# Economics 326
# Methods of Empirical Research in Economics

# Lecture 4: Properties of OLS

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# The OLS estimators are random variables

- The model

$$\begin{aligned} Y_i &= \alpha + \beta X_i + U_i, \\ E\left(U_i | X_1, \ldots, X_n\right) &= 0. \end{aligned}$$

Conditioning on $X$ in $E\left(U_i | X_1, \ldots, X_n\right) = 0$ allows us to treat all $X$'s as fixed, but $Y$ is still random.

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- The estimators

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\hat{\beta} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right) Y_i}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2} \text{ and } \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}
$$

  are random because they are functions of random data.

# The estimators are linear

- Since $\hat{\beta} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$, we can write $\hat{\beta} = \sum_{i=1}^{n} w_i Y_i$, where

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- For $\hat{\alpha}$,
$$\begin{aligned} \hat{\alpha} &= \bar{Y} - \hat{\beta}\bar{X} \\ &= \frac{1}{n}\sum_{i=1}^{n} Y_i - \left(\sum_{i=1}^{n} w_i Y_i\right)\bar{X} \end{aligned}$$

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&= \sum_{i=1}^{n}\left(\frac{1}{n} - \bar{X} w_i\right) Y_i \\
&= \sum_{i=1}^{n}\left(\frac{1}{n} - \bar{X}\frac{X_i - \bar{X}}{\sum_{l=1}^{n}(X_l - \bar{X})^2}\right) Y_i.
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\hat{\beta} = \beta + \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right) U_i}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.
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- Once we condition on $X_1, \ldots, X_n$, all $X$'s in

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$$\begin{aligned} E\left(U_1|X_1,\ldots,X_n\right) &= E\left(U_1|X_1\right), \\ E\left(U_2|X_1,\ldots,X_n\right) &= E\left(U_2|X_2\right) \text{ and etc.} \end{aligned}$$

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- The OLS estimator is in general biased if the strong exogeneity assumption is violated.

# Variance of $\hat{\beta}$

- If $Y_i = \alpha + \beta X_i + U_i$, $E\left(U_i | X_1, \ldots, X_n\right) = 0$, and
$$E\left(U_i^2 | X_1, \ldots, X_n\right) = \sigma^2 = \text{constant},$$

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- The assumption $E\left(U_i^2 | X_1, \ldots, X_n\right) = \sigma^2$ =constant is called (conditional) homoskedasticity.

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  Then
  $$Var\left(\hat{\beta} | X_1, \ldots, X_n\right) = \frac{\sigma^2}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.$$
- The assumption $E\left(U_i^2 | X_1, \ldots, X_n\right) = \sigma^2$ =constant is called (conditional) homoskedasticity.
- The assumption $E\left(U_i U_j | X_1, \ldots, X_n\right) = 0$ for $i \neq j$ can be replaced by the assumption that the observations are independent.

# Variance of $\hat{\beta}$

$$Var\left(\hat{\beta}|X_1,\ldots,X_n\right) = \frac{\sigma^2}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.$$

- The variance of $\hat{\beta}$ is positively related to the variance of the errors $\sigma^2 = Var\left(U_i\right)$.

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- The variance of $\hat{\beta}$ is positively related to the variance of the errors $\sigma^2 = Var\left(U_i\right)$.
- The variance of $\hat{\beta}$ is smaller when $X$'s are more dispersed.

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$$Var\left(\hat{\beta}\right) = E\left[\left(\hat{\beta} - E\hat{\beta}\right)^2\right]$$

# Variance of $\hat{\beta}$

- We are going to condition on $X$'s and will treat them as constants. All expectations below are implicitly conditional on $X$'s.
- We have $\hat{\beta} = \beta + \frac{\sum_{i=1}^{n}(X_i - \bar{X})U_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$ and $E\hat{\beta} = \beta$,

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\begin{aligned}
Var\left(\hat{\beta}\right) &= E\left[\left(\hat{\beta} - E\hat{\beta}\right)^2\right] \\
&= E\left[\left(\frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)U_i}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}\right)^2\right]
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&= \left(\frac{1}{\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}\right)^2 E\left[\left(\sum_{i=1}^n \left(X_i - \bar{X}\right) U_i\right)^2\right]
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- $$\begin{aligned} & \left( \sum_{i=1}^{n} (X_i - \bar{X}) U_i \right)^2 \\ = & \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - \bar{X}) (X_j - \bar{X}) U_i U_j \end{aligned}$$

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$$= \sum_{i=1}^{n} (X_i - \bar{X})^2 U_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} (X_i - \bar{X})(X_j - \bar{X}) U_i U_j.$$

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$$\begin{aligned} E\left[ \left( \sum_{i=1}^{n} (X_i - \bar{X}) U_i \right)^2 \right] &= \sum_{i=1}^{n} (X_i - \bar{X})^2 E U_i^2 + 0 \\ &= \sum_{i=1}^{n} (X_i - \bar{X})^2 \sigma^2 . \end{aligned}$$

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We have

$$Var\left(\hat{\beta}\right) = \left(\frac{1}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}\right)^2 E\left[\left(\sum_{i=1}^{n}\left(X_i - \bar{X}\right)U_i\right)^2\right],$$

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- Conditional on $X_1, \ldots, X_n$

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\begin{aligned}
\hat{\beta} &\sim N\left(E\hat{\beta}, Var\left(\hat{\beta}\right)\right) \\
&\sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}\right).
\end{aligned}
$$