Economics 326
Methods of Empirical Research in Economics

Lecture 3: Simple Linear Regression and OLS

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May 5, 2010

Introduction

- The simple linear regression modelis used to study the relationship between two variables.
- It has many limitations, but nevertheless there examples in the literature where the simple linear regression is applied (e.g. stock returns predictability).
- It is also a good starting point to learning the regression technique.

Definitions

TABLE 2.1

Terminology for Simple Regression

y	x
Dependent variable	Independent variable
Explained variable	Explanatory variable
Response variable	Control variable
Predicted variable	Predictor variable
Regressand	Regressor

Sample and population

- The econometrician observes random data:

observation	dependent variable	regressor
1	Y_1	X_1
2	Y_2	X_2
$\vdots$	$\vdots$	$\vdots$
n	Y_n	X_n

- A pair X_i, Y_i is called an observation.
- Sample: $\{(X_i, Y_i) : i = 1, \ldots, n\}$.
- The population is the joint distribution of the sample.

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- Slope: β measures the effect of a unit change in X on Y :

$$\begin{aligned} \beta &= E\left(Y_i | X_i = x + 1\right) - E\left(Y_i | X_i = x\right) \\ &= \left[\alpha + \beta(x+1)\right] - \left[\alpha + \beta x\right]. \end{aligned}$$

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$$\beta = \frac{dE\left(Y_i|X_i\right)}{dX_i}.$$

- The effect is the same for all x!

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- The model:

$$\begin{aligned} Y_i &= \alpha + \beta X_i + U_i, \\ E(U_i|X_i) &= 0. \end{aligned}$$

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- The most popular function is $\log$.

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- Example: Cobb-Douglas production function: $Y = \alpha K^{\beta_1} L^{\beta_2} \Longrightarrow \log Y = \log \alpha + \beta_1 \log K + \beta_2 \log L$ (two regressors log of capital and log of labour).

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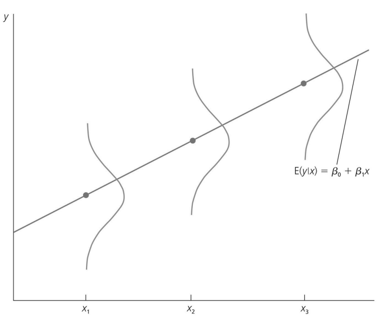
The model

$$Y_i = \underbrace{\alpha + \beta X_i}_{\text{Predicted by } X} + \underbrace{U_i}_{\text{Orthogonal to } X}$$

FIGURE 2.1

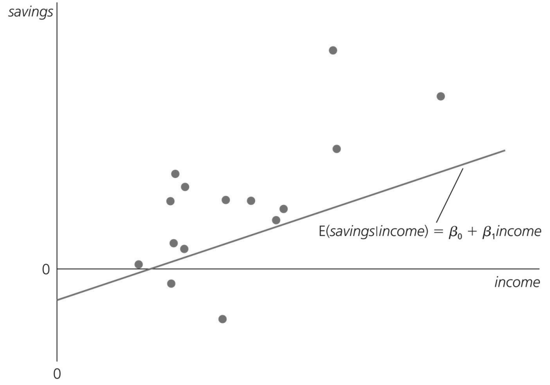
E($y|x$) as a linear function of x.



Estimation problem

FIGURE 2.2

Scatterplot of savings and income for 15 families, and the population regression $E(\textit{savings}|\textit{income}) = \beta_0 + \beta_1 \textit{income}$.



Problem: estimate the unknown parameters α and β using the data (n observations) on Y and X.

Method of moments

- We assume that

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- An estimator is a function of the observable data, it can depend only on observable X and Y. Let $\hat{\alpha}$ and $\hat{\beta}$ denote the estimators of α and β.
- **Method of moments:** Replace expectations with averages. Normal equations:

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right) &= 0. \\ \frac{1}{n}\sum_{i=1}^{n}X_i\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right) &= 0. \end{aligned}$$

Solution

- Let $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ (averages).

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\bar{Y} - \hat{\alpha} - \hat{\beta}\bar{X} &= 0.
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  The fitted regression line goes through the averages.

- $$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}.$$

Solution

- $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$ and therefore

$$0 = \frac{1}{n}\sum_{i=1}^{n} X_i \left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right)$$

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$$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i \left(Y_i - \bar{Y} \right)}{\sum_{i=1}^{n} X_i \left(X_i - \bar{X} \right)}.$$
- Since
$$\sum_{i=1}^{n} X_i \left(Y_i - \bar{Y} \right) = \sum_{i=1}^{n} \left(X_i - \bar{X} \right) \left(Y_i - \bar{Y} \right) = \sum_{i=1}^{n} \left(X_i - \bar{X} \right) Y_i$$

