d_{12} & d_{13} & \dots \\ r_2 & = 0. & d_{21} & d_{22} & d_{23} & \dots \\ r_3 & = 0. & d_{31} & d_{32} & d_{33} & \dots \\ \vdots & & \vdots & & & \end{array}$$

where each $d_{ij} \in \{0, 1, \dots, 9\}$, and no expansion ends in all nines. We will now show that there is a real number in $(0,1)$ that is not in the list. Let $x^* = 0.d_1^* d_2^* d_3^* \dots$ where d_n^* is chosen such that $d_n^* \neq d_{nn}$ and x^* is sure not to end in all nines. There are many possibilities, but to be concrete, let's set

$$d_n^* = \begin{cases} d_{nn} + 1 \text{ if } d_{nn} < 8 \\ 0 \text{ if } d_{nn} \geq 8 \end{cases}.$$

x^* is in $(0,1)$, but $x^* \neq r_n$ for any n because $d_n^* \neq d_{nn}$. Thus, we have a contradiction, and there cannot be a onto mapping from $\mathbb{N}$ to $(0,1)$. If there is no surjective mapping from $\mathbb{N}$ to $(0,1)$, there can be no surjective mapping from $\mathbb{N}$ to $\mathbb{R}$ since $(0,1) \subset \mathbb{R}$. □

Countable sets are said to have cardinality $\aleph_0$ ("aleph null"). Note that an implication of theorem 1.1 is that $\aleph_0$ is the smallest infinite cardinal number. The real numbers have cardinality of the continuum, sometimes written $2^{\aleph_0}$ or $\mathbf{c}$. You might be wondering whether there are larger cardinal numbers. The answer is yes. The set of all subsets of a set, A, called the **power set** of A, always has larger cardinality $2^{|A|}$ (the proof of this is similar to the proof that the real numbers are uncountable).

A final question to ask yourself is whether there are sets with cardinality between $\aleph_0$ and $2^{\aleph_0}$. The answer to that question is whatever you want it to be. The conjecture that there are no cardinal numbers between $\aleph_0$ and $2^{\aleph_0}$ is known as the continuum hypothesis. It was proposed by Cantor in the 1870s. In 1900, Hilbert made a famous list of 23 important unsolved problems in mathematics. The continuum hypothesis was the first. In 1940, Gödel showed that the continuum hypothesis cannot be disproved from the standard axioms that lie at the foundation of mathematics. In 1963, Cohen showed that the continuum hypothesis cannot be proved from the standard axioms. This is an example of Gödel's incompleteness theorem, a very interesting result that we won't be able to cover in this course. Loosely speaking, Gödel's incompleteness theorem says that for any non-trivial set of assumptions and system of logic, you can make statements consistent with the system of logic that cannot be proven or disproven from the assumptions.

2. Relations

There is not much more we can say about generic sets. Fortunately, sets used in economics and mathematics typically have some additional properties that we can utilize. Basic mathematics studies real numbers. Numbers have many properties: they are ordered, they can be added and multiplied, etc. Most of the abstract sets that we will study will have some, but not all, of the properties of real numbers. We will begin by studying ordered sets.

Orderings, or, more generally, relations, are important in economics because they can be used to represent preferences. Relations are things like =, <, ≤, and ⊂. Formally,

Definition 2.1 (Relation)**.** A **relation** on two sets A and B is any subset of $A \times B$, $R \subseteq A \times B$. We usually denote relations by $a \overset{R}{\sim} b$ if $(a, b) \in R$ (where $\overset{R}{\sim}$ could be some other symbol).

Although we define relations in terms of a subset of the product set, it's usually easier to just think about relations as a rule expressing the relationship between elements of A and B. For most relations, $A = B$, and then we say $\overset{R}{\sim}$ is a relation on A.

Example 2.1. Let $A = B = \mathbb{R}$. Then $<$ is associated with $R_< = \{(a, b) \in \mathbb{R}^2 : a < b\}$.

Relations usually have some of following properties.

Definition 2.2 (Properties of relations)**.** A relation $\overset{R}{\sim}$ on A is

- **reflexive** if $a \overset{R}{\sim} a \; \forall a \in A$,
- **transitive** if $a \overset{R}{\sim} b$ and $b \overset{R}{\sim} c$ implies $a \overset{R}{\sim} c$,
- **symmetric** if $a \overset{R}{\sim} b$ implies $b \overset{R}{\sim} a$,
- **antisymmetric** if $a \overset{R}{\sim} b$ and $b \overset{R}{\sim} a$ implies $a = b$,
- **complete** if either $a \overset{R}{\sim} b$ or $b \overset{R}{\sim} a$ or both $\forall a, b \in A$.

