4. Sentential Connectives

Some English words can be used to connect two English sentences together, forming a third. One such word is ‘and’. For example, the two sentences “Socrates was a philosopher” and “Socrates lived in Athens” can be connected to form the single sentence:

Socrates was a philosopher and Socrates lived in Athens

This longer sentence is a little clumsy, perhaps, but it does make sense. The word ‘or’ can join two sentences in the same way, although the resulting sentence has a different meaning. We then get:

Socrates was a philosopher or Socrates lived in Athens

A third such connective is the word ‘not’, although it is a bit trickier to use. For now, let us note that we can form a new sentence from “Socrates was a philosopher”, or any other sentence, by putting the phrase “It is not the case that” on the front. So we form, for example:

It is not the case that Socrates was a philosopher

This sentence happens to be false, but the point is that it makes sense. In this section we shall examine how these three connectives work, and see what they mean exactly.

4.1 Conjunction

Consider a pair of propositions now, not sentences – call them P and Q. The idea is that, in some cases at least, there exists some third proposition, known as the conjunction of P and Q, and written ‘P ∧ Q’. Which proposition is this?

The intuitive idea of P ∧ Q is that it contains all the information in P, and all the information in Q, and nothing else. Conjunction is a matter of putting the information (in P and in Q) together in a single proposition, called P ∧ Q. More precisely, P ∧ Q entails P, and P ∧ Q entails Q, and it is the weakest proposition that does this. P ∧ Q is the weakest proposition that entails both P and Q. The reason we take the weakest proposition is so that it has no extra information, on top of what is in P and Q together.

**Definition** P ∧ Q, the conjunction of P with Q, is the weakest proposition X such that X ⇒ P and X ⇒ Q.
In the conjunction \( P \land Q \), the propositions \( P \) and \( Q \) are known as the *conjuncts*.

This is all rather trivial, really. Let \( P = \text{‘Socrates was a philosopher’} \) and \( Q = \text{‘Socrates lived in Athens’} \). Then \( P \land Q \) is ‘Socrates was a philosopher who lived in Athens’. This entails that Socrates was a philosopher. It also entails that Socrates lived in Athens. It doesn’t tell us anything else, however. Other propositions which entail both \( P \) and \( Q \) are stronger, i.e. they contain extra information. For example ‘Socrates was a snub-nosed philosopher who lived in Athens’ entails both \( P \) and \( Q \), but is not the weakest such proposition.

What happens if \( P \) and \( Q \) are inconsistent? In that case, there is no proposition that entails them both, and so there is no weakest such proposition. It follows that the conjunction \( P \land Q \) does not exist, or is undefined. Two propositions cannot be conjoined unless they are consistent. The sentence ‘\( P \land Q \)’ still exists, however. It simply has no meaning.

What happens if \( P \) entails \( Q \)? (or the other way around.) In this case \( Q \) has no information to contribute to the conjunction. Everything it says is also said by \( P \). Therefore, if \( P \Rightarrow Q \) then \( P \land Q = P \). A conjunction resolves to the *stronger* conjunct. You can check that this is true, using the definition above. \( P \Rightarrow P \), and \( P \Rightarrow Q \), and \( P \) is clearly the weakest proposition that does both.

It should be noted that the meaning of ‘\( \land \)’ is not always the same as the English word ‘and’. For example, the proposition \( P \land Q \) is always identical to \( Q \land P \). (We say that ‘\( \land \)’ is *symmetric.*) An English sentence involving ‘and’, on the other hand, does not always keep the same meaning when the conjuncts are switched around.

**Example**

Consider the sentence “Janet got married and got pregnant”. Most people would understand it to mean something different from “Janet got pregnant and got married”. Why are these read differently? It’s because ‘and’ often means ‘and then’, or ‘and so’. We perhaps read the first sentence as “Janet got married and then got pregnant”, or the second one as “Janet got pregnant and so got married”. These stronger connectives are not symmetric, i.e. they cannot be switched around without changing the meaning.

Another tricky point is that sentences containing adverbs cannot always be expressed as conjunctions. Consider, for example, “Betty is a fast reader”. This is *not* the conjunction of ‘Betty is fast’ and ‘Betty is a reader’. The first sentence, ‘Betty is fast’, does not specify what Betty is fast *at*, so it would have to be read as ‘Betty is fast at something’. The second sentence merely says that Betty can read. So, conjoining them, we get that Betty is fast at something, and knows how to read. In other words, this conjunction sentence does not tell us that Betty is fast at reading, and so fails to express the proposition we want.