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- $$0=\sum_{i=1}^{n} X_i\left(Y_i-\bar{Y}\right)-\hat{\beta}\sum_{i=1}^{n} X_i\left(X_i-\bar{X}\right) \text{ or}$$
  $$\hat{\beta}=\frac{\sum_{i=1}^{n} X_i\left(Y_i-\bar{Y}\right)}{\sum_{i=1}^{n} X_i\left(X_i-\bar{X}\right)}.$$
- Since
  $$\sum_{i=1}^{n} X_i\left(Y_i-\bar{Y}\right) = \sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)=\sum_{i=1}^{n}\left(X_i-\bar{X}\right)Y_i$$

Solution

- $$0 = \sum_{i=1}^{n} X_i \left(Y_i - \bar{Y}\right) - \hat{\beta} \sum_{i=1}^{n} X_i \left(X_i - \bar{X}\right) \text{ or}$$
  $$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i \left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n} X_i \left(X_i - \bar{X}\right)}.$$
- Since
  $$\begin{aligned} \sum_{i=1}^{n} X_i \left(Y_i - \bar{Y}\right) &= \sum_{i=1}^{n} \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right) = \sum_{i=1}^{n} \left(X_i - \bar{X}\right) Y_i \text{ and} \\ \sum_{i=1}^{n} X_i \left(X_i - \bar{X}\right) &= \sum_{i=1}^{n} \left(X_i - \bar{X}\right)\left(X_i - \bar{X}\right) = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 \end{aligned}$$

Solution

- $$0 = \sum_{i=1}^{n} X_i \left(Y_i - \bar{Y}\right) - \hat{\beta} \sum_{i=1}^{n} X_i \left(X_i - \bar{X}\right) \text{ or}$$

  $$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i \left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n} X_i \left(X_i - \bar{X}\right)}.$$

- Since

  $$\sum_{i=1}^{n} X_i \left(Y_i - \bar{Y}\right) = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right) = \sum_{i=1}^{n} \left(X_i - \bar{X}\right) Y_i \text{ and}$$

  $$\sum_{i=1}^{n} X_i \left(X_i - \bar{X}\right) = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)\left(X_i - \bar{X}\right) = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$$

  we can also write

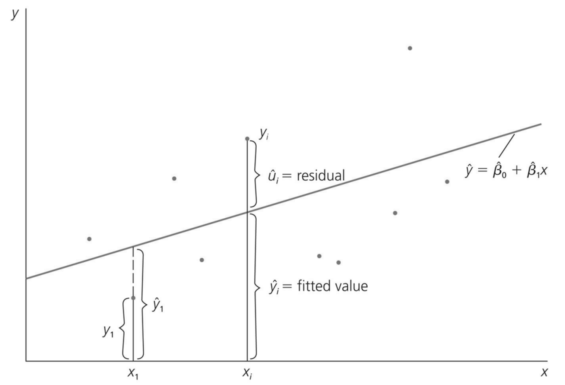
  $$\hat{\beta} = \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}\right) Y_i}{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}.$$

Fitted line

- Fitted values: $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i$.
- Fitted residuals: $\hat{U}_i = Y_i - \hat{\alpha} - \hat{\beta} X_i$.

FIGURE 2.4

Fitted values and residuals.

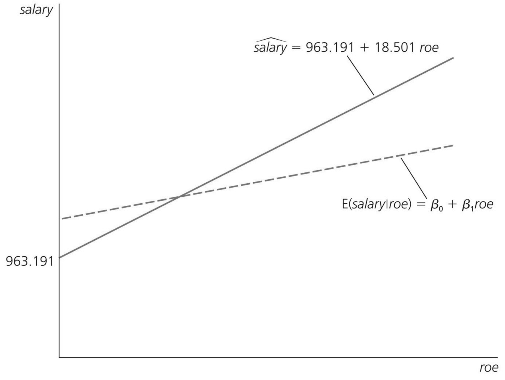


True line and fitted line

- **True:** $Y_i = \alpha + \beta X_i + U_i$, $EU_i = EX_iU_i = 0$.
- **Fitted:** $Y_i = \hat{\alpha} + \hat{\beta} X_i + \hat{U}_i$, $\sum_{i=1}^n \hat{U}_i = \sum_{i=1}^n X_i \hat{U}_i = 0$.

FIGURE 2.5

The OLS regression line $\widehat{salary} = 963.191 + 18.501\ roe$ and the (unknown) population regression function.



Ordinary Least Squares (OLS)

- Minimize $Q(a, b) = \sum_{i=1}^{n} (Y_i - a - bX_i)^2$ w.r.t. a and b.

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- Derivatives:

$$\begin{aligned}\frac{dQ\left(a,b\right)}{da} &= -2\sum_{i=1}^{n}\left(Y_i-a-bX_i\right).\\ \frac{dQ\left(a,b\right)}{db} &= -2\sum_{i=1}^{n}\left(Y_i-a-bX_i\right)X_i.\end{aligned}$$

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- First-order conditions:

$$0 = \sum_{i=1}^{n}\left(Y_i - \hat{\alpha} - \hat{\beta}X_i\right) = \sum_{i=1}^{n}\hat{U}_i.$$

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- Method of moments = OLS

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \text{ and } \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}.$$