Exercise 2.1. As an exercise, you may want to work out which of the above properties =, <, and ≤ on $\mathbb{R}$ have.

Slightly confusingly, symmetric and antisymmetric are not opposites. The usual equality in $\mathbb{R}$, = is both symmetric and antisymmetric.

Example 2.2 (Preference relation)**.** A consumer's preference relation, $\succ$, is a relation on her consumption set, X. $x \succ y$ means that the consumer likes the bundle of goods x at least as much of the bundle of goods y. We will assume that preference relations are complete, transitive.

Exercise 2.2. Show that any complete and transitive order is also reflexive.

2.1. **Equivalence relations.** An **equivalence relation** is a relation that is reflexive, transitive, and symmetric. If $\sim$ is an equivalence relation on X, then the **equivalence class** of x is $\sim (x) = \{a \in X : a \sim x\}$. Since $x \sim x$, all $x \in X$ must be in some equivalence class. Also, since equivalence relations are symmetric, each $x \in X$ is in only one equivalence class.

Example 2.3 (Indifference)**.** Let $\succ$ be a preference relation on X. Then we can define an equivalence relation by $x \sim y$ if $x \succ y$ and $y \succ x$. This relation is called the indifference relation. The equivalence classes of $\sim$ are called indifference classes. You are probably familiar with graphs of indifference curves. Indifference curves are indifference classes.

Example 2.4 (Isoquants). Consider a production function $f : \mathbb{R}^n \to \mathbb{R}$. We can define an equivalence relation on $\mathbb{R}^n$ by $x \sim z$ iff $f(x) = f(z)$. The equivalence classes of this preference relation are the isoquants of the production function.

2.2. **Order relations.** A relation that is transitive and reflexive but not symmetric is called an **order**. Preference relations are orders. Any order $\preceq$ induces another relation $\prec$ defined by $x \prec y$ if $x \preceq y$ and $y \npreceq x$. The reflexive order $\preceq$ is often called a non-strict order, and the non-reflexive $\prec$ is called a strict order.

The usual order on $\mathbb{R}$, $\leq$ has two additional properties that a generic order need not have. First, any set in $\mathbb{R}$ with an upper bound has a least upper bound. y is an upper bound of $A \subseteq \mathbb{R}$, if $y \geq x$ for all $x \in A$. y is the least upper bound of A if there is no $z < y$ that is also an upper bound of A. Least upper bounds in $\mathbb{R}$ are unique. Among other things, uniqueness of least upper bounds is important for ensuring that optimization problems are well-defined. A **partial order** is a relation that is transitive, reflexive, and antisymmetric. Not all elements of a partially ordered set need to be comparable.

Example 2.5. Let S be set. The power set of S (set of all subsets) is partially ordered by $\subseteq$.

Example 2.6. A partial order on $\mathbb{R}^n$ can be defined by $x \leq y$ if $x_i \leq y_i$ for all i.

A preference relation is not a partial order because it allows indifference between goods that are not the same.

Another aspect of the usual $\leq$ order on $\mathbb{R}$ is that all elements are comparable, i.e. it is complete. A relation that is complete, transitive, and reflexive is called a (non-strict) **weak order** (or sometimes just an ordering). Preference relations are weak orders.[6]

2.3. **Pareto order.** [7]In this subsection we will derive an interesting economic result — Arrow's impossibility theorem — purely by thinking about orders. Suppose we have n individuals each with some preference, $\succ_i$ on set X. Think of X as some set of outcomes for society. For example, each element of X could specify how much consumption to give each individual. The Pareto order on X is defined as

$$x \succ^P y \text{ if } x \succ_i y \text{ for all } i = 1, \ldots, n$$

We say that x Pareto dominates y if $x \succ^P y$. x is Pareto optimal (or Pareto efficient) if there is no y such that $y \succ^P x$.

Exercise 2.3. Which of the properties of a relation does the Pareto order have?

Remember that any relation on X can be represented by $R \subseteq X \times X$. If R_i is the set associated with $\succ_i$, i.e. $R_i = \{(x, y) : x \succ_i y\}$, then $x \succ^P y$ if $(x, y) \in \cap_{i=1}^n R_i$. Similarly,

[6]The "weak" refers to the fact that $x \preceq y$ and $y \preceq x$ does not imply $x = y$. This is somewhat confusing because in economics we might read $x \preceq y$ as "y is weakly preferred to x", where weakly refers to allowing indifference, not the fact that preference relations are weak orders.

[7]Based on example 1.4.3 of Carter (2001).

$x \succeq^P y$ if $(x, y) \in \cap_{i=1}^n R_i^c$. If (x, y) is in neither of these two intersections, then x and y are not Pareto comparable.

The Pareto order depends on the n individual preferences. In other words the Pareto order is a function of individual preferences. Since each individual preference can be represented by a subset of $X \times X$, we can think of the Pareto order as a function $\mathcal{P}(X \times X)^n \to \mathcal{P}(X \times X)$. Thinking of orders as subsets of $X \times X$ (equivalently elements of $\mathcal{P}(X \times X)$) may not feel natural, but it is essential to remember that the Pareto order is a function of individual preferences.

The Pareto order will usually not be complete. For example take $n = 2$. Let elements of X be consumption bundles for person 1 and 2, (x_1, x_2), such that $x_1 + x_2 = c$, for some constant c. Then we would expect that $(c, 0) \succ_1 (0, c)$ and $(0, c) \succ_2 (c, 0)$, so $(c, 0)$ and $(0, c)$ are not Pareto comparable.

Social choice theory is about how best to combine individual preferences to make social decisions. It would be great if we could somehow complete the Pareto order, because then we could just choose the maximal outcome under the completed Pareto order. Arrow (1950) showed that this is impossible to do in an appealing manner. To precisely state Arrow's result, we first need a couple more definitions.

A **social choice rule** is a rule for combining individual preferences $\succ_i$ into a single weak order $\succeq$. In other words, a social choice rule is a function from the set of n individual preferences to the set of weak orders. Just like the Pareto order, a social choice rule is a function of individual preferences. That is, a social choice rule is a function : $\mathcal{P}(X \times X)^n \to \mathcal{P}(X \times X)$. Unlike the Paretor order, a social choice rule is required to be complete (weak orders are complete).

We will denote the social ordering by $F(\succ_1, ..., \succ_n) = \succeq$. A social choice rule F completes the Pareto order if $\succeq$ is complete and for all x, y such that $x \succ^P y$, the social order agrees $x \succeq y$. Although not made explicit by this notation, it is important to keep in mind that both the Pareto order, $\succ^P$, and the social order, $\succeq$, are functions of individual preferences.

A social choice rule satisfies the **independence of irrelevant alternatives** (IIA) if for every $A \subseteq X$, if $\succ_i$ and $\succ_i'$ are two sets of individuals preferences that agree on A,

$$x \succ_i y \iff x \succ_i' y \; \forall x, y \in A \text{ for each } i$$

then $\succeq = F(\succ_1, ..., \succ_n)$ and $\succeq' = F(\succ_1', ..., \succ_n')$ also agree on A. Independence of irrelevant alternatives says that the question of whether society likes apples better than bananas should not depend on what anyone thinks about the comparison between apples and oranges (or oranges and kiwis, etc). It is an intuitively desirable feature of a social choice rule.

One social choice rule that completes the Pareto order is the dictatorial ordering. That is, we let any one person's preference be society's preference. Of course, this is not a very desirable social ordering.

Theorem 2.1 (Arrow's impossibility theorem)**.** *It is impossible to complete the non-strict Pareto ordering in a way that is not dictatorial and is independent of irrelevant alternatives when $|X| \geq 4$.*

Feldman (1974) contains a very concrete proof for when $n = 2$ and $|X| = 3$. We will prove the general theorem, but if you find the general argument confusing (and it is a bit difficult), then reading Feldman (1974) might be helpful.

Proof. Suppose we have a social choice rule, F that is consistent with the Pareto order and obeys the irrelevance of independent alternatives. We will prove the theorem in three steps, but first a definition will be useful.

A group of individuals, $S \subseteq \{1, \cdots, n\}$ is **decisive** over $x, y \in X$ if for any individual preference relations if $x \succ_i y$ for all $i \in S$ implies $x \succ y$, where $\succ = F(\succ_1, ..., \succ_n)$. Being decisive is a property of the subgroup S and the decision rule F. It is something that holds for all possible individual preferences.

(1) *Field expansion lemma*: Suppose $|X| \geq 4$ and that there is one pair of alternatives, x and y such that some group S is decisive over x, y. We will show that S must also be decisive over all other pairs of alternatives. Let $z, w \in X$ and suppose that for all $i \in S$, $w \succ_i z$. We want to show that $w \succ z$.

By the irrelevance of independent alternatives, as long as we hold all individuals preferences over w and z constant, the social choice rule must give the same ordering between w and z. This means that if for any configuration of individual preferences over other outcomes, we can show $w \succ z$, then we know $w \succ z$ for any set of preferences. In other words, for any individual preferences $\succ'_i$ such that $w \succ'_i z$ if and only if $w \succ_i z$ for all i, it must be that $w \succ z$ if and only if $w \succ' z$, where $\succ' = F(\succ'_1, ..., \succ'_n)$ and $\succ = F(\succ_1, ..., \succ_n)$. This means that we can freely change preferences among alternatives as long as we hold the preference between w and z fixed.

Consider preferences such that $w \succ'_j x$ and $y \succ'_j z$ for all j, and $x \succ'_i y$ for all $i \in S$. Then $y \succ'^P z$ and $w \succ'^P x$. Since the social order must agree with the Pareto order, we also have $y \succ' z$ and $w \succ' x$. By assumption S is decisive over x and y, so $x \succ' y$. By transitivity $w \succ' z$. Like we argued earlier, by the independence of irrelevant alternatives, it must also be that $w \succ z$, i.e. S is also decisive over w and z

(2) *Group contraction lemma*: We will show that if S is decisive and $|S| \geq 2$, then there exists $S' \subset S$ such that S' is decisive.

Let $z \in X$, $z \neq x, y$. Partition S into two groups. That is, let $S_1 \cup S_2 = S$ where $S_1 \cap S_2 = \emptyset$, and S_1 and S_2 are not empty. Consider preferences $\succ_i$ such that $x \succ_i y$ for all $i \in S$. S is decisive, so then, $x \succ y$ as well.

If S_1 is not decisive, then from the previous part, S_1 is not decisive over any pair. In particular, S_1 is not decisive over z and x. This means that we can find preferences $x \succ_i z$ for all $i \in S_1$ such that the social order $\succ = F(\succ_1, ..., \succ_2)$ prefers z to x, $z \succeq x$. Moreover, by the independence of irrelevant alternatives, we can make these preferences be such that $x \succ_i y \succ_i z$ for all $i \in S_1$.

If S_2 is also not decisive, then S_2 is not decisive over any pair, including z and y. Thus, far we have placed assumptions about preferences between z and x and x and y. We have not said anything about preferences between z and y. By the independence of irrelevant alternatives, we are free to specify preferences between z and y which will still satisfy all the conditions about z and x above. This means that we can find preferences such that $z \succ_i y$ for all $i \in S_2$ but the social order (weakly) prefers y to z, $y \succeq z$. To summarize, assuming neither S_1 nor S_2 is decisive, we have constructed preferences such that

- for all $i \in S_1$, $x \succ_i y \succ_i z$, but $z \succeq x$, and
- for all $i \in S_2$, $z \succ_i y$ and $x \succ_i y$, but $y \succeq z$

However, S being decisive implies $x \succ y$, which contradicts $y \succeq z \succeq x$. Therefore, either S_1 or S_2 must be decisive.

(3) To conclude the proof, notice that the set S of everyone is decisive for any social ordering that agrees with the Pareto ordering. By the group contraction lemma, there is a decisive proper subset S'. We can apply the group contraction lemma again and again until we get to a decisive set with one element, i.e. a dictator.

□

So what does this theorem mean? One implication is that any non-dictatorial social choice rule that agrees with the Pareto order must either (i) not actually be a weak order (usually not trnsitive) or (ii) violate the independence of irrelevant alternatives.

Example 2.7 (Binary voting)**.** Suppose n is odd (to avoid ties) and the social choice rule is determined by majority vote. That is,

$$x \succeq y \iff x \succeq_i y \text{ for at least } (n+1)/2 \text{ individuals} .$$

Voting is complete, agrees with the Pareto order, and is non-dictatorial. Voting also obeys the independence or irrelevant alternatives. Whether $x \succeq y$ only depends on individual preferences between x and y. So, how can this be possible? Well, voting does not always lead to a transitive order, so it does not qualify as a social choice rule. One of the requirements of a social choice rule is that it is a weak order (given any individual preferences), so it must be transitive. Transitivity was essential at the end of step 2 of our proof of the impossibility theorem.

We could also try to construct social choice rules through other voting arrangements.

Example 2.8 (Majority voting)**.** Suppose the social choice rule is to let each person vote on their most preferred outcome. Then outcomes are ranked by the number of votes received. This social choice rule is transitive and complete (so it is a weak order), but it does not obey the independence of irrelevant alternatives. We see this often in elections with 3 candidates, where the presence of a third fringe candidate can influence which of two mainstream candidates wins.

Appendix A. Numbers

We have been assuming familiarity with the natural numbers, integers, rationals, and real numbers. This section explores some properties of these sets of numbers and heuristically describes how these sets of numbers are constructed. It may appear silly and slightly confusing to try to be "rigorous" about something like real numbers that we already feel like we understand. Much of mathematics is about finding and describing patterns that apply to abstract objects. Many of the abstract objects that we will study are similar to the real numbers in some ways, but different in others. Examples of things that are similar to the real numbers include complex numbers, vector spaces, matrices, and sets of functions. Some of these things we will be able to add and multiple just like real numbers, but not all of them. A natural sort of question is: this class of objects shares properties X, Y, and Z with the real numbers; what theorems that we know about the real numbers will also be true of this class of objects? Before answering this sort of question we have to be precise about what properties the real numbers have.

We will take for granted that we understand what the natural numbers are. Note, however, that it is possible to rigorously construct the natural numbers from a simple list of assumptions using logic or set theory. We will also take for given that we know how to add and multiply natural numbers. Addition has the following nice properties.

1 *Closure* if $a, b \in \mathbb{N}$, so is $a + b$
2 *Associative* $a + (b + c) = (a + b) + c$.

If we demand that addition also has

3 *Identity* $\exists 0$ s.t. $a + 0 = a$,
4 *Inverse* $\forall a, \exists b$ s.t. $a + b = 0$

then we must expand the natural numbers to include the integers, $\mathbb{Z}$. Multiplication also satisfies these four analogous properties:

1′ *Closure* if $a, b \in A$, so is ab
2′ *Associative* $a(bc) = (ab)c$.
3′ *Identity* $\exists 1$ s.t. $a1 = a$,
4′ *Inverse* $\forall a \neq 0, \exists b$ s.t. $ab = 1$

However, if we want multiplicative inverses to exist for all $z \in \mathbb{Z}$, then we must further expand our set of numbers to the rationals, $\mathbb{Q}$. Addition and multiplication are also

5 *Commutative* $a + b = b + a$
6 *Distributive* $a(b + c) = ab + ac$

To summarize: if we start with the natural numbers, and then demand that multiplication and addition have these six properties, we end up with the rational numbers.

More generally, we could study a set A combined with one or two operations that satisfy certain properties. The branch of mathematics that studies these sort of objects is abstract algebra. We will not be studying algebra in detail, but it may be useful to be familiar with some basic terms. A **group** is a set and operation, $(A, \oplus)$ such that A is closed under $\oplus$, $\oplus$ is associative, there exists an identity, and inverses exist under $\oplus$ (i.e. properties 1-4). If $\oplus$ is also commutative, we call $(A, \oplus)$ an abelian (or commutative) group. Examples of groups include $(\mathbb{Z}, +)$ and $(\mathbb{Q}, \cdot)$. A **ring** is a set with two operations, $(A, \oplus, \odot)$ such that $(A, \oplus)$ is a group, and $\odot$ has properties 1-3 and 6. $(\mathbb{Z}, +, \cdot)$ is a ring. One ring that will come up repeatedly in this course is the set of all n by n matrices with the usual matrix addition and multiplication. A **field** is a set with two operations such that 1-6 hold for both operations. $(\mathbb{Q}, +, \cdot)$ is a field. Another field that you may have encountered is the complex numbers with the usual addition and multiplication. If you're interested you may want to verify that the integers modulo any number is a ring, and the integers modulo any prime number if a field.

A.1. **Real numbers.** The rational numbers are pretty nice; they're a field with the six properties listed above. However, $\mathbb{Q}$ does not contain all the numbers that we think it should. For example,

Theorem A.1. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = p/q$ where p and q are not both even. If we square both sides, we get

$$2 = p^2/q^2$$
$$2q^2 = p^2.$$

Hence, p^2 must be even. From the review, then p must also be even, say $p = 2m$. Then we have

$$2q^2 = 2(2m^2)$$
$$q^2 = 2m^2,$$

which means q must also be even, contrary to our starting assumption. □

Apparently, the rationals have some holes in them that we should fill in. To do so in a unique way, we need to define another property of the rational numbers. A **totally ordered set** is a set, A, and a relation, $<$, such that (i) (total) $\forall a, b \in A$ either $a < b$ or $a = b$ or $a > b$; and (ii) (transitive) if $a < b$ and $b < c$ then $a < c$. An **ordered field** is a field that is a totally ordered set and addition and multiplication preserve the ordering in that (i) if $b < c$ then $a + b < a + c$ (ii) if $a > 0$ and $b > 0$ then $ab > 0$.

We need one more definition. Simon and Blume state that one property of real numbers that will be used throughout the book is the least upper bound property. It turns out that this property is not only useful; it lies at the foundation of the real numbers. Let S be an ordered set and $A \subset S$. $s \in S$ is an **upper bound** of A if $s \geq a \forall a \in A$. s is a **least upper bound** (aka supremum) of A if s is an upper bound of A and if $r < s$, then r is not an upper bound of A. S has the **least-upper-bound property** (aka complete or Dedekind complete) if whenever $A \subset S$ has an upper bound, A has a least upper bound. Given that $\sqrt{2} \notin \mathbb{Q}$, it should not be surprising that the rational numbers are not complete.

Theorem A.2 (Real numbers)**.** *There exists an ordered field, $\mathbb{R}$, that has the least upper bound property. $\mathbb{R}$ contains $\mathbb{Q}$. Moreoever, $\mathbb{R}$ is "unique".*

The proof of this is surprisingly long, so we will not go over it in detail. Existence can be proven by construction. One method involves constructing real numbers as Dedekind cuts. A Dedekind cut is a nonempty subset of the rationals, $A \subset \mathbb{Q}$, such that (i) if $p \in A$, $q \in \mathbb{Q}$, and $q < p$, then $q \in A$ and (ii) if $p \in A$ then $p < r$ for some $r \in A$ (i.e. A has no greatest element. For example, the Dedekind cut associated with $\sqrt{2}$ would be $\{p \in \mathbb{Q} : p^2 < 2\}$). We would then define addition, multiplication, and ordering of these cuts in the natural way and verify that all the properties above are satisfied. See Rudin (1976) for details if you are interested.

The "uniqueness" is harder to prove. $\mathbb{R}$ is unique in the sense that any two ordered fields with the least-upper-bound property are isomorphic (there exists a bijection between them that preserves multiplication, addition, and ordering). The proof proceeds by supposing that $\mathbb{R}$ and $\mathbb{F}$ are two ordered fields with the least-upper-bound property and then shows that there is an isomorphism between them.

References

Arrow, Kenneth J. 1950. "A Difficulty in the Concept of Social Welfare." *Journal of Political Economy* 58 (4):pp. 328–346. URL `http://www.jstor.org/stable/1828886`.

Carter, Michael. 2001. *Foundations of mathematical economics*. MIT Press.

Corbae, Dean, Maxwell B Stinchcombe, and Juraj Zeman. 2009. *An introduction to mathematical analysis for economic theory and econometrics*. Princeton University Press.

De la Fuente, Angel. 2000. *Mathematical methods and models for economists*. Cambridge University Press.

Feldman, Allan M. 1974. "A VERY UNSUBTLE VERSION OF ARROW'S IMPOSSIBILITY THEOREM." *Economic Inquiry* 12 (4):534–546. URL `http://dx.doi.org/10.1111/j.1465-7295.1974.tb00420.x`.

Ok, Efe A. 2007. *Real analysis with economic applications*, vol. 10. Princeton University Press.

Rudin, Walter. 1976. *Principles of Mathematical Analysis (International Series in Pure & Applied Mathematics)*. McGraw-Hill Publishing Co.

Tao, Terence. 2003. "Math 131AH." URL `http://www.math.ucla.edu/~tao/resource/general/131ah.1.03w/`.

———. 2006. *Analysis*, vol. 1. Springer.