Some sentences with adverbs *can* be expressed as conjunctions. For example ‘Betty is an English grandmother’ is the conjunction of ‘Betty is English’ and ‘Betty is a grandmother’.
Exercise 4.1

1. For each of the following sentences, say whether or not it can be expressed as a conjunction. If it can, then write down the two conjuncts in good English.

a. Ralph is a sore loser
b. Vancouver is a Canadian city
c. Dumbo is a small elephant
d. Fluffy is a large mouse
e. I bought an aluminum bicycle
f. John and Cathy work at the same office
g. John works for BC Hydro, and so does Cathy
h. Vijay is a criminal lawyer
i. Dubya is a Republican president

2. For each row of the table of propositions below, write down (in good English) a sentence that expresses the conjunction of P with Q. Use the shortest sentence you can, i.e. do not repeat information. E.g. if P = ‘Klein is a right-wing politician” and Q = “Klein is the Alberta premier”, then P \land Q is “Klein is a right-wing Alberta premier”. Once you’ve said that Klein is a provincial premier, there’s no need to add that he’s a politician, as this can be inferred.

N.B. If the conjunction doesn’t exist, then just say this.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>Janet is a rock climber</td>
<td>Janet is a mother</td>
</tr>
<tr>
<td>(ii)</td>
<td>Janet is a rock climber</td>
<td>Janet is a rock climber and a mother</td>
</tr>
<tr>
<td>(iii)</td>
<td>X = 2</td>
<td>X = 4</td>
</tr>
<tr>
<td>(iv)</td>
<td>X = 2</td>
<td>X &lt; 6</td>
</tr>
<tr>
<td>(v)</td>
<td>No poor people read The Economist</td>
<td>No readers of The Economist are poor</td>
</tr>
<tr>
<td>(vi)</td>
<td>All swans are white</td>
<td>At least some swans are white</td>
</tr>
<tr>
<td>(vii)</td>
<td>Chris is male</td>
<td>Chris is the father of Rachel</td>
</tr>
<tr>
<td>(viii)</td>
<td>Simpson is overweight</td>
<td>Simpson is not underweight</td>
</tr>
<tr>
<td>(ix)</td>
<td>Smith is a banker</td>
<td>Wilson is a journalist</td>
</tr>
<tr>
<td>(x)</td>
<td>Smith is an Albertan</td>
<td>Smith is an Albertan farmer</td>
</tr>
<tr>
<td>(xi)</td>
<td>Fred is a son of Janet</td>
<td>Fred is not a lawyer</td>
</tr>
<tr>
<td>(xii)</td>
<td>Zhang is over six feet tall</td>
<td>Zhang is five-foot-seven tall</td>
</tr>
<tr>
<td>(xiii)</td>
<td>Smith has been sent to jail</td>
<td>Smith has committed a crime</td>
</tr>
</tbody>
</table>
4.2 Disjunction

The proposition $P \lor Q$, read “$P$ or $Q$”, is called the disjunction of $P$ with $Q$. The disjunction, roughly speaking, is the “common content” of $P$ and $Q$, or the strongest point of agreement between them. $P \lor Q$ contains all the information that is in common between $P$ and $Q$.

Example

Suppose Fred thinks that Kant was Austrian, whereas Jim says that Kant was Prussian. Clearly, they disagree about Kant’s nationality. But they do at least agree that Kant was European. Indeed, though it sounds a little odd, they also agree that Kant was either Austrian or Prussian. Since this is the strongest point of agreement between these claims, it is also the disjunction of the claims. If $P = ‘\text{Kant was Austrian}’$, and $Q = ‘\text{Kant was Prussian}’$, then $P \lor Q$ is ‘Kant was either Austrian or Prussian’.

More precisely, we have the following definition:

Definition $P \lor Q$, the disjunction of $A$ with $B$, is the strongest $X$ such that $P \Rightarrow X$ and $Q \Rightarrow X$.

In the disjunction $P \lor Q$, $P$ and $Q$ are known as the disjuncts.

This definition is also easy to understand. Since $P \lor Q$ contains only what is in common between $P$ and $Q$, it must be true that $P \Rightarrow P \lor Q$. Similarly, $Q \Rightarrow P \lor Q$. Furthermore, in order to get all the common content of $P$ and $Q$, we require that $P \lor Q$ be the strongest such proposition.

You may have noticed that conjunction and disjunction, while different, are defined in a similar way. This similarity is shown on the following diagram:

```
     P \land Q
    /    \    \
   /      \  /
  /        \/
 P   \    /\   Q
    /   \ / \
   /     /  \
  /     /   \
 P \lor Q
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This diagram shows that $P \land Q$ entails both $P$ and $Q$, while $P \lor Q$ is entailed by them.
What happens to \( P \lor Q \) when \( P \) entails \( Q \)? In this case, the proposition \( Q \) is something that \( P \) and \( Q \) agree on. Also, since \( Q \) is all of the content of \( Q \), it is clearly the strongest such proposition. Thus, if \( P \Rightarrow Q \), then \( P \lor Q = Q \). A disjunction resolves to the *weaker* disjunct. You can check that this is true using the definition above.

In some disjunctions \( P \lor Q \), the disjuncts \( P \) and \( Q \) are consistent. Suppose, for example, that Fred claims to have aced his biology final. A sceptic remarks that Fred is either a good biologist or a good liar. This is the disjunction of \( B \) (= ‘Fred is a good biologist’) with \( L \) (= ‘Fred is a good liar’), which we can write as \( B \lor L \). These propositions are consistent, as Fred could be multi-talented; he might be good at biology and at lying. Thus in this case the conjunction \( B \land L \) exists. Is this conjunction \( B \land L \) consistent with the disjunction \( B \lor L \)? It must be, since in fact \( B \land L \) entails \( B \), and so it entails \( B \lor L \). So, when a disjunction is believed, one does not rule out the possibility that both disjuncts are true.

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**Exercise 4.2**

1. For each of the pairs of propositions from the previous exercise, Question 2, write down the disjunction \( P \lor Q \) in good English, as briefly as possible. Watch out for cases where one disjunct entails the other, and simplify accordingly.

2. What is the disjunction of ‘Ralph’s car cost less than $5000’ with ‘Ralph’s car cost at least $3000’, in its simplest form?

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**4.3 Negation**

The negation of \( P \), written \( \neg P \), is a proposition that disagrees with \( P \). (\( \neg P \) is inconsistent with \( P \).) There are many propositions that disagree with \( P \), however, so which one is it? It is actually the *weakest* of all the propositions that are inconsistent with \( P \).

**Example**

Let \( P \) be ‘Socrates lived in Athens (only)’. There are many other propositions that are inconsistent with this one, such as ‘Socrates lived in Edmonton’, ‘Socrates lived in East Van’, and so on. Each of these goes beyond mere disagreement with \( P \), however, and offers an alternative place of residence. The weakest way to disagree with \( P \) would be to say ‘Socrates did not live in Athens’, or ‘Socrates lived somewhere other than in Athens’. This weak disagreement with \( P \) is the negation of \( P \). The negation of \( P \) merely says that \( P \) has got it wrong, and nothing more. Note that every proposition inconsistent with \( P \) entails \( \neg P \). For example, “Socrates lived in Edmonton” entails that Socrates did not live in Athens.

**Definition** \( \neg P \), the *negation* of \( P \), is the weakest proposition that is inconsistent with \( P \).
Note that $P$ and $\neg P$ share the same presuppositions. The negation doesn’t contradict any of $P$’s assumptions, but only the things $P$ actually says. For example, if $P$ is “The Loch Ness Monster weighs at least 40 tons”, then $\neg P$ says that the Loch Ness Monster weighs less than 40 tons. $\neg P$ does not question the existence of the Loch Ness Monster. It makes exactly the same background assumptions.

Also note that, if $P$ says several things, then $\neg P$ doesn’t have to disagree with all of them, or even with any particular one of them. For example, if $P$ says that Smith is an Albertan farmer, then $\neg P$ doesn’t deny that Smith is farmer; nor does it deny that Smith is from Alberta. It merely denies that Smith is both an Albertan and a farmer. It allows that Smith be an Albertan trucker, or an Okanagan farmer, for example. (Or a Bay Street broker, of course.) So $\neg P$ says that Smith is either not an Albertan, or not a farmer (or neither).

Exercise 4.3

1. For each of the propositions in Exercise 4.1 Qu.2, write down a sentence that expresses the negation of that proposition. Use the shortest, most natural English sentence you can think of. E.g. write ‘Smith is not a farmer’, rather than ‘It is not the case that Smith is a farmer’.

2. For each of the following propositions, first write down any assumption(s) it makes that is contentious, then write down the negation of the proposition.

   (i) The inhabitants of Mars are no taller than humans.
   (ii) Noah’s ark came to rest on Mt. Ararat, in eastern Turkey.
   (iii) The reason women can’t do logic is that their brains are smaller than men’s.

3. For each of the following sentences, write down another that expresses its negation, in good English.

   (i) All Canadians are polite.
   (ii) No logic student is lazy.
   (iii) Fred might get a job as a waiter.
   (iv) Betty is a good student.
   (v) If Betty does all her logic assignments, then she is a good student.

4. For which pairs of propositions (in Exercise 4.1 Qu.2) does $P$ entail the negation of $Q$, i.e. $P \implies \neg Q$? Do you notice anything here? What is another way to express the claim that $P \implies \neg Q$?

5. If $P$ entails $\neg Q$, then does $Q$ always entail $\neg P$?

6. For which pairs of propositions does $\neg Q$ entail $\neg P$? What is another way to express the fact that $\neg Q \implies \neg P$?
7. Let $A$ be ‘Janet is a mother’. What is the negation of the negation of $A$, i.e. $\neg(\neg A)$? (Negate $A$, and then negate the result.)

8. For $D = ‘Smith is a dentist’$ and $H = ‘Smith is a home owner’$, write down $D \lor H$, and then $\neg(D \lor H)$, in good English. Can you express this proposition $\neg(D \lor H)$ using only ‘$D$’, ‘$H$’, ‘$\neg$’ and ‘$\land$’?

9. For the $D$, $H$ in Qu.6, write down $D \land H$, and then $\neg(D \land H)$, in good English. Can you express the proposition $\neg(D \land H)$ using only ‘$D$’, ‘$H$’, ‘$\neg$’ and ‘$\lor$